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WAVE FRONT TRACKING IN SYSTEMS OF CONSERVATION LAWS*

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Abstract. This paper contains several recent results about nonlinear systems of hyperbolic conservation laws obtained through the technique of Wave Front Tracking.

Keywords: conservation laws, Wave Front Tracking

MSC 2000: 35L65, 35A35

1. INTRODUCTION

Wave Front Tracking is a set of techniques for constructing approximate solutions to hyperbolic conservation laws in 1 space dimension, i.e. to first order quasilinear systems of partial differential equations of the form

$$\partial_t u + \partial_x f(u) = 0 \tag{1.1}$$

with $f\colon \Omega \to \mathbb{R}^n$ smooth and Ω an open subset of $\mathbb{R}^n$, $n \geqslant 1$, $t \in [0, +\infty[$ and $x \in \mathbb{R}$. Wave Front Tracking was first introduced by Dafermos [46]. Recently, its use has grown thanks to several extensions [10], [13], [15], [18], [25], [26], [36], [38], [57], [58], [64].

These equations state the conservation of the observables described by the densities $u \equiv (u_1, \ldots, u_n)$. More precisely, they state that any variation in time in the quantity of each observable contained in a segment $[a, b]$ is due to the inflow at a and to the outflow at b. In symbols,

$$\int_a^b u(t_2, x)\,\mathrm{d}x - \int_a^b u(t_1, x)\,\mathrm{d}x = \int_{t_1}^{t_2} f(u(t, a))\,\mathrm{d}t - \int_{t_1}^{t_2} f(u(t, b))\,\mathrm{d}t. \tag{1.2}$$

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The paradigm for conservation laws is the Euler system for a compressible non viscous gas that, in Eulerian coordinates, reads

$$\partial_t \begin{bmatrix} \varrho \\ \varrho v \\ \frac{1}{2}\varrho v^2 + \varrho e \end{bmatrix} + \partial_x \begin{bmatrix} \varrho v \\ \varrho v^2 + p \\ v\left(\frac{1}{2}\varrho v^2 + \varrho e + p\right) \end{bmatrix} = 0 \tag{1.3}$$

where ϱ is the mass density, v the gas speed, p the pressure and e the internal energy density. (1.3) is closed by the state equation of the gas considered.

In the isentropic case with Lagrangian coordinates, (1.3) is the p-system

$$\partial_t \begin{bmatrix} \tau \\ v \end{bmatrix} + \partial_x \begin{bmatrix} -v \\ p(\tau) \end{bmatrix} = 0, \tag{1.4}$$

τ being the specific volume and v the (Lagrangian) gas speed. Equation (1.4) provides a very useful computable example of a conservation law.

Other applications of conservation laws deal with traffic flow, chromatography, phase transitions, combustion, ...

Wave Front Tracking is not the only technique to study conservation laws. Other fruitful tools are: Glimm scheme, which provided the first existence proof for global weak solutions to (1.1) [52], [59]; generalized characteristics, which allow to obtain fine properties of solutions [27], [47], [49], [50], [72]; piecewise Lipschitz approximations, which provided the first well-posedness proof when $n > 2$ [28] and compensated compactness, see [63] and the references therein. More recently, an entirely new technique based on viscous approximations appeared in [16].

In the present notes, proofs are omitted. Most of them can be found in several books on conservation laws that recently appeared, see [48], [53], [56], [67] and in particular [22]. Other surveys on this subject are [20], [21], [23].

2. Weak solutions and admissibility conditions

System (1.1) is *strictly hyperbolic* as soon as it satisfies

(SH) For all $u \in \Omega$, $\mathrm{D}f(u)$ admits n real and distinct eigenvalues.

For $i = 1, \dots, n$, $\lambda_i(u)$ is the ith eigenvalue of $\mathrm{D}f(u)$. The eigenvalues are numbered so that $\lambda_{i-1} < \lambda_i$ for $i = 2, \dots, n$. r_i is the right eigenvector corresponding to λ_i. The following stronger (uniform) condition is also of use below:

(USH) For $i = 2, \dots, n$, $\sup_\Omega \lambda_{i-1}(u) < \inf_\Omega \lambda_i(u)$.

Clearly, (USH) $\Rightarrow$ (SH), while the converse holds in general only locally, due to the smoothness of f.

In the *linear* case $f(u) = Au$ (with $A \in \mathbb{R}^{n\times n}$), thanks to (SH), the following procedure allows to obtain a solution $u = u(t,x)$ to the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0,x) = \bar{u}(x) \end{cases} \tag{2.1}$$

for (1.1), with $\bar{u} \in \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$:

1. diagonalise (1.1) obtaining n decoupled equations $\partial_t v_i + \lambda_i \partial_x v_i = 0$;
2. let each component $\bar{v}_i$ of the initial datum translate with speed λ_i and superimpose: $u(t,x) = \sum_i \bar{v}_i(x - \lambda_i t) r_i$.

In the *non*linear case none of the previous steps remains doable:

1. the eigenvectors $r_1, \dots, r_n$ depend on u and in general no decoupling of (1.1) can be achieved;
2. the eigenvalues λ_i depend on u and the above construction may lead to a multivalued function.

As a consequence, the solution to (1.1) may develop singularities independently of the smoothness of the initial datum. Indeed, (1.1) admits the integral formulation (1.2) which is meaningful provided u is merely integrable. The following formulations are more usual.

Definition 2.1. Let $f\colon \Omega \to \mathbb{R}^n$ be smooth and $\bar{u}\colon \mathbb{R} \to \mathbb{R}^n$ be in $\mathbf{L}^1_{\mathrm{loc}}$. A measurable $u\colon [0,+\infty[\times \mathbb{R} \to \Omega$ is a *distributional solution* of (2.1) in Ω if for every $\mathbf{C}^1$ function $\varphi\colon [0,+\infty[\times \mathbb{R} \to \mathbb{R}^n$ with compact support

$$\int_0^{+\infty} \int_{\mathbb{R}} [u(t,x)\partial_t \varphi(t,x) + f(u(t,x))\partial_x \varphi(t,x)] \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} \bar{u}(x)\varphi(0,x) \,\mathrm{d}x = 0.$$

u is a *weak solution* of (2.1) in Ω if

(W1) $u\colon [0,+\infty[\to \mathbf{L}^1_{\mathrm{loc}}(\mathbb{R})$ is continuous in $\mathbf{L}^1_{\mathrm{loc}}$;

(W2) $u(0) = \bar{u}$;

(W3) $\int_0^{+\infty}\int_{\mathbb{R}} [u(t,x)\partial_t \varphi(t,x) + f(u(t,x))\partial_x \varphi(t,x)] \,\mathrm{d}x \,\mathrm{d}t = 0$ for any $\mathbf{C}^1$ map $\varphi\colon]0,+\infty[\times \mathbb{R} \to \mathbb{R}^n$ with compact support.

(Above and in the sequel, we use the fact that any function $u = u(t,x)$ of two variables identifies a map $u = u(t)$ attaining values in a suitable function space.) Note that a distributional solution can be arbitrarily modified at a countable set of times on all $\mathbb{R}$ still remaining a distributional solution. The usual relations between classical and weak solutions apply: any classical solution is also a weak solution while a smooth weak solution is also a classical solution.

The simplest example of a possible non smooth solution to (1.1) is

$$u(t,x) = \begin{cases} u^l & \text{if } x < \Lambda t, \\ u^r & \text{if } x > \Lambda t \end{cases} \tag{2.2}$$

where $u^l, u^r \in \Omega$ and $\Lambda \in \mathbb{R}$. Applying (1.2) or Definition 2.1, we have

Lemma 2.2. *Let $f\colon \Omega \to \mathbb{R}^n$ be smooth. The function u in (2.2) is a weak solution to (1.1) if and only if*

$$\Lambda \cdot (u^l - u^r) = f(u^l) - f(u^r). \tag{2.3}$$

The vector relation (2.3) is known as the *Rankine-Hugoniot condition.* In general, it does not single out a unique solution to (2.1).

Example 2.3. Fix an arbitrary $N \in \mathbb{N}$, $N > 0$ and choose $\omega_0, \dots, \omega_{N+1}$ in $[0,1]$ with $\omega_0 = 0$, $\omega_{N+1} = 1$ and $\omega_{h-1} \leqslant \omega_h$ for $h = 1, \dots, N+1$. Then

$$u(t,x) = \begin{cases} 0 & \text{if } x \in \,]-\infty, \omega_1 t/2[, \\ \omega_n & \text{if } x \in \,](\omega_{h-1} + \omega_h)t/2, (\omega_h + \omega_{h+1})t/2[\quad \text{for } h = 1, \dots, N, \\ 1 & \text{if } x \in \,]\omega_{N+1} t/2, +\infty[\end{cases}$$

is a weak solution to Burgers' equation $\partial_t u + \partial_x(u^2/2) = 0$, since the Rankine-Hugoniot condition (2.3) is satisfied along any jump.

Several criteria have been devised to single out a unique solution to (2.1). One, motivated by physical considerations, is based on the concept of entropy and on the second principle of thermodynamics.

Definition 2.4. A pair of $\mathbf{C^1}$ functions (η, q) with $\eta, q\colon \Omega \to \mathbb{R}$ is an *entropy-entropy flux pair* if $\mathrm{D}\eta(u)\mathrm{D}f(u) = \mathrm{D}q(u)$ for all $u \in \Omega$.

Hence, any smooth solution u to (1.1) satisfies also $\partial_t \eta(u) + \partial_x q(u) = 0$. However, a non smooth solution may violate this latter conservation law, giving a contribution with varying sign along discontinuities.

Definition 2.5. A weak solution u to (1.1) is *entropy admissible* if for any entropy-entropy flux pair (η, q) with η convex, the inequality

$$\partial_t \eta(u) + \partial_x q(u) \leqslant 0 \tag{2.4}$$

holds in distributional sense.

Note that, besides physical systems, entropy-entropy flux pairs certainly exist if $n = 1$ or $n = 2$. Another criterion arises from *viscous* approximations.

Definition 2.6. A weak solution u to (1.1) is *admissible in the sense of viscosity* if there exists a positive sequence ε_n with $\varepsilon_n \to 0$ for $n \to +\infty$ such that the solutions u_n to $\partial_t u_n + \partial_x f(u_n) = \varepsilon_n \partial^2_{xx} u_n$ converge in $\mathbf{L}^1_{\mathrm{loc}}$ to u as $n \to +\infty$.

The next admissibility condition is extremely useful in connection with Wave Front Tracking, but only under the assumption (GNL/LD) below.

Definition 2.7. Let $u\colon [0, +\infty[\times \mathbb{R} \to \Omega$ be a weak solution to (1.1). The solution u satisfies the *Lax entropy inequalities* if for any $(\tau, \xi) \in [0, T] \times \mathbb{R}$ such that there exist states $u^l, u^r \in \Omega$ and a speed $\lambda \in \mathbb{R}$ with

$$\lim_{\varrho \to 0+} \frac{1}{\varrho^2} \int_{\tau-\varrho}^{\tau+\varrho} \int_{\xi-\varrho}^{\xi+\varrho} \|u(t,x) - U(t,x)\| \, \mathrm{d}x \, \mathrm{d}t = 0$$

where

$$U(t,x) = \begin{cases} u^l & \text{if } x - \xi < \lambda \cdot (t - \tau), \\ u^r & \text{if } x - \xi > \lambda \cdot (t - \tau), \end{cases}$$

then, for an $i \in \{1, \ldots, n\}$, $\lambda_i(u^l) \geqslant \lambda \geqslant \lambda_i(u^r)$.

Note that the discontinuities in Example 2.3 do not satisfy Definition 2.7. Conservation laws admit a symmetry group.

Lemma 2.8. *Fix a positive ϱ, constants $\tau, \xi \in \mathbb{R}$, and a function $u\colon [0, +\infty[\times \mathbb{R} \to \Omega$. Define $w(t,x) = u(\tau + \varrho t, \xi + \varrho x)$. Then,*

1. *if u is a weak (or distributional) solution to (1.1), then so is w;*
2. *if u is entropy admissible, then so is w;*
3. *if u is admissible in the sense of viscosity, then so is w.*

The *hyperbolic rescaling* is the transformation $t \to \varrho t$ and $x \to \varrho x$. A function $u\colon [0, +\infty[\times \mathbb{R} \to \mathbb{R}^n$ is *self similar* if for all $\varrho > 0$, $u(\varrho t, \varrho x) = u(t, x)$.

Proposition 2.9. *Fix two functions $u_l, u_r\colon [0, +\infty[\times \mathbb{R} \to \Omega$ and a continuous map $\psi\colon [0, +\infty[\to \mathbb{R}$ such that $\lim_{x \to \psi(t)-} u_l(t,x) = \lim_{x \to \psi(t)+} u_r(t,x)$ for a.e. $t \geqslant 0$ and define*

$$w(t,x) = \begin{cases} u_l(t,x) & \text{if } x < \psi(t), \\ u_r(t,x) & \text{if } x > \psi(t). \end{cases}$$

Then,

1. *if u_l, u_r are weak (distributional) solutions to (1.1) in Ω, then so is w;*
2. *if u_l, u_r are entropy admissible, then so is w.*

Note that 1. above requires also the existence of the limits. If $u_l(t)$, $u_r(t)$ are in $\mathrm{BV}(\mathbb{R})$ for all t, both the limits certainly exist.

Corollary 2.10. *Fix $u_* \in \Omega$, a function $u\colon [0,+\infty[\times \mathbb{R} \to \Omega$ and a continuous map $\psi\colon [0,\infty[\to \mathbb{R}$ such that $\lim_{x\to\psi(t)-} u(t,x) = u_*$ for a.e. $t \geqslant 0$. Define*

$$w(t,x) = \begin{cases} u(t,x) & \text{if } x < \psi(t), \\ u_* & \text{if } x > \psi(t). \end{cases}$$

Then,

1. *if u is a weak (or distributional) solution to (1.1) in Ω, then so is w;*
2. *if u is entropy admissible, then so is w.*

3. The Riemann problem

The Riemann problem for (1.1) is the following particular Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0,x) = \begin{cases} u^l & \text{if } x < 0, \\ u^r & \text{if } x > 0. \end{cases} \end{cases} \tag{3.1}$$

Note that this problem is self similar, in the sense that the hyperbolic rescaling leaves it unchanged.

If $n > 1$, a solution to (3.1) is constructed using the eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $r_1, \dots, r_n$ of $\mathrm{D}f$. The term "*i-characteristic field*" often refers to both the maps $u \mapsto r_i(u)$ and $u \mapsto \lambda_i(u)$. Below, we require that (USH) holds and choose the eigenvectors so that $\|r_i(u)\| = 1$ for all i and u.

Definition 3.1. The i-characteristic field is *genuinely nonlinear* if

$$\nabla\lambda_i(u) \cdot r_i(u) \neq 0$$

for all u. It is *linearly degenerate* if $\nabla\lambda_i(u) \cdot r_i(u) = 0$ for all u.

The following assumption greatly simplifies the necessary techniques:

(GNL/LD) each characteristic field is either *genuinely nonlinear* or *linearly degenerate.*

If the i-characteristic field is genuinely nonlinear, we choose the ith eigenvector oriented so that $\nabla\lambda_i(u) \cdot r_i(u) > 0$ for all $u \in \Omega$. Within the framework of Wave Front

Tracking, the results in [9], [10] allow to relax the above assumption, at the price of heavy technicalities. The recent paper [16], through entirely different and new techniques, proves the well-posedness of (2.1) only under assumption (SH), without even requiring (1.1) being in conservation form.

3.1. Rarefaction waves

Lemma 3.2. *If the i-characteristic field is genuinely nonlinear, then for all $u_o \in \Omega$ there exists a positive σ_o and a smooth curve $\sigma \mapsto R_i(u_o, \sigma)$ defined for $\sigma \in [-\sigma_o, \sigma_o]$ with the properties:*

(R1) $\mathrm{d}/\mathrm{d}\sigma R_i(u_o, \sigma) = r_i(R_i(u_o, \sigma))$;

(R2) $R_i(u_o, 0) = u_o$;

(R3) *σ can be chosen so that $\lambda_i(R_i(u_o, \sigma)) = \lambda_i(u_o) + \sigma$, i.e. $\nabla\lambda_i(u) \cdot r_i(u) = 1$.*

The curve $\sigma \mapsto R_i(u_o, \sigma)$ is the *i-rarefaction* through u_o. It solves the Cauchy problem $\begin{cases} u' = r_i(u), \\ u(0) = u_o. \end{cases}$ The choice of σ to parameterise R_i is arbitrary, (R3) facilitates some estimates.

Proposition 3.3. *Assume that the i-characteristic field is genuinely nonlinear and the i-rarefaction curve is parametrized as in* [(R3), *Lemma* 3.2]. *If there exists a $\sigma_i \in [0, \sigma_o]$ such that $u^r = R_i(u^l, \sigma_i)$, then the function*

$$u(t,x) = \begin{cases} u^l & \text{if } x < \lambda_i(u^l)t, \\ R_i(u^l, \sigma) & \text{if } x = \lambda_i(R_i(u^l, \sigma))t \text{ for } \sigma \in [0, \sigma_i], \\ u^r & \text{if } x > \lambda_i(u^r)t \end{cases} \tag{3.2}$$

1. *is a weak solution to* (3.1);
2. *is continuous and self similar;*
3. *is entropy admissible (see Definition* (2.5)*) with equality in* (2.4), *if* (1.1) *admits an entropy-entropy flux pair;*
4. *is such that for all $t > 0$, the map $x \mapsto u(t,x)$ is Lipschitz with the constant $1/[(\sup_u \|\nabla\lambda_i \cdot r_i\|)t]$.*

The solution (3.2) is a *(centered) rarefaction wave* with *strength* σ_i. Note that if $u^r = R_i(u^l, \sigma)$ with $\sigma < 0$, the above construction (3.2) is not possible. The case $\sigma_i = 0$ of a null rarefaction is considered for completeness.

3.2. Shock waves

Lemma 3.4. *If the i-characteristic field is genuinely nonlinear, then for all $u_o \in \Omega$ there exists a positive σ_o, a smooth curve $\sigma \mapsto S_i(u_o, \sigma)$ defined for $\sigma \in [-\sigma_o, \sigma_o]$ and a function $\Lambda_i(u_o, \cdot)\colon [-\sigma_o, \sigma_o] \to \mathbb{R}$ with the properties*

(S1) $f(S_i(u_o,\sigma)) - f(u_o) = \Lambda_i(u_o,\sigma)\cdot(S_i(u_o,\sigma) - u_o)$;
(S2) $S_i(u_o,0) = u_o$, $\mathrm{d}/\mathrm{d}\sigma S_i(u_o,0) = r_i(u_o)$ *and* $\Lambda_i(u_o,0) = \lambda_i(u_o)$;
(S3) *the parameter* σ *can be chosen so that* $\lambda_i(S_i(u_o,\sigma)) = \lambda_i(u_o) + \sigma$.

The curve $\sigma \mapsto S_i(u_o,\sigma)$ is the *i-shock* through u_o. It is the unique curve of solutions to the Rankine-Hugoniot condition (2.3) exiting u_o tangent to $r_i(u_o)$.

Proposition 3.5. *Assume that the i-characteristic field is genuinely nonlinear and the i-shock curve is parametrized as in* [(S3), *Lemma* 3.4]. *If there exists a* $\sigma_i \in [-\sigma_o, \sigma_o]$ *such that* $u^r = S_i(u^l,\sigma_i)$, *then the function*

$$u(t,x) = \begin{cases} u^l & \text{if } x < \Lambda_i(u^l,\sigma_i)t, \\ u^r & \text{if } x > \Lambda_i(u^l,\sigma_i)t \end{cases} \tag{3.3}$$

1. *is a weak solution to* (3.1);
2. *is self similar;*
3. *is entropy admissible (see Definition* 2.5*), provided* $\sigma \leqslant 0$ *and* (1.1) *admits an entropy-entropy flux pair;*
4. *is such that for all* $t > 0$, *the maps* $x \mapsto u(t,x)$ *and* $x \mapsto \lambda_i(u(t,x))$ *have a jump discontinuity at* $\Lambda_i(u^l,\sigma_i)t$.

The solution (3.3) is a *shock*. To stress the admissibility in the case $\sigma \leqslant 0$, "*entropic shock*" is often used. If $\sigma_i > 0$ then, differently from the case of rarefactions, the solution (3.3) is still well defined and it is a weak solution to (3.1), see Example 2.3. However, it is not entropic. Furthermore, it leads to a construction which is not *consistent*, according to Definition 3.12.

Proposition 3.6. *Assume the i-characteristic field is genuinely nonlinear and the i-shock curve is parametrized as in* [(S3), *Lemma* 3.4]. *The following statements are equivalent:*

1. *the weak solution* (3.3) *is entropy admissible, if* (1.1) *admits an entropy-entropy flux pair;*
2. *the weak solution* (3.3) *is admissible in the sense of viscosity;*
3. $\lambda_i(u^l) > \Lambda_i(u^l,\sigma_i) > \lambda_i(u^r)$;
4. $\sigma_i < 0$.

Remark that the last statement above depends on the choice of the orientation of r_i. Condition 2 is far from immediate, see [67, Chapter 7]. The inequalities at 3 are known as *Lax inequalities*, see Definition 2.7.

3.3. Contact discontinuities

Lemma 3.7. *If the i-characteristic field is linearly degenerate, then for all $u_o \in \Omega$ there exists a positive σ_o and a smooth curve $\sigma \mapsto \mathcal{L}_i(u_o, \sigma)$ defined for $\sigma \in [-\sigma_o, \sigma_o]$ with the properties*

(CD1) $\mathrm{d}/\mathrm{d}\sigma \mathcal{L}_i(u_o, \sigma) = r_i(\mathcal{L}_i(u_o, \sigma))$;

(CD2) $f(\mathcal{L}_i(u_o, \sigma)) - f(u_o) = \lambda_i(u_o) \cdot (\mathcal{L}_i(u_o, \sigma) - u_o)$;

(CD3) $\mathcal{L}_i(u_o, 0) = u_o$;

(CD4) $\lambda_i(\mathcal{L}_i(u_o, \sigma)) = \lambda_i(u_o)$, *for all* $\sigma \in [-\sigma_o, \sigma_o]$;

(CD5) *the arc-length can be chosen as the parameter* σ.

In the linearly degenerate case, shock and rarefaction curves coincide.

Proposition 3.8. *Assume the i-characteristic field is linearly degenerate and the i-shock curve is parametrized as in Lemma 3.7. If there exists a $\sigma_i \in [-\sigma_o, \sigma_o]$ such that $u^r = \mathcal{L}_i(u^l, \sigma_i)$, then the function*

$$u(t,x) = \begin{cases} u^l & \text{if } x < \lambda_i(u^l)t, \\ u^r & \text{if } x > \lambda_i(u^l)t \end{cases} \tag{3.4}$$

1. *is a weak solution to* (3.1);
2. *is self similar;*
3. *is entropy admissible (see Definition* 2.5*) with equality in* (2.4), *if* (1.1) *admits an entropy-entropy flux pair;*
4. *satisfies Lax inequalities (see Definition* 2.7*), with equality signs;*
5. *is such that for all $t > 0$, the map $x \mapsto u(t,x)$ has a jump discontinuity at $\lambda_i(u^l)t$, while the map $x \mapsto \lambda_i(u(t,x))$ is constant.*

3.4. Lax curves

Lemma 3.9. *If the i-characteristic field is genuinely nonlinear, let us parametrize shock and rarefaction curves so that*

$$\begin{aligned} \lambda_i(R_i(u_o,\sigma)) &= \lambda_i(u_o) + \sigma \qquad \partial_\sigma \lambda_i(R_i(u_o,\sigma)) = 1, \\ \lambda_i(S_i(u_o,\sigma)) &= \lambda_i(u_o) + \sigma \qquad \partial_\sigma \lambda_i(S_i(u_o,\sigma)) = 1. \end{aligned} \tag{3.5}$$

For all $u_o \in \Omega$, there exists a constant C such that for all $\sigma \in [-\sigma_o, \sigma_o]$,

$$\|R_i(u_o,\sigma) - S_i(u_o,\sigma)\| \leqslant C \cdot |\sigma|^3. \tag{3.6}$$

Definition 3.10. If the i-characteristic field is either genuinely nonlinear or linearly degenerate, through each $u_o \in \Omega$ let us define the *i-Lax curve*

$$\mathcal{L}_i(u_o, \sigma) = \begin{cases} S_i(u_o, \sigma) & \text{if } \sigma < 0, \\ R_i(u_o, \sigma) & \text{if } \sigma \geqslant 0. \end{cases} \tag{3.7}$$

If the i-field is linearly degenerate, then the i-Lax curve is defined in Lemma 3.7.

Theorem 3.11. *Under assumptions* (SH) *and* (GNL/LD), *any u_o in Ω has a neighbourhood $\mathcal{U}$ such that*

1. (USH) *holds in $\mathcal{U}$;*
2. *for any $\bar{u}$ in $\mathcal{U}$ there exists $\bar{\sigma} > 0$ such that for $i = 1, \dots, n$ the i-Lax curve $\mathcal{L}_i$ through $\bar{u}$ is defined on $[-\bar{\sigma}, \bar{\sigma}]$;*
3. *for any two points u^l, u^r in $\mathcal{U}$ there exists a unique n-tuple $(\sigma_1, \dots, \sigma_n)$ such that there are $n+1$ states $u_0, \dots, u_n$ in Ω satisfying $u_0 = u^l, \dots, u_1 = \mathcal{L}_1(u_0, \sigma_1), \dots, u_i = \mathcal{L}_i(u_{i-1}, \sigma_i), \dots, u_n = u^r$;*
4. *for any two points u^l, u^r in $\mathcal{U}$, the Riemann problem* (3.1) *admits a weak solution obtained as the juxtaposition (for $i = 1, \dots, n$) of*
   4.1. *an i-rarefaction* (3.2), *if the i-field is genuinely nonlinear and $\sigma_i \geqslant 0$;*
   4.2. *an i-shock* (3.3), *if the i-field is genuinely nonlinear and $\sigma_i < 0$;*
   4.3. *an i-contact discontinuity* (3.4), *if the i-field is linearly degenerate;*
5. *the solution so constructed satisfies Lax inequalities, Definition* 2.7 *and, if* (1.1) *admits an entropy-entropy flux pair, is entropy admissible;*
6. *any two weak solutions to* (2.1) *valued in $\mathcal{U}$ and consisting of the juxtaposition of rarefactions, entropic shocks or contact discontinuities, coincide.*

A *Riemann Solver* is a map that with the initial data u^l, u^r in (3.1) associates a self similar weak solution to (3.1), computed at, say, time $t = 1$. The Riemann solver defined in Theorem 3.11 is the *Lax* Riemann solver.

The ordering in the wave speeds induced by (SH) and the Lax inequalities lead to introduce the following property, enjoyed by the Lax Riemann solver.

Definition 3.12. The Riemann solver $\mathcal{R}$ is *consistent* if the following holds:

$$\left.\begin{aligned} \mathcal{R}(u^l, u^m)(\bar{x}) &= u^m \\ \mathcal{R}(u^m, u^r)(\bar{x}) &= u^m \end{aligned}\right\} \Rightarrow \mathcal{R}(u^l, u^r) = \begin{cases} \mathcal{R}(u^l, u^m) & \text{if } x < \bar{x}, \\ \mathcal{R}(u^m, u^r) & \text{if } x \geqslant \bar{x}, \end{cases} \tag{C1}$$

$$\mathcal{R}(u^l, u^r)(\bar{x}) = u^m \Rightarrow \begin{cases} \mathcal{R}(u^l, u^m) = \begin{cases} \mathcal{R}(u^l, u^r) & \text{if } x \leqslant \bar{x}, \\ u^m & \text{if } x > \bar{x}, \end{cases} \\ \mathcal{R}(u^m, u^r) = \begin{cases} u^m & \text{if } x < \bar{x}, \\ \mathcal{R}(u^l, u^r) & \text{if } x \geqslant \bar{x} \end{cases} \end{cases} \tag{C2}$$

Both these properties are enjoyed by the Lax Riemann solver. Essentially, (C1) states that whenever two solutions to two Riemann problems can be placed side by side, then their juxtaposition is again a solution to a Riemann problem, see Fig. 1. Condition (C2) is the vice-versa.

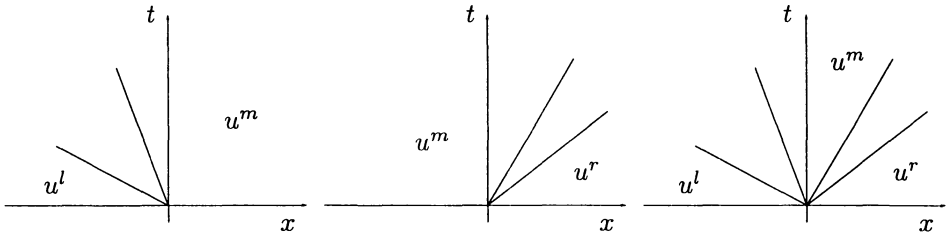

Figure 1. Consistency of a Riemann solver.

4. Existence of solutions to the Cauchy problem

Proposition 2.9 and Definition 3.12 show that solutions to conservation laws can be obtained through a suitable "*gluing*" of other known solutions. The previous section allows to solve Riemann problems. It is thus natural to try to construct solutions to (2.1) through the juxtaposition of solutions to Riemann problems.

Approximate $\bar{u}$ in (2.1) through a piecewise constant function $\bar{u}^\varepsilon$ such that

$$\lim_{\varepsilon \to 0} \|\bar{u} - \bar{u}^\varepsilon\|_{\mathbf{L}^1} = 0. \tag{4.1}$$

It is now natural to proceed by solving the Riemann problems at the points of jump of $\bar{u}$ and gluing the various solutions. However, as soon as a Riemann problem is solved by means of a rarefaction, the approximate solution $u^\varepsilon(t,\cdot)$ to (2.1) so constructed ceases to be piecewise constant at $t = 0+$. Therefore, it is useful to define an ε-approximate Riemann solver $\mathcal{R}^\varepsilon$ so that $\mathcal{R}^\varepsilon(u^l, u^r)$ is piecewise constant. Contact discontinuities and (entropic) shocks are not approximated. Concerning rarefactions, fix $\varepsilon > 0$ and assume that $u^r = \mathcal{L}_i(u^l, \sigma_i)$ with $\sigma_i > 0$. Then, split the rarefaction into $p_i = \lfloor \sigma_i/\varepsilon \rfloor + 1$ waves (here, $\lfloor \xi \rfloor$ is the integer part of ξ) and define the intermediate states $w_j = \mathcal{L}_i(u_o, j\sigma_i/p_i)$ for $j = 0, \dots, p_i$. Finally, define the ε-approximate rarefaction as

$$u^\varepsilon(t,x) = \begin{cases} u^l & \text{if } x < \lambda_i(w_1)t, \\ w_j & \text{if } x \in]\lambda_i(w_j)t, \lambda_i(w_{j+1})t[\text{ and } j = 1, \dots, p_i - 1, \\ u^r & \text{if } x > \lambda_i(u^r)t. \end{cases} \tag{4.2}$$

The ε-approximate Riemann solver $\mathcal{R}^\varepsilon$ can now be defined as the Lax Riemann solver, substituting the exact rarefactions (3.2) by the ε-approximate ones (4.2). By

means of $\mathcal{R}^\varepsilon$, the Riemann problems at the points of jumps of $\bar{u}^\varepsilon$ are approximately solved and an ε-approximate solution $t \mapsto u^\varepsilon(t,\cdot)$ to (2.1) is defined up to the first time, say t_1, at which two discontinuities collide, see Fig. 2. At time t_1, a new Riemann problem arises where the discontinuities meet. Thus, $\mathcal{R}^\varepsilon$ can again be applied and the ε-approximate solution u^ε can be extended up to the next interaction point. Note that this procedure can be applied also if more than 2 waves collide. Two difficulties may stop this construction.

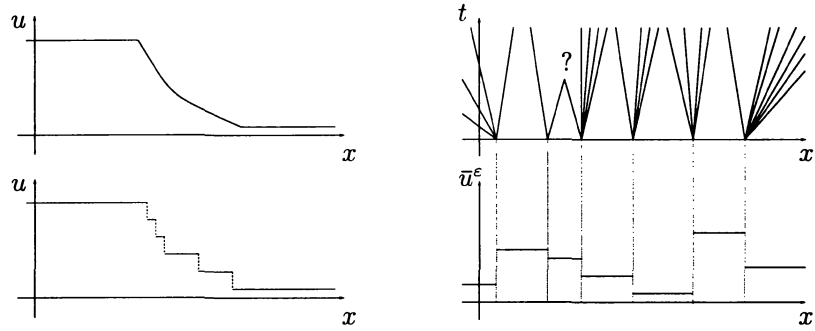

Figure 2. Left, an exact (above) and an approximated (below) rarefaction. Right, the beginning of wave front tracking.

(D1) The solution to a Riemann problem with data in $\mathcal{U}$ need not attain values only in $\mathcal{U}$. Hence the Riemann problems arising in the iteration of the above procedure may well be unsolvable, see Fig. 3, left.

(D2) A Riemann solver may not prolong the ε-solution u^ε, as for instance in case there exists a point (t_*, x_*) such that u^ε suffers a discontinuity at each (t_*, x_n) with $\lim_{n\to+\infty} x_n = x_*$ and $x_n \neq x_*$, see Fig. 3, right.

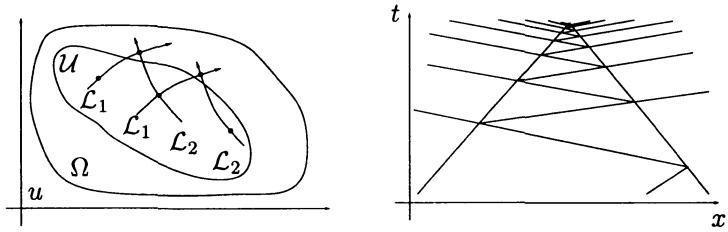

Figure 3. Left, the recursive solution to Riemann problems may lead to exiting $\mathcal{U}$. Right, a cluster point of interaction points.

Suitable *a priori* estimates on the approximate solution allow to ensure that the range of u^ε remains where Riemann problems can be solved. More precisely, an upper bound uniform in ε for the total variation $\mathrm{TV}(u^\varepsilon(t))$ will be obtained through Glimm functionals. This bound, since $u^\varepsilon(t) \in \mathbf{L}^1$ for all t, gives an estimate for the

diameter of $u^\varepsilon(t, \mathbb{R})$. Note also that these bounds also allow to use Helly's theorem to obtain a convergent (sub)sequence of approximate solutions.

On the contrary, (D2) requires technical modifications in the algorithm above. Indeed, a cluster point of interaction points as shown in Fig. 3, right, can indeed arise, see [14], [54] or [48, §13.9].

4.1. Glimm functionals

Write the ε-solution at time t constructed following the above procedure as

$$u^\varepsilon(x) = \sum_\alpha u^\alpha \chi_{]x_\alpha, x_{\alpha+1}]}(x) \quad \text{with} \tag{4.3}$$
$$u^{\alpha+1} = \mathcal{L}^n(\dots \mathcal{L}^i(\dots \mathcal{L}^1(u^\alpha, \sigma_{1,\alpha}), \dots \sigma_{i,\alpha}), \dots \sigma_{n,\alpha}),$$

i.e. $\sigma_{i,\alpha}$ is the (total) size of the i-wave in the solution to the Riemann problem at x_α. For any fixed initial datum $\bar{u}^\varepsilon$, a somewhat "*intrinsic*" measure of $\mathrm{TV}(u^\varepsilon)$ is given by the total strength of waves

$$V = \sum_\alpha \sum_{i=1}^n |\sigma_{i,\alpha}|. \tag{4.4}$$

Note that V is a functional defined on all piecewise constant functions attaining values in the set $\mathcal{U}$ where Riemann problems can be solved.

Let an interaction take place at time t_*. To estimate the variation $\Delta V(t_*) = V(u^\varepsilon(t_*+)) - V(u^\varepsilon(t_*-))$ of V, the following interaction estimates are essential. Remark that interactions take place also in the linear case. But there, waves simply cross each other. In the nonlinear case, any interaction may cause the birth of new waves, see the remarks about Temple systems in Paragraph 4.2.2.

Lemma 4.1. *Assume as in* Fig. 4 *that two waves of different (right) or of the same (left) family interact and that all states on the sides of the discontinuities are in $\mathcal{U}$. Then there exists a constant C such that, in both the cases, the following estimates hold:*

$$|\sigma_i^+ - \sigma_i^-| + |\sigma_j^+ - \sigma_j^-| + \sum_{l \neq i,j} |\sigma_l^+| \leqslant C \cdot |\sigma_i^- \sigma_j^-| \tag{4.5}$$
$$|\sigma_i^+ - (\sigma_i' + \sigma_i'')| + \sum_{l \neq i} |\sigma_l^+| \leqslant C \cdot |\sigma_i' \sigma_i''| \cdot (|\sigma_i'| + |\sigma_i''|).$$

The constant C in (4.5) depends on a compact set containing all states on the sides of the interacting waves. Moreover, if the interacting waves are sufficiently

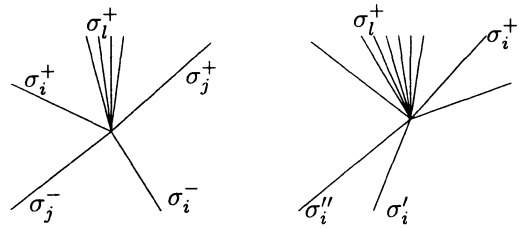

Figure 4. Interactions between waves of different families, left, and of the same family, right.

small, these estimates show that waves do not change sign at interactions. A further consequence of (4.5) is that, in both cases, the function V may well increase at any interaction. In an interaction between waves of different families

$$\Delta V(t_*) = |\sigma_i^+| + |\sigma_j^+| + \sum_{l \neq i,j} |\sigma_l^+| - (|\sigma_i^-| + |\sigma_j^-|) \leqslant C \cdot |\sigma_i^- \sigma_j^-| \tag{4.6}$$

holds, while for interacting waves of the same family we have

$$\Delta V(t_*) = |\sigma_i^+| + \sum_{l \neq i} |\sigma_l^+| - (|\sigma_i'| + |\sigma_i''|) \leqslant C \cdot |\sigma_i' \sigma_i''| \cdot (|\sigma_i'| + |\sigma_i''|).$$

The increase in V is at most *quadratic* in the sizes of the interacting waves.

On the other hand, (SH) ensures that any two waves "*may interact at most once*". Following Glimm [52], we thus introduce the *Glimm interaction potential*

$$Q = \sum_{(\sigma_{i,\alpha}, \sigma_{j,\beta}) \in \mathcal{A}} |\sigma_{i,\alpha} \sigma_{j,\beta}|, \tag{4.7}$$

$\mathcal{A}$ being the set of the approaching waves, i.e. waves that "*may potentially interact*".

Definition 4.2. The waves $\sigma_{i,\alpha}$ and $\sigma_{j,\beta}$ are *approaching* either if $x_\alpha < x_\beta$ and $i > j$, or if $i = j$, the i-field is genuinely nonlinear and $\min\{\sigma_{i,\alpha}, \sigma_{j,\beta}\} < 0$.

The above definition is motivated by the fact that two adjacent rarefactions or contact discontinuities may not interact.

If the interacting waves belong to different families (Fig. 4, left), then

$$\begin{aligned} \Delta Q(t_*) = {} & -|\sigma_i^- \sigma_j^-| + (|\sigma_i^+| - |\sigma_i^-|) \sum_{\sigma_{k,\beta}\,:\,(\sigma_i^+, \sigma_{k,\beta}) \in \mathcal{A}} |\sigma_{k,\beta}| \\ & + (|\sigma_j^+| - |\sigma_j^-|) \sum_{\sigma_{k,\beta}\,:\,(\sigma_j^+, \sigma_{k,\beta}) \in \mathcal{A}} nit|\sigma_{k,\beta}| + \sum_{l \neq i,j} |\sigma_l^+| \sum_{\sigma_{k,\beta}\,:\,(\sigma_l^+, \sigma_{k,\beta}) \in \mathcal{A}} |\sigma_{k,\beta}| \\ \leqslant {} & (-1 + C \cdot V(t_*-)) \cdot |\sigma_i^- \sigma_j^-| \end{aligned} \tag{4.8}$$

and a similar result holds for interacting waves of the same family.

Theorem 4.3. *There exists a positive constant δ such that $B(0,\delta) \subseteq \mathcal{U}$ and*

- (Υ1) *any Riemann problem with data in $B(0,\delta)$ admits a unique ε-solution;*
- (Υ2) *there exists a constant C such that the estimates (4.5) hold whenever the interacting waves separate states in $B(0,\delta)$;*
- (Υ3) *the functional $\Upsilon = V + 3C \cdot Q$ with V as in (4.4) and Q as in (4.7) is such that for any initial datum $\bar{u}^\varepsilon$ satisfying $\Upsilon(\bar{u}^\varepsilon) < \delta$, the functional $t \mapsto \Upsilon(u^\varepsilon(t))$ is non-increasing along the ε-solution constructed above;*
- (Υ4) *Υ is uniformly equivalent to the total variation, i.e. there exists a constant K such that for any piecewise constant function with values in $B(0,\delta)$ we have*

$$\frac{1}{K} \cdot \Upsilon(u) \leqslant \mathrm{TV}(u) \leqslant K \cdot \Upsilon(u).$$

Above, $B(0,\delta)$ is the open sphere centered at u with radius δ.

Remark 4.4. The proof of the above result is here described in the case that at most two waves may interact at a single interaction point (t_*, x_*). In general, this is not true and different ways to bypass this difficulty have been devised. One possibility is to change by an "*arbitrarily small*" quantity the speed of waves so that no more than two waves may interact at a single point, see [22]. This allows to prove the above result, but the approximations so obtained fail to depend Lipschitz continuously upon the initial data, see [25, Example 1].

We have assumed above that at most one interaction takes place at any t_*. The case of more interactions at the same t_* does not require a specific treatment, due to the *finite propagation speed* displayed by (1.1) and by the present algorithm.

4.2. Control on the number of interactions

The usual way to prevent the formation of cluster points of discontinuities (difficulty (D2) above) is to bound the number of interaction points. More precisely, interaction points are proved to be finite on any compact subset of $[0, +\infty[\times \mathbb{R}$.

The bound on the total variation [(Υ3), Theorem 4.3] shows that the number of interaction points can be bounded once the number of *small* waves can be controlled. To this aim, several techniques have been considered. We consider first the "*general case*", i.e. a construction that works under the assumptions

$$\text{(SH)}, \quad \text{(GNL/LD)}, \quad n \geqslant 1, \quad \mathrm{TV}(\bar{u}) \text{ small}.$$

Separately, we consider the case of Temple systems, where a geometrical assumption is required, but neither (GNL/LD) nor $\mathrm{TV}(\bar{u})$ small are necessary. Finally, we present a construction that works only in the case $n = 2$ but can be extended to various other situations, such as systems with phase transitions [36].

4.2.1. The general case

In the case $n \geqslant 1$ with initial data having small total variation, various techniques in literature [13], [22], [53], [64], [65] lead to the suppression of sufficiently small waves. Here, we follow the construction from [22, Chapter 7], see also [23].

The first simplification is achieved avoiding further splitting of rarefactions. When a rarefaction hits a wave of another family, its size slightly varies, see the first estimate in (4.5). Strict application of the above procedure would lead to the splitting of this rarefaction into waves having size at most ε. To reduce the number of waves (and, hence, of interactions) we convene not to split any rarefaction after its birth. Indeed, it holds that the size $\sigma(t)$ of a rarefaction born at time t_o is uniformly bounded, i.e. $|\sigma(t)| \leqslant K\varepsilon$ for all times $t \geqslant t_o$, the constant K depending only on the total variation of the initial data.

Let $\hat{\lambda}$ be an upper bound for all characteristic speeds and let us fix a threshold ϱ with, say, $\varrho \ll \varepsilon$. Use the *Accurate Riemann solver* at time $t = 0$ and whenever the product of the interacting waves is in absolute value greater than ϱ.

When two waves σ_i, σ_j with $|\sigma_i \sigma_j| \leqslant \varrho$ interact, use the following *Simplified Riemann solver*: prolong the incoming waves with waves of the same family and size. Then, introduce a further *non-physical* wave to adjust the states on the right, see Fig. 5, left. More precisely, we have

$$\begin{aligned} &\text{Before:} \quad u_-^m = \mathcal{L}_i(u^l, \sigma_i), \quad u^r = \mathcal{L}_j(u_-^m, \sigma_j); \\ &\text{After:} \quad u_+^m = \mathcal{L}_j(u^l, \sigma_j), \quad u_+^r = \mathcal{L}_i(u_+^m, \sigma_i). \end{aligned}$$

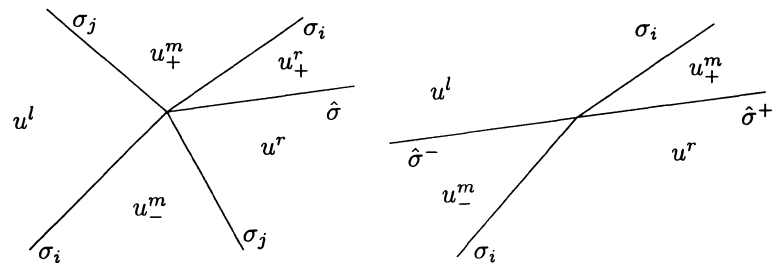

Figure 5. Left, the simplified Riemann solver and a non-physical wave. Right, a non-physical wave hits an i-wave.

The non-physical wave $\hat{\sigma}$ separates the states u_+^r and u^r, is assigned the size $\hat{\sigma} = \|u_+^r - u^r\|$ and is considered to belong to a fictitious linearly degenerate $(n+1)$st family. Moreover, whenever a non-physical wave $\hat{\sigma}$ hits a physical one σ_i, the former proceeds with unchanged size, while the size of the latter needs to be slightly adjusted,

see Fig. 5, right:

$$\begin{aligned} &\text{Before:} && \|u_-^m - u^l\| = \hat{\sigma}^-, \quad u^r = \mathcal{L}_i(u_-^m, \sigma_i); \\ &\text{After:} && u_+^m = \mathcal{L}_i(u^l, \sigma_i), \quad \|u^r - u_+^m\| = \hat{\sigma}^+. \end{aligned}$$

This modification makes the derivation of the bounds on the total variation much more intricate: at any step, various cases need to be considered depending on the nature (physical or non-physical) of the waves considered. A key estimate in this whole procedure is that if ϱ is sufficiently smaller than ε, then the total strength of all non-physical waves is bounded by ε:

$$\sum_{\sigma_\alpha \text{ non-physical}} \|\sigma_\alpha\| = \mathcal{O}(1) \cdot \varepsilon. \tag{4.9}$$

4.2.2. Temple systems

Definition 4.5. System (1.1) is a *Temple system* if the following holds:

(T1) assumption (USH);

(T2) shock and rarefaction curves coincide;

(T3) there exist coordinates w such that $\partial u/\partial w_i$ is the ith right eigenvector of $\mathrm{D}f$.

(In literature, this definition is subject to *variazioni*, see [12], [15], [66], [67], 69]).

A typical property of Temple systems is a sort of decoupling, in the sense that solving any Riemann problem is equivalent to solving n scalar Riemann problems. Besides, in Temple systems interactions have a "*linear*" behaviour. Indeed, if the waves $\sigma_i^1, \dots, \sigma_i^{m_i}$ of the i-family interact, for $i = 1, \dots, n$, the i-waves exiting the interaction have total size $\sigma_i^+ = \sum_{l=1}^{m_i} \sigma_i^l$, using the parametrization defined in Lemmas 3.2, 3.4 and 3.7. Note however that in an interaction involving only 2 waves of different families, the waves that exit the interaction have the same size as those entering it, but possibly different speed. Moreover, if no i-wave enters the interaction point, then no i-wave exits it.

As a consequence, the Wave Front Tracking algorithm in [12], [15] defines solutions attaining values on a *fixed* grid with mesh size ε. Hence, no wave can have size smaller than ε. This, together with a careful use of the decrease of Q, allows to prove that there is a *finite* number of waves on all $[0, +\infty[\times \mathbb{R}$. Another consequence of the introduction of the ε-grid is that all the functionals V, Q and Υ formally depend also on ε.

The results in [15] allow to remove (GNL/LD) in Temple systems. Morever, the total variation of the initial data is not required to be small.

4.2.3. 2×2 systems

In the case $n = 2$, a careful definition of the approximate solution implies that all adjacent small waves of the same family are parallel. Indeed, when small waves are involved, the flow f is approximated essentially through a Temple system and adjacent waves of the same family having size σ with $|\sigma| < \varepsilon$ may be parallel. In this approximation, it is essential to substitute the Lax curves (3.7) by the approximations (see [25])

$$\mathcal{L}_i^\varepsilon(u_o, \sigma) = \begin{cases} S_i(u_o, \sigma) & \text{if } \sigma < -2\sqrt{\varepsilon}, \\ \varphi\Big(\dfrac{\sigma}{\sqrt{\varepsilon}}\Big) S_i(u_o, \sigma) + \Big(1 - \varphi\Big(\dfrac{\sigma}{\sqrt{\varepsilon}}\Big)\Big) R_i(u_o, \sigma), & \\ & \text{if } \sigma \in [-2\sqrt{\varepsilon}, -\sqrt{\varepsilon}], \\ R_i(u_o, \sigma) & \text{if } \sigma > -\sqrt{\varepsilon} \end{cases} \tag{4.10}$$

with φ being a $\mathbf{C}^\infty$ function such that

$$\begin{cases} \varphi(\sigma) = 1 & \text{if } \sigma \in]-\infty, -2], \\ \varphi'(\sigma) \in [-2, 0] & \text{if } \sigma \in]-2, -1[, \\ \varphi(\sigma) = 0 & \text{if } \sigma \in [-1, +\infty[. \end{cases}$$

The ε-approximate rarefactions are then obtained cutting the rarefactions along a fixed grid of size ε. Note that also all shocks with size $|\sigma| < 2\sqrt{\varepsilon}$ are approximated, due to the interpolation (4.10), and (3.6) provides bounds for this error.

The speeds of these waves are assigned so that the coordinates along the approximate rarefactions (4.10) are *exact* solutions to scalar conservation laws with piecewise linear flux function, see [25, Section 2] for the explicit formulæ. Hence, adjacent small waves propagate parallelly and less interactions take place.

With this construction, it is then possible to show directly that no compact set may contain a cluster point of interaction points. This step depends essentially on the assumption $n = 2$.

An inductive procedure allows to extend estimates of the type (4.5) to the case of multiple interactions. Hence, the "*arbitrarily small*" change of the wave speeds to avoid multiple interaction is not necessary and the approximate solutions depend Lipschitz continuously on the initial data.

Note that when the initial data is a perturbation of a *large* jump, then the number of interactions may well be infinite over all $[0, +\infty[\times \mathbb{R}$, see [26].

If (1.1) is a Temple system, the present algorithm provides the same solutions as those specifically constructed for Temple systems, the only difference being that here approximate solutions do not need take values in a fixed ε-grid.

4.3. The limit

Before passing to the limit for $\varepsilon \to 0$, we estimate how far the ε-approximate solution is from being an exact solution to (2.1). First, there is an error due to the initial datum, which vanishes as $\varepsilon \to 0$ thanks to (4.1).

Then, another error is due to the fact that not all discontinuities in u^ε satisfy the Rankine-Hugoniot conditions [(S1), Lemma 3.4]. In the general case (Paragraph 4.2.1), rarefaction waves and non-physical waves violate it. Let $\Delta u^\varepsilon(t, x_\alpha) = u^\varepsilon(t, x_\alpha+) - u^\varepsilon(t, x_\alpha-)$ and define $\Delta f(u^\varepsilon(t, x_\alpha))$ similarly. The Rankine-Hugoniot conditions along a rarefaction wave supported at x_α with propagation speed $\dot{x}_\alpha$ are missed with an error

$$|\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon \cdot \sigma_\alpha,$$

σ_α being the rarefaction at x_α. This second order estimate is a consequence of (3.6). Summing over all rarefactions, if $\Upsilon(\bar{u}) < \delta$, the total error is

$$\sum_{\sigma_\alpha \text{ rarefaction}} |\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon \cdot \delta$$

and converges to 0 as $\varepsilon \to 0$. If a non-physical wave is supported at x_α, then

$$|\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot |\sigma_\alpha|$$

and the bound is now only of the first order. On the other hand, when summing over all non-physical waves, by (4.9) we obtain

$$\sum_{\sigma_\alpha \text{ non-physical}} |\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon.$$

The entropy inequalities are yet another source of error: neither approximate rarefactions nor non-physical waves satisfy them. The corresponding bounds are similar to those for Rankine-Hugoniot conditions, see [4] and [5, Theorem 5.1].

Before the final limit, note that ε-solutions are Lipschitz with respect to t.

Lemma 4.6. *With the above definitions of ε-solution, there exist constants δ, L such that for all piecewise constant initial data $u \in \mathbf{L^1}$ with $\Upsilon(u) < \delta$, the ε-solution u^ε satisfies $\|u^\varepsilon(t'') - u^\varepsilon(t')\|_{\mathbf{L^1}} \leqslant L \cdot |t'' - t'|$ uniformly in ε.*

Above, L depends on δ and on the maximal characteristic speed.

The existence of solutions to (2.1) is now at hand. Theorem 4.3 provides a bound uniform in ε on the total variation of the ε-approximate solution $u^\varepsilon(t, \cdot)$ at any time

$t \geqslant 0$. Fix a positive sequence ε_n with $\lim_{n\to+\infty} \varepsilon_n = 0$. A slight modification of the classical Helly's Compactness Theorem [22, Theorem 2.4] allows to extract from ε_n a subsequence ε_{n_k} such that there exists a function $u\colon [0,+\infty[\to \mathbf{L^1}(\mathbb{R})$ with the properties

1. $u^{\varepsilon_{n_k}} \to u$ in $\mathbf{L^1_{loc}}([0,+\infty[\times \mathbb{R})$ as $k \to +\infty$;
2. $\|u(t'') - u(t')\|_{\mathbf{L^1}} \leqslant L \cdot |t'' - t'|$;
3. $\mathrm{TV}(u(t)) \leqslant K\delta$ with K as in [(Υ4), Theorem 4.3].

The above properties lead to the proof that u is a weak entropic solution.

Remark 4.7. The existence of solutions was obtained by means of a *compactness* argument. Hence, neither uniqueness nor continuous dependence are directly available through the same method.

In Temple systems, the limiting procedure is somewhat simpler, since non-physical waves are absent. In the 2×2 case, also small shocks cause some error, due to the interpolation (4.10), and need to be considered separately.

5. Stability

This section is devoted to the proof of continuous dependence on the initial data. In the case of conservation laws, this proof usually preceeds that of uniqueness. For the stability with respect to the flow f, see [17].

The results in the previous sections prove the following theorem.

Theorem 5.1. *Let f be smooth and satisfy* (SH), (GNL/LD). *Then there exist positive δ, K, L and, for all small ε, a map $S^\varepsilon\colon [0,+\infty[\times \mathcal{D}^\varepsilon \to \mathcal{D}^\varepsilon$ satisfying*

1. *the domain $\mathcal{D}^\varepsilon = \{u \in \mathbf{L^1}\colon u$ piecewise constant and $\Upsilon(u) < \delta\}$ is invariant with respect to S^ε;*
2. *$\mathcal{D}^\varepsilon \supseteq \{u \in \mathbf{L^1}\colon u$ piecewise constant and $\mathrm{TV}(u) < K\delta\}$;*
3. *if $n = 2$ or if* (1.1) *is a Temple system, then S^ε is a semigroup;*
4. *along a physical discontinuity at, say, $x_\alpha(t)$, the Rankine-Hugoniot condition* [(S1), *Lemma* 3.4] *is approximately satisfied:*

$$|\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon \cdot |\sigma_\alpha|,$$

while along a non-physical discontinuity we have

$$|\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot |\sigma_\alpha|;$$

5. *if an entropy-entropy flux pair exists along a physical discontinuity at* $x_\alpha(t)$, *the error in the entropy condition* (2.4) *is bounded by*

$$|\dot{x}_\alpha \cdot \Delta\eta(u^\varepsilon(t, x_\alpha)) - \Delta q(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon \cdot |\sigma_\alpha|,$$

while in the case of a nonphysical discontinuity we have

$$|\dot{x}_\alpha \cdot \Delta\eta(u^\varepsilon(t, x_\alpha)) - \Delta q(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot |\sigma_\alpha|;$$

6. *the total size of non-physical waves is bounded as in* (4.9);
7. S^ε *is* $\mathbf{L^1}$*-Lipschitz with respect to time:* $\|S^\varepsilon_{t_1} u - S^\varepsilon_{t_2} u\|_{\mathbf{L^1}} \leqslant L \cdot \|t_2 - t_1\|$.

Note that in the general case, due to the presence of the Simplified Riemann solver, S^ε is *not* a semigroup. Indeed, assume that an interaction takes place at time t_*. The definition of S^ε after t_* depends on the *history* before t_*. In the general case, the semigroup property is recovered in the limit as a consequence of the uniqueness of the limit of the approximate semigroups.

Our next target is the regularity of S^ε as a function of the initial data. Two entirely different techniques are available, the former in the general case, while the latter works for Temple and 2×2 systems.

5.1. The functional

This section follows the ideas introduced in [34], [60], [61], [62].

The limit semigroup is proved Lipschitz once a functional $\Phi\colon \mathcal{D}^\varepsilon \times \mathcal{D}^\varepsilon \to [0, +\infty[$ with the following properties, uniform in ε, is introduced:

(Φ1) for a suitable positive κ, $1/\kappa \cdot \|u - w\|_{\mathbf{L^1}} \leqslant \Phi(u, w) \leqslant \kappa \cdot \|u - w\|_{\mathbf{L^1}}$;

(Φ2) for any two ε-solutions u and w, $\Phi(u(t), w(t)) \leqslant \Phi(u(0), w(0)) + \mathcal{O}(1) \cdot \varepsilon \cdot t$.

To define Φ, introduce first a sort of distance in $\mathcal{U}$ by means of (1.1). In the next lemma and in all this paragraph, we use the term "*i-shock*" also for the i-Lax curves of linearly degenerate families. Moreover, also the non entropic parts of shock curves are used due to the interpolation (4.10).

Lemma 5.2. *Any* $\bar{u} \in \Omega$ *admits a neighborhood* $\mathcal{U}$ *such that*

1. *the conclusions of Theorem* 3.11 *hold in* $\mathcal{U}$;
2. *for any two points* u, w *in* $\mathcal{U}$, *there exists a unique* n*-tuple* $(q_1, \dots, q_n)$ *such that there are* $n+1$ *states* $u_0, \dots, u_n$ *in* Ω *satisfying* $u_0 = u, u_1 = \mathcal{L}_1(u_0, \sigma_1), \dots, u_i = S_i(u_{i-1}, q_i), \dots, u_n = w$;
3. *there exists a positive* c *such that* $1/c \cdot \|w - u\| \leqslant \sum_{i=1}^{n} |q_i| \leqslant c \cdot \|w - u\|$.

In other words, the q_is are the sizes of the i-shocks that solve the Riemann problem (3.1) with $u^l = u$ and $u^r = w$ having the minimum number of (possibly non

entropic) shocks. The last statement above amounts to say that the q_is provide a measure of the distance in $\mathcal{U}$ equivalent to the Euclidean one. However, setting $d(u,w) = \sum_i |q_i|$, d is not a distance for, in general, since neither the triangular inequality nor the symmetric property $d(u,w) = d(w,u)$ hold.

Theorem 5.3. *Let f satisfy* (SH) *and* (GNL/LD). *Let $q_1(x), \dots, q_n(x)$ be the shock sizes defined in Lemma* 5.2 *with reference to $u(x)$ and $w(x)$, for $u, w \in \mathcal{D}^\varepsilon$. Reducing the δ in* [1, *Theorem* 5.1], *if necessary, it is possible to define weights $W_1, \dots, W_n \colon \mathbb{R} \to [0, +\infty[$ so that*

$$\Phi(u,w) = \int_{\mathbb{R}} \sum_{i=1}^{n} |q_i(x)| W_i(x)\, \mathrm{d}x$$

satisfies (Φ1) *and* (Φ2).

Note that (Φ1) holds as soon as the weights W_i are uniformly bounded, say $W_i(x) \in [1,2]$ for all x. All difficulties in the proof of the well-posedness of (1.1) are thus reduced to the search for these weights, whose explicit definition is far from immediate. For any u in $\mathcal{D}^\varepsilon$, written as in (4.3), let $\mathcal{J}(u)$ be the set of jumps in u. Define, for $i = 1, \dots, n$,

$$W_i(x) = 1 + \kappa_1 A_i(x) + \kappa_1 \kappa_2 (Q(u) + Q(w)) \tag{5.1}$$

and, if the i-field is linearly degenerate, set

$$\begin{aligned} A_i(x) = {} & \sum \{ |\sigma_{j,\alpha}| \colon \alpha \in \mathcal{J}(u) \cup \mathcal{J}(w),\ x_\alpha < x,\ j \in \{i+1, \dots, n\} \} \\ & + \sum \{ |\sigma_{j,\alpha}| \colon \alpha \in \mathcal{J}(u) \cup \mathcal{J}(w),\ x_\alpha > x,\ j \in \{1, \dots, i-1\} \} \end{aligned}$$

where $\sum \{\sigma \colon \sigma \in A\} = \sum_{\sigma \in A} \sigma$. If the i-field is genuinely nonlinear, let

$$\begin{aligned} A_i(x) = {} & \sum \{ |\sigma_{j,\alpha}| \colon \alpha \in \mathcal{J}(u) \cup \mathcal{J}(w),\ x_\alpha < x,\ j \in \{i+1, \dots, n\} \} \\ & + \sum \{ |\sigma_{j,\alpha}| \colon \alpha \in \mathcal{J}(u) \cup \mathcal{J}(w),\ x_\alpha > x,\ j \in \{1, \dots, i-1\} \} \\ & + \begin{cases} \sum \{ |\sigma_{i,\alpha}| \colon \alpha \in \mathcal{J}(u),\ x_\alpha < x \} + \sum \{ |\sigma_{i,\alpha}| \colon \alpha \in \mathcal{J}(w),\ x_\alpha > x \}, \\ \qquad\qquad \text{if } q_i(x) < 0, \\ \sum \{ |\sigma_{i,\alpha}| \colon \alpha \in \mathcal{J}(u),\ x_\alpha > x \} + \sum \{ |\sigma_{i,\alpha}| \colon \alpha \in \mathcal{J}(w),\ x_\alpha < x \} \\ \qquad\qquad \text{if } q_i(x) > 0. \end{cases} \end{aligned}$$

Note that non-physical waves have no role in the construction of the weights.

With this definition, Theorem 5.3 can be proved. The particular above choice of the weights is partly motivated as follows. Rewrite Φ as

$$\begin{aligned}\Phi(u(t),w(t)) = &\int_{\mathbb{R}}\left(\sum_{j=1}^{n}|q_j(x)|\right)\mathrm{d}x\\ &+\sum_{\sigma_{i,\alpha}\in\mathcal{J}(u)\cup\mathcal{J}(w)}\sum_{j=1}^{n}|\sigma_{i,\alpha}|\int_{\mathcal{I}_{i,\alpha}^{j}}|q_j(x)|\,\mathrm{d}x\\ &+\kappa_1\kappa_2(Q(u)+Q(w))\int_{\mathbb{R}}\left(\sum_{j=1}^{n}|q_j(x)|\right)\mathrm{d}x\end{aligned}$$

where $\mathcal{I}_{i,\alpha}^{j}$ is the subset of $\mathbb{R}$ where $q_j(x)$ approaches $\sigma_{i,\alpha}$ in the sense of Definition 4.2. The first term is essentially the $\mathbf{L}^1$ norm, see [3, Lemma 5.2]. The last term exploits the fact that the interaction potential decreases and is used to ensure the decrease of Φ at interaction times. The second term is the key point. Call u_j and w_j the coordinate of u and w along the j-Lax curve. Each summand in this sum is the size of a wave $\sigma_{i,\alpha}$ in u or in w, multiplied by the area selected by u_j and w_j over $\mathcal{I}_{i,\alpha}^{j}$. This set, a union of intervals, is defined as the place where $q_j(x)$ approaches $\sigma_{i,\alpha}$ in the sense of Definition 4.2.

Consider for example the jth summand in the first sum defining A_i, in the case $i > j$ with the i-field genuinely nonlinear. Then $\int_{\mathcal{I}_{i,\alpha}^{j}}|q_j(x)|\,\mathrm{d}x$ is a measure of the area between u_j and w_j to the right of x_α. Due to (SH), this area decreases in time, see Fig. 6, since $\lambda_i > \lambda_j$. For analogous geometric interpretations of the other summands in A_i, see [23, Chapter 5].

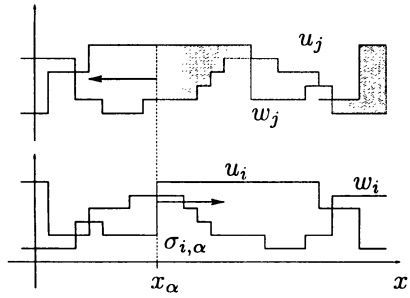

Figure 6. The dashed area between u_j and w_j to the right of x_α diminishes.

Technically, to achieve the proof of the key estimate ($\Phi 2$), the first step is taking the derivative of $\Phi(u(t),w(t))$. Denote $q_i^{\alpha\pm} = q_i(x_\alpha\pm)$ and similarly for W_i, while $\dot{x}_\alpha$ denotes the speed of the wave at x_α (at each x_α there is a unique such wave outside

interaction times). Then

$$\frac{\mathrm{d}\Phi(u(t),w(t))}{\mathrm{d}t} = \sum_{\alpha}\sum_{i=1}^{n}\left(|q_i^{\alpha-}|W_i^{\alpha-} - |q_i^{\alpha+}|W_i^{\alpha+}\right)\cdot\dot{x}_\alpha$$
$$= \sum_{i,\alpha}\left(|q_i^{\alpha+}|W_i^{\alpha+}(\lambda_i^{\alpha+} - \dot{x}_\alpha) - |q_i^{\alpha-}|W_i^{\alpha-}(\lambda_i^{\alpha-} - \dot{x}_\alpha)\right)$$

where $\lambda_i^{\alpha\pm} = \lambda_i(u(x_\alpha\pm))$. In the last line above, the equality $|q_i^{\alpha-}|W_i^{\alpha-}\lambda_i^{\alpha-} = |q_i^{(\alpha-1)+}|W_i^{(\alpha-1)+}\lambda_i^{(\alpha-1)+}$ was used. Then, upper bounds on this quantity are derived through suitable interaction estimates. Indeed, by considering various cases, one shows that for any physical wave $\sigma_{i,\alpha}$, for any j and α,

$$|q_j^{\alpha+}|W_j^{\alpha+}(\lambda_j^{\alpha+} - \dot{x}_\alpha) - |q_j^{\alpha-}|W_j^{\alpha-}(\lambda_j^{\alpha-} - \dot{x}_\alpha) = \mathcal{O}(1)\cdot\varepsilon\cdot|\sigma_{i,\alpha}|$$

while if $\sigma_{i,\alpha}$ is a non-physical wave

$$|q_j^{\alpha+}|W_j^{\alpha+}(\lambda_j^{\alpha+} - \dot{x}_\alpha) - |q_j^{\alpha-}|W_j^{\alpha-}(\lambda_j^{\alpha-} - \dot{x}_\alpha) = \mathcal{O}(1)\cdot|\sigma_{i,\alpha}|.$$

Summing over all i and α, we get $\mathrm{d}\Phi(u(t),w(t))/\mathrm{d}t \leqslant c\cdot\varepsilon\cdot\delta$ for a $c > 0$ and for all times when no interaction occur. A suitable choice of κ_2 in (5.1) implies that $\Delta\Phi(u(t),w(t)) \leqslant 0$ at any interaction time, completing the proof of (Φ2).

Concerning the final limiting procedure, choose a positive sequence ε_ν with $\varepsilon_\nu \to 0$ as $\nu \to +\infty$ and approximate the initial data $\bar{u}$ by means of a sequence $\bar{u}_\nu$ with $\lim_{\nu\to+\infty}\|\bar{u} - \bar{u}_\nu\|_{\mathbf{L}^1} = 0$. Then, by ($\Phi$1) and ($\Phi$2),

$$\|S_t^{\varepsilon_{\nu_1}}\bar{u}_{\nu_1} - S_t^{\varepsilon_{\nu_2}}\bar{u}_{\nu_2}\|_{\mathbf{L}^1} \leqslant \kappa\cdot\Phi(S_t^{\varepsilon_{\nu_1}}\bar{u}_{\nu_1}, S_t^{\varepsilon_{\nu_2}}\bar{u}_{\nu_2})$$
$$\leqslant \kappa\cdot\Phi(\bar{u}_{\nu_1},\bar{u}_{\nu_2}) + \mathcal{O}(1)\cdot\max\{\varepsilon_{\nu_1},\varepsilon_{\nu_2}\}\cdot t$$
$$\leqslant \kappa^2\cdot\|\bar{u}_{\nu_1} - \bar{u}_{\nu_2}|_{\mathbf{L}^1} + \mathcal{O}(1)\cdot\max\{\varepsilon_{\nu_1},\varepsilon_{\nu_2}\}\cdot t,$$

which shows the existence of the limit by a completeness argument.

Remark 5.4. The proof of the Lipschitz dependence of the approximate solution allows also to obtain an existence proof entirely independent of the previous one. Moreover, the compactness argument is now substituted by a completeness argument. These observations apply also to the other technique exposed below.

5.2. Pseudopolygons

In the present section, we follow [25].

In Temple and 2×2 systems, the ε-approximate semigroups are $\mathbf{L^1}$-Lipschitz with a Lipschitz constant bounded uniformly in ε. To prove it, any two initial data u,

w in $\mathcal{D}^\varepsilon$ are interpolated through a curve γ and the whole curve evolves with S^ε, leading to $S_t^\varepsilon \circ \gamma$. An estimate of $\|S_t^\varepsilon u - S_t^\varepsilon w\|_{\mathbf{L}^1}$ is obtained through the length of $S_t^\varepsilon \circ \gamma$. In turn, this is achieved by means of an estimate of how the vector tangent to $S_t^\varepsilon \circ \gamma$ varies with time.

The key idea is to define γ shifting the locations of the jumps in the initial data u at constant rates, and then studying the rates at which the jumps in the corresponding solution $S_t^\varepsilon u$ are shifted, for any fixed $t > 0$.

Definition 5.5. Let $]a, b[$ be an open interval. An *elementary path* is a map $\gamma\colon]a, b[\to \mathcal{D}^\varepsilon$ of the form $\gamma(\theta) = \sum_{\alpha=1}^{N} \omega_\alpha \cdot \chi_{]x_{\alpha-1}^\theta, x_\alpha^\theta]}$, where $x_\alpha^\theta = \bar{x}_\alpha + \xi_\alpha \theta$ with $x_{\alpha-1}^\theta < x_\alpha^\theta$ for all $\theta \in]a, b[$ and $\alpha = 1, \dots, N$.

For each θ, $\gamma(\theta)$ is piecewise constant with bounded support. As θ varies, the values ω_α remain constant while the locations of the jumps x_α^θ shift with constant speeds ξ_α leaving the ordering of the x_α^θ unchanged.

Definition 5.6. A continuous map $\gamma\colon [a, b] \to \mathcal{D}^\varepsilon$ is a *pseudopolygon* if there exist countably many disjoint open intervals $J_h \subset [a, b]$ such that the restriction of γ to each J_h is an elementary path and the set $[a, b] \setminus \bigcup_{h \geqslant 1} J_h$ is countable.

Every couple of initial conditions u, w in $\mathcal{D}^\varepsilon$ can be joined by a pseudopolygon. A remarkable property of the algorithm described in Paragraphs 4.2.2 and 4.2.3 is that it preserves pseudopolygons.

Proposition 5.7. *Let γ be a pseudopolygon. Then, for all $t > 0$, the path $S_t^\varepsilon \circ \gamma$ is also a pseudopolygon. Furthermore, there exist countably many open intervals J_h such that $[a, b] \setminus (\bigcup_h J_h)$ is countable and for any h, the functions $(S_t^\varepsilon \circ \gamma)(\theta)$, $\theta \in J_h$, all have the same number of waves, interacting at the same number of points in $[0, t] \times \mathbb{R}$. As θ varies, these waves and the interaction points are shifted with constant speeds in the (t, x)-plane.*

The $\mathbf{L}^1$-length of a pseudopolygon γ is the sum of the lengths of the elementary paths obtained by restricting γ to each subinterval J_h. It is thus sufficient to study the case where γ is an elementary path and the wave configuration of the solution $S_t^\varepsilon \circ \gamma$ on the strip $[0, T] \times \mathbb{R}$ is fixed for all $\theta \in]a, b[$. For a fixed $t \in [0, T]$, let $u^\theta(t) = (S_t^\varepsilon \circ \gamma)(\theta)$ have the form $u^\theta(t, x) = \sum_{\alpha=1}^{N} u^\alpha \chi_{]x_{\alpha-1}^\theta(t), x_\alpha^\theta(t)]}(x)$ where $x_\alpha^\theta(t) = \bar{x}_\alpha(t) + \xi_\alpha \theta$. The $\mathbf{L}^1$-length of the path $S_t^\varepsilon \circ \gamma$ is then measured by $\|S_t^\varepsilon \circ \gamma\| = \int_a^b \sum_\alpha |\partial_\theta x_\alpha^\theta| \, \|\Delta u^\theta(x_\alpha)\| \, \mathrm{d}\theta$. In order to relate the length of $S_t^\varepsilon \circ \gamma$ with the length of the path γ interpolating the initial conditions, for any given $\theta \in]a, b[$ we study how

the sum $\sum_\alpha |\partial_\theta x_\alpha^\theta| \, \|\Delta u^\theta(x_\alpha)\|$ varies in time. Clearly, this sum can change only at those times where an interaction takes place.

Consider, for instance, an interaction as in Fig. 4, left, between two waves σ_j^-, σ_i^- located on the lines $x_j^{\theta-}(t) = x_j^-(t) + \xi_j^-\theta$ and $x_i^{\theta-}(t) = x_i^-(t) + \xi_i^-\theta$. As θ varies, the waves before the interaction shift at the rates $\xi_j^- = \partial_\theta x_j^{\theta-}(t)$, $\xi_i^- = \partial_\theta x_i^{\theta-}(t)$. Assume that the interaction produces n_k waves of the k-family for $k = 1, \dots, n$, having sizes $\sigma_{k,1}^+, \dots, \sigma_{k,n_k}^+$. The interaction point $P^\theta = (\bar{t}^\theta, \bar{x}^\theta)$ shifts at a constant rate, as do the locations of the outgoing waves, say $x_{k,\ell}^{\theta+}(t) = x_{k,\ell}^+(t) + \xi_{k,\ell}^+\theta$ for $\ell = 1, \dots, n_k$ and $k = 1, \dots, n$. The next lemma provides the basic estimate on strengths and shift rates of waves before and after an interaction, generalizing (4.5) to the case of shifting interactions.

Lemma 5.8. *There exists a constant C independent of ε such that, whenever two waves interact, the quantities introduced above satisfy*

$$\sum_{k=1}^{n} \sum_{\ell=1}^{n_k} |\sigma_{k,\ell}^+ \xi_{i,\ell}^+| \leqslant |\sigma_i^- \xi_i^-| + |\sigma_j^- \xi_j^-| + C \cdot |\sigma_i^- \sigma_j^-| (|\xi_i^-| + |\xi_j^-|).$$

The $\mathbf{L^1}$-length of $S_t^\varepsilon \circ \gamma$ may well increase in time. Lemma 5.8 allows to control this increase by means of an interaction potential. Indeed, we will introduce on $\mathcal{D}^\varepsilon$ a metric equivalent to the $\mathbf{L^1}$-distance and such that with respect to it, the semigroup S^ε turns out to be contractive.

We define the *weighted length* of an elementary path $\gamma\colon\ ]a, b[\to \mathcal{D}^\varepsilon$ by

$$\|\gamma\|_\varepsilon = \sum_\alpha \sum_{i=1}^{n} (b-a) \cdot \|\sigma_{i,\alpha} \xi_{i,\alpha}| \cdot W_{i,\alpha}$$

for suitable weights $W_{i,\alpha}$. In the more general case where γ is a pseudopolygon, we define its *weighted length* $\|\gamma\|_\varepsilon$ as the sum of the weighted lengths of its elementary paths. As soon as the weigths $W_{i,\alpha}$ satisfy uniform bounds

$$1 \leqslant W_{i,\alpha} \leqslant c, \tag{5.2}$$

it is immediate to prove the equivalence between the $\mathbf{L^1}$-length and the weighted length, i.e. there exists a constant $\tilde{c}$ such that $1/\tilde{c} \cdot \|\gamma\|_{\mathbf{L^1}} \leqslant \|\gamma\|_\varepsilon \leqslant \tilde{c} \cdot \|\gamma\|_{\mathbf{L^1}}$.

The next proposition contains the key difficulties of the present approach, namely the definition of the weights $W_{i,\alpha}$.

Proposition 5.9. *There exist constants δ and c such that for all small ε, for any elementary path $\gamma\colon [a,b] \to \mathcal{D}^\varepsilon$ with $S_t^\varepsilon \circ \gamma$ having a fixed wave configuration on $[0,T]$, there exist weights $W_{i,\alpha}$ satisfying* (5.2) *uniformly in ε and such that the map $t \mapsto \|S_t^\varepsilon \circ \gamma\|_\varepsilon$ is non-increasing for $t \in [0,T]$.*

Remark that in the case $n = 2$, the weights $W_{i,\alpha}$ can be explicitely defined, by quite different expressions depending on the specific situation, see [25], [26], [36], [38], [44]. On the contrary, in the case of Temple systems with $n > 2$, only an implicit (backward) recursive construction is currently available [12], [15].

On $\mathcal{D}^\varepsilon$ define now the metric

$$d_\varepsilon(u,w) = \inf\{\|\gamma\|_\varepsilon\colon\ \gamma \text{ a pseudopolygon in } \mathcal{D}^\varepsilon \text{ joining } u \text{ with } w\}. \tag{5.3}$$

Note that d_ε does not fit in the set of metrics considered in [70]. Apply now Proposition 5.9 to obtain the following lemma.

Lemma 5.10. *There exists a $\delta > 0$ such that, restricted to $\mathcal{D}^\varepsilon$, the distance d_ε in* (5.3) *is uniformly equivalent to the $\mathbf{L^1}$-distance. Moreover, S^ε is contractive with respect to d_ε: $d_\varepsilon(S_t^\varepsilon u, S_t^\varepsilon w) \leqslant d_\varepsilon(u,w)$ for all $t \geqslant 0$ and all u, w in $\mathcal{D}^\varepsilon$.*

In terms of the $\mathbf{L^1}$-metric, we obtain $\|S_t^\varepsilon u - S_t^\varepsilon w\|_{\mathbf{L^1}} \leqslant L \cdot \|u - w\|_{\mathbf{L^1}}$ for all $t \geqslant 0$, $u, w \in \mathcal{D}^\varepsilon$ and for some L independent of ε. To complete the proof of the $\mathbf{L^1}$-Lipschitz dependence, choose $\varepsilon_\nu = 2^{-\nu}$, $\nu \in \mathbb{N}$, and define S as $S_t u = \lim_{\nu \to +\infty} S_t^{\varepsilon_\nu} u_\nu$, with $u_\nu \in \mathcal{D}^{\varepsilon_\nu}$ and $\|u_\nu - u\|_{\mathbf{L^1}} \leqslant \varepsilon_\nu$. The existence of the previous limit follows from a *completeness* argument based on Lemma 6.2. The domain of S is $\mathcal{D} = \{u \in \mathbf{L^1}\colon\ \exists u_\nu \in \mathcal{D}^{\varepsilon_\nu} \text{ and } \|u_\nu - u\|_{\mathbf{L^1}} \leqslant \varepsilon_\nu\}$.

6. Uniqueness

A uniqueness result consists in the selection of a class such that

(U1) an existence result provides solutions in this class;

(U2) two solutions in this class coincide.

In the case of conservation laws, two different kinds of uniqueness results can be considered: the one referring to the *semigroup* constructed above and another one referring to the *single solution* to (2.1). The first result in this direction, in the case $n = 2$, was obtained in [26]. In the general case, see [29], [31].

6.1. Uniqueness of the semigroup

The key difficulty in proving the uniqueness of the semigroup S constructed above consists in the selection of the properties of S that define the class where S is unique. This class was selected by Bressan in [19] through the following definition.

Definition 6.1. The system (1.1) generates a *Standard Riemann Semigroup* S: $[0,+\infty[\times \mathcal{D} \to \mathcal{D}$ if there exist positive constants δ and L such that

(SRS1) $\mathcal{D} \supseteq \{u \in \mathbf{L^1}\colon u(\mathbb{R}) \subseteq \Omega$ and $\mathrm{TV}(u) \leqslant \delta\}$;

(SRS2) S is a semigroup: $S_0 = \mathrm{Id}$ and $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$;

(SRS3) S is Lipschitz: $\|S_{t_2}u_2 - S_{t_1}u_1\|_{\mathbf{L^1}} \leqslant L(|t_2 - t_1| + \|u_2 - u_1\|_{\mathbf{L^1}})$;

(SRS4) if $u \in \mathcal{D}$ is piecewise constant, then for t small, $S_t u$ is the gluing of the Lax solutions to Riemann problems at the points of jump in u.

Note that no reference is made to the fact that the orbits of S yield solutions to (1.1). Indeed, the semigroups constructed in Section 5 enjoy all the above properties and, moreover, the map $t \mapsto S_t\bar{u}$ is a weak entropic solution to (2.1). Hence, a further byproduct of Theorem 6.3 is that any semigroup satisfying (SRS1), ..., (SRS4) also yields weak entropic solutions to (1.1).

The admissibility conditions considered in Section 2 appear in Definition 6.1 only through the Lax [55] solution to Riemann problems. In other words, the choice of the Lax Riemann solver uniquely determines also the solutions to Cauchy problems, once Lipschitz continuous dependence on the initial data is required.

The starting point for the uniqueness of the semigroup is the following abstract result, presented here at the level of metric spaces, see [22], [39].

Lemma 6.2. *Let $(\mathcal{D}, d)$ be a complete metric space, S: $[0,+\infty[\times \mathcal{D} \to \mathcal{D}$ a Lipschitz semigroup with Lipschitz constant L and w: $[0,T] \to \mathcal{D}$ a Lipschitz map. Then $d(w(T), S_T w(0)) \leqslant L \cdot \int_0^T \liminf_{h\to 0+} d(w(t+h), S_h w(t))/h \,\mathrm{d}t$.*

The above lemma reduces the problem of computing the distance between the orbits of different semigroups to that of computing the difference between the "*tangent vectors*" [30], [33]. This fits particularly well with conservation laws, since the tangent vector to solutions is essentially characterized by the Lax Riemann solver, at least in the case of a piecewise constant initial datum.

Theorem 6.3. *Let S: $[0,+\infty[\times \mathcal{D} \to \mathcal{D}$ be the semigroup constructed above through Wave Front Tracking. Let $\tilde{S}$: $[0,+\infty[\times \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$ be another* SRS *generated by* (1.1) *with $\tilde{\mathcal{D}} \supseteq \mathcal{D}$. Then for all $u \in \mathcal{D}$ and all $t \geqslant 0$, $S_t u = \tilde{S}_t u$.*

In the proof of Theorem 6.3, Wave Front Tracking approximations have a key role, thanks to their relation with Lax solutions to Riemann problems.

6.2. Uniqueness of the single solution

To meet the requirement (U1) above, it is now necessary to find such properties of the solutions yielded by the SRS such that single solutions to Cauchy problems

could be fully characterized. As a first intermediate step, assumed the existence of a SRS, we seek those conditions on a solution u implying that $u(t) = S_t u(0)$.

Let $u_*\colon [0,T] \to \mathcal{D}$ be a weak solution to (2.1). Define $U^\sharp_{(u_*,\tau,\xi)}$ as the solution to the Riemann problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & t \geqslant \tau,\ x \in \mathbb{R}, \\ u(\tau,x) = \begin{cases} \lim\limits_{x\to\xi-} u_*(\tau,x) & \text{if } x < \xi, \\ \lim\limits_{x\to\xi+} u_*(\tau,x) & \text{if } x > \xi. \end{cases} \end{cases}$$

Let $U^\flat_{(u_*,\tau,\xi)}$ be the solution to the linear Cauchy problem

$$\begin{cases} \partial_t u + \mathrm{D}f(u(\tau,\xi))\partial_x u = 0, & t \geqslant \tau,\ x \in \mathbb{R}, \\ u(\tau,x) = u_*(\tau,x). \end{cases}$$

By means of $U^\flat$ and $U^\sharp$, the first property of the semigroup is singled out.

Proposition 6.4. *Let $S\colon [0,+\infty[\times \mathcal{D} \to \mathcal{D}$ be the* SRS *constructed above by means of Wave Front Tracking. Let $\hat\lambda$ be an upper bound for the moduli of all characteristic speeds. Then for all $u \in \mathcal{D}$ and for all $\tau \geqslant 0$*

1. *for all $\xi \in \mathbb{R}$, $\lim\limits_{h\to 0+} 1/h \int_{\xi-h\hat\lambda}^{\xi+h\hat\lambda} \|(S_{\tau+h}u)(x) - U^\sharp_{(u,\tau,\xi)}(h,x)\|\,\mathrm{d}x = 0$;*
2. *there exists a $C > 0$ such that for all $\xi \in]a,b[$ and $h \in]0,(b-a)/(2\hat\lambda)[$, $1/h \int_{a+h\hat\lambda}^{b-h\hat\lambda} \|(S_{\tau+h}u)(x) - U^\flat_{(u,\tau,\xi)}(h,x)\|\,\mathrm{d}x \leqslant C \cdot \big[\mathrm{TV}\big(u(\tau)|_{]a,b[}\big)\big]^2$.*

We now obtain a uniqueness result suited to the above properties.

Theorem 6.5. *Let $S\colon [0,+\infty[\times \mathcal{D} \to \mathcal{D}$ be the* SRS *constructed above. Let $\hat\lambda$ be a finite upper bound for the moduli of all characteristic speeds. If $u_*\colon [0,T] \to \mathcal{D}$ be an $\mathbf{L^1}$-Lipschitz continuous weak solution to* (2.1) *such that*

1. *for all $\xi \in \mathbb{R}$, $\lim\limits_{h\to 0+} 1/h \int_{\xi-h\hat\lambda}^{\xi+h\hat\lambda} \|u_*(\tau+h,x) - U^\sharp_{(u_*,\tau,\xi)}(h,x)\,\mathrm{d}x = 0$,*
2. *there exists a $C > 0$ such that for all $\xi \in]a,b[$ and $h \in]0,(b-a)/(2\hat\lambda)[$, $1/h \int_{a+h\hat\lambda}^{b-h\hat\lambda} \|u_*(\tau+h,x) - U^\flat_{(u_*,\tau,\xi)}(h,x)\|\,\mathrm{d}x \leqslant C \cdot \big[\mathrm{TV}\big(u(\tau)|_{]a,b]}\big)\big]^2$.*

Then u coincides with a semigroup trajectory: $u(t) = S_t(u(0))$.

Note that the existence of the SRS S and the fact that u_* attains values in the domain $\mathcal{D}$ of S are essential. Indeed, they are unavoidable also in the next result, where the integral bounds provided by $U^\sharp$ and $U^\flat$ are substituted by the Lax entropy inequalities and a bound on the total variation.

Theorem 6.6. *Let system* (1.1) *satisfy* (SH), (GNL/LD) *and generate an* SRS $S\colon [0,+\infty[\times \mathcal{D} \to \mathcal{D}$. *Let* $u\colon [0,T] \to \mathcal{D}$ *be such that*

1. u *is a weak solution of* (2.1);
2. u *is* $\mathbf{L^1}$*-continuous;*
3. u *satisfies the Lax entropy condition, see Definition* 2.7;
4. *there exists a* $\delta > 0$ *such that for every Lipschitz curve* $\gamma\colon [a,b] \to [0,+\infty[$ *with Lipschitz constant* δ, *the map* $t \mapsto u(\gamma(x),x)$ *has bounded variation.*

Then for all $t \in [0,T]$, $u(t) = S_t\bar{u}$.

7. Other problems

Below we briefly consider (1.1) with boundary and models displaying phase transitions. Other areas in which Wave Front Tracking has been successful are: balance laws, see [5], [6], [45]; control problems, see [7], [8], [24], [40], [42]; the structural stability of (1.1), see [32] and the dependence of the solutions to (1.1) on f, see [17].

7.1. The Initial Boundary Value Problem

Different approaches to the Initial Boundary Value Problem (IBVP) for (1.1) are present in the current literature. First, if the range of the speed $\dot{\Psi}$ of the boundary is separated from that of the eigenvalues, i.e.

$$\exists i_*\colon \sup_{u\in\Omega} \lambda_{i_*}(u) < \inf_{t\in[0,+\infty[} \dot{\Psi}(t) \quad \text{and} \quad \sup_{t\in[0,+\infty[} \dot{\Psi}(t) < \inf_{u\in\Omega} \lambda_{i_*+1}(u), \tag{7.1}$$

then $n-i$ scalar conditions can be imposed along the boundary, leading to a *non-characteristic* problem

$$\begin{cases} \partial_t u + \partial_x[f(u)] = 0, & (t,x) \in [0,+\infty[\times [\Psi(t),+\infty[, \\ u(0,x) = \bar{u}(x), & x \in [\Psi(0),+\infty[, \\ b(u(t,\Psi(t))) = g(t), & t \in [0,+\infty[\end{cases} \tag{7.2}$$

where the boundary profile $\Psi\colon [0,+\infty[\to \mathbb{R}$ is assumed continuous. Denote $E = \{(t,x) \in [0,+\infty[\times \mathbb{R}\colon x \in [\Psi(t),+\infty[\}$. In the case (7.1), $b\colon \Omega \to \mathbb{R}^{n-i_*}$ and $g\colon [0,+\infty[\to \mathbb{R}^{n-i_*}$ essentially fix $n-i_*$ components of u. As the *solution* to (7.2) we choose a solution to (1.1) for (t,x) in the interior of E satisfying the boundary condition in the sense of the trace, i.e. $\lim_{x\to\Psi(t)+} b(u(t,\Psi(t))) = g(t)$. The non-characteristic problem (7.2) is considered in [2], [3], [4], [7].

In the case of a *characteristic* boundary, i.e. when (7.1) is violated, a more general definition of solution was proposed in [51]. This definition is not suitable for

applications to gas dynamics, since it does not consider the boundary layer effects due to viscosity. Nevertheless, it is fully *intrinsic*, for it is based on the solution to Riemann problems. Indeed, consider the IBVP

$$\begin{cases} \partial_t u + \partial_x [f(u)] = 0, & (t,x) \in [0,+\infty[\times [\Psi(t),+\infty[, \\ u(0,x) = \bar{u}(x), & x \in [\Psi(0),+\infty[, \\ u(t,\Psi(t)) = \tilde{u}(t), & t \in [0,+\infty[. \end{cases} \tag{7.3}$$

As in [51], we introduce the following definitions.

Definition 7.1. The *characteristic Riemann problem with boundary* is (7.3) with $\Psi(t) = mt$ and $\bar{u}$, $\tilde{u}$ constant. Its solution is the restriction to E of the Lax solution to the standard Riemann problem

$$\begin{cases} \partial_t u + \partial_x [f(u)] = 0, \\ u(x,0) = \begin{cases} \tilde{u} & \text{if } x < 0, \\ \bar{u} & \text{if } x > 0. \end{cases} \end{cases} \tag{7.4}$$

Definition 7.2. Let $u\colon E \to \mathcal{U}$ be such that $x \mapsto u(t,x)$ is in BV for a.e. t. Let w be the Lax solution to the Riemann problem (with a jump at $(\tau, \Psi(\tau))$)

$$\begin{cases} \partial_t w + \partial_x [f(w)] = 0, \\ w(\tau,x) = \begin{cases} \tilde{u}(\tau) & \text{if } x < \Psi(\tau), \\ u(\tau, \Psi(\tau)+) & \text{if } x > \Psi(\tau). \end{cases} \end{cases}$$

u is defined to be a solution of (7.3) if

(i) it is a weak entropic solution to (7.3) for $t > 0$ and $x > \Psi(\tau)$,

(ii) it coincides with $\bar{u}$ at time $t = 0$,

(iii) it satisfies the boundary condition, i.e. for all but countably many $\tau \geqslant 0$, $u(\tau, \Psi(\tau)+) = w(t,x)$ for all $(t,x) \in E$ such that $x - \Psi(\tau) > D_-\Psi(\tau) \cdot (t - \tau)$ with $t > \tau$.

Above, $D_-\Psi(t) = \liminf_{s \to t^-} (\Psi(s) - \Psi(t))/(s-t)$. According to the above definition, and differently from the non-characteristic case (7.2), a solution u to (7.3) may well have a trace $u(t, \Psi(t)+)$ that differs from the boundary data $\tilde{u}(t)$, but only if the Riemann problem defined by $u(t, \Psi(t)+)$ and $\tilde{u}(t)$ is solved by waves supported outside E, i.e. by waves slower than the boundary.

In the two cases

- (SH), (GNL/LD), small total variation, $n = 2$;
- (SH), Temple system, $n \geqslant 1$,

both the problems (7.2) and (7.3) are well posed in $\mathbf{L}^1$, in the sense that they define a unique $\mathbf{L}^1$-Lipschitz solution operator. Moreover, characterizations similar to those provided by Theorems 6.3, 6.5 and 6.6 apply, see [2], [3], [4], [41].

7.2. Phase transitions and combustion

The analytical techniques described above seem particularly useful when dealing with phase transitions. In that case, Ω is the disjoint union of 2 (or more) sets, to be referred as *phases*, say $\Omega = \Omega_1 \cup \Omega_2$. At time t, the system described by (1.1) is in phase i at x if and only if $u(t,x) \in \Omega_i$. A *phase transition* is a jump discontinuity in a solution to (1.1) between states belonging to different phases. The *phase boundary* is its support.

Extensions of the classical Lax solution to (3.1) applicable in presence of phase transitions to specific physical models have been considered, see [1], [11], [68], [71], [73] and [36] for a more abstract approach. The usual difficulty that arises when phase transitions are present is the lack of uniqueness, bypassed by introducing further physical information, either through an admissibility function or through constraints on the structure of the solution.

As a first example, consider the p-system (1.4) closed through the pressure law $p = p(\tau)$ where $p\colon\ \Omega_1 \cup \Omega_2 \to]0, +\infty[$. Ω_1 and Ω_2 are disjoint real intervals representing the liquid and the vapor phase, respectively. Reasonable qualitative properties of this function p are in Fig. 7, left. Assume that u^l and u^r in (3.1) are in different phases.

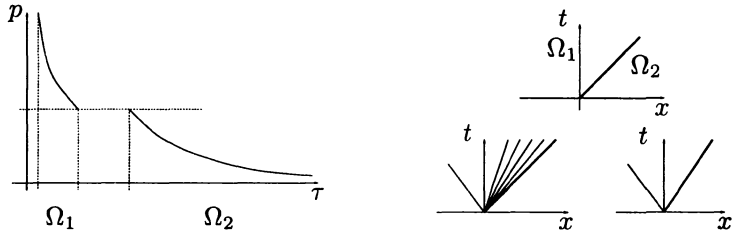

Figure 7. Left, the pressure p as a function of the specific volume τ. Right, configurations in the solution to a Riemann problem for Chapman-Jouguet detonations: above the *unperturbed* one.

In the *subsonic* case (3.1) is well known to be underdetermined. In the case of the p-system, by subsonic we mean that the modulus of the Rankine-Hugoniot speed of the phase boundary is lower than the modulus of the characteristic speeds on the sides of the discontinuity. Two usual criteria used to single out a unique solution are based on the entropy rate dissipation or on visco-capillarity approximation. From a more abstract point of view, it suffices to require that the admissibility condition

$$\Psi(u^l, u^r) = 0 \quad \text{with} \quad \Psi\colon\ (\Omega_1 \times \Omega_2) \cup (\Omega_2 \times \Omega_1) \to \mathbb{R} \tag{7.5}$$

be satisfied by any pair u^l, u^r on the sides of a phase boundary. Within this framework, various analytical results can be obtained [36], [37], [43].

Note that the same mathematical structure (1.3)–(7.5) applies also to models of phase transitions in solids [1], [73].

The structure provided by a system of the form (1.1) on a disconnected domain Ω provides a good model also for Chapman-Jouguet detonations, see [71]. This well known combustion model consists in unburnt gas filling the half line $x > 0$ and burnt gas filling $x < 0$ at time $t = 0$. The chemical reaction is instantaneous. A mathematical description is provided by two Euler systems, one for the burnt and one for the unburnt gas, coupled through a free boundary which models the location of the reaction:

$$\begin{array}{ll} \text{Burnt gas (1)} & \text{Unburnt gas (2)} \\ \begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t \Big(e + \dfrac{v^2}{2}\Big) + \partial_x (pv) = 0, \end{cases} & \begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t \Big(e + \dfrac{v^2}{2}\Big) + \partial_x (pv) = 0, \end{cases} \\ S_1 = c_v^1 \ln e + \dfrac{R}{\mu_1} \ln \tau & S_2 = c_v^2 \ln(e - Q) + \dfrac{R}{\mu_2} \ln \tau. \end{array}$$

S is the entropy, e the internal energy, Q the energy to be dissipated through combustion, c_v^i and μ^i are respectively the specific heat and the molar weight of the ith gas.

Chapman-Jouguet strong detonations are characterized by the flame having approximately the sound speed of the burnt gas next to it. We are thus driven to consider the *sonic* situation. As above, we need to introduce a criterion that singles out a unique solution to Riemann problems. Differently from above, here we do not introduce any admissibility condition (7.5) but rather impose restrictions on the *structure* of the solution. The solution to the Riemann problem with the (unperturbed) Chapman-Jouguet data consists of a single sonic phase transition (alias the combustion flame), see Fig. 7, right. Perturbing this data, the solution may contain either a 1-wave, a 2-contact discontinuity and a subsonic phase transition; or a 1-wave, a 2-contact discontinuity and a 3-rarefaction adjacent to the phase transition. With this choice, the Cauchy problem obtained by adding a BV small perturbation to a Chapman-Jouguet Riemann data still admits a solution, see [37] for details. For the sake of completeness, we mention here that in the case of Chapman-Jouguet deflagrations, a unique solution to the Riemann problem is chosen through the introduction of both an admissibility condition (7.5) and a constraint on the structure of the solution, see [38].

A model describing phase transitions in traffic flow is considered in [35].

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WAVE FRONT TRACKING IN SYSTEMS OF CONSERVATION LAWS*

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Abstract. This paper contains several recent results about nonlinear systems of hyperbolic conservation laws obtained through the technique of Wave Front Tracking.

Keywords: conservation laws, Wave Front Tracking

MSC 2000: 35L65, 35A35

1. Introduction

Wave Front Tracking is a set of techniques for constructing approximate solutions to hyperbolic conservation laws in 1 space dimension, i.e. to first order quasilinear systems of partial differential equations of the form

$$\partial_t u + \partial_x f(u) = 0 \tag{1.1}$$

with $f\colon \Omega \to \mathbb{R}^n$ smooth and Ω an open subset of $\mathbb{R}^n$, $n \geqslant 1$, $t \in [0, +\infty[$ and $x \in \mathbb{R}$. Wave Front Tracking was first introduced by Dafermos [46]. Recently, its use has grown thanks to several extensions [10], [13], [15], [18], [25], [26], [36], [38], [57], [58], [64].

These equations state the conservation of the observables described by the densities $u \equiv (u_1, \dots, u_n)$. More precisely, they state that any variation in time in the quantity of each observable contained in a segment $[a, b]$ is due to the inflow at a and to the outflow at b. In symbols,

$$\int_a^b u(t_2, x)\,\mathrm{d}x - \int_a^b u(t_1, x)\,\mathrm{d}x = \int_{t_1}^{t_2} f(u(t, a))\,\mathrm{d}t - \int_{t_1}^{t_2} f(u(t, b))\,\mathrm{d}t. \tag{1.2}$$

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The paradigm for conservation laws is the Euler system for a compressible non viscous gas that, in Eulerian coordinates, reads

$$\partial_t \begin{bmatrix} \varrho \\ \varrho v \\ \frac{1}{2}\varrho v^2 + \varrho e \end{bmatrix} + \partial_x \begin{bmatrix} \varrho v \\ \varrho v^2 + p \\ v\big(\frac{1}{2}\varrho v^2 + \varrho e + p\big) \end{bmatrix} = 0 \tag{1.3}$$

where ϱ is the mass density, v the gas speed, p the pressure and e the internal energy density. (1.3) is closed by the state equation of the gas considered.

In the isentropic case with Lagrangian coordinates, (1.3) is the p-system

$$\partial_t \begin{bmatrix} \tau \\ v \end{bmatrix} + \partial_x \begin{bmatrix} -v \\ p(\tau) \end{bmatrix} = 0, \tag{1.4}$$

τ being the specific volume and v the (Lagrangian) gas speed. Equation (1.4) provides a very useful computable example of a conservation law.

Other applications of conservation laws deal with traffic flow, chromatography, phase transitions, combustion, ...

Wave Front Tracking is not the only technique to study conservation laws. Other fruitful tools are: Glimm scheme, which provided the first existence proof for global weak solutions to (1.1) [52], [59]; generalized characteristics, which allow to obtain fine properties of solutions [27], [47], [49], [50], [72]; piecewise Lipschitz approximations, which provided the first well-posedness proof when $n > 2$ [28] and compensated compactness, see [63] and the references therein. More recently, an entirely new technique based on viscous approximations appeared in [16].

In the present notes, proofs are omitted. Most of them can be found in several books on conservation laws that recently appeared, see [48], [53], [56], [67] and in particular [22]. Other surveys on this subject are [20], [21], [23].

2. Weak solutions and admissibility conditions

System (1.1) is *strictly hyperbolic* as soon as it satisfies

(SH) For all $u \in \Omega$, $\mathrm{D}f(u)$ admits n real and distinct eigenvalues.

For $i = 1, \dots, n$, $\lambda_i(u)$ is the ith eigenvalue of $\mathrm{D}f(u)$. The eigenvalues are numbered so that $\lambda_{i-1} < \lambda_i$ for $i = 2, \dots, n$. r_i is the right eigenvector corresponding to λ_i. The following stronger (uniform) condition is also of use below:

(USH) For $i = 2, \dots, n$, $\sup_\Omega \lambda_{i-1}(u) < \inf_\Omega \lambda_i(u)$.

Clearly, (USH) $\Rightarrow$ (SH), while the converse holds in general only locally, due to the smoothness of f.

In the *linear* case $f(u) = Au$ (with $A \in \mathbb{R}^{n\times n}$), thanks to (SH), the following procedure allows to obtain a solution $u = u(t,x)$ to the Cauchy problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0,x) = \overline{u}(x) \end{cases} \tag{2.1}$$

for (1.1), with $\overline{u} \in \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\mathbb{R})$:

1. diagonalise (1.1) obtaining n decoupled equations $\partial_t v_i + \lambda_i \partial_x v_i = 0$;
2. let each component $\overline{v}_i$ of the initial datum translate with speed λ_i and superimpose: $u(t,x) = \sum_i \overline{v}_i(x - \lambda_i t) r_i$.

In the *non*linear case none of the previous steps remains doable:

1. the eigenvectors $r_1, \dots, r_n$ depend on u and in general no decoupling of (1.1) can be achieved;
2. the eigenvalues λ_i depend on u and the above construction may lead to a multivalued function.

As a consequence, the solution to (1.1) may develop singularities independently of the smoothness of the initial datum. Indeed, (1.1) admits the integral formulation (1.2) which is meaningful provided u is merely integrable. The following formulations are more usual.

Definition 2.1. Let $f\colon \Omega \to \mathbb{R}^n$ be smooth and $\overline{u}\colon \mathbb{R} \to \mathbb{R}^n$ be in $\mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}$. A measurable $u\colon [0,+\infty[\times \mathbb{R} \to \Omega$ is a *distributional solution* of (2.1) in Ω if for every $\mathbf{C^1}$ function $\varphi\colon [0,+\infty[\times \mathbb{R} \to \mathbb{R}^n$ with compact support

$$\int_0^{+\infty} \int_{\mathbb{R}} [u(t,x)\partial_t \varphi(t,x) + f(u(t,x))\partial_x \varphi(t,x)] \,\mathrm{d}x \,\mathrm{d}t + \int_{\mathbb{R}} \overline{u}(x)\varphi(0,x) \,\mathrm{d}x = 0.$$

u is a *weak solution* of (2.1) in Ω if

(W1) $u\colon [0,+\infty[\to \mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}(\mathbb{R})$ is continuous in $\mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}$;
(W2) $u(0) = \overline{u}$;
(W3) $\int_0^{+\infty}\int_{\mathbb{R}} [u(t,x)\partial_t \varphi(t,x) + f(u(t,x))\partial_x \varphi(t,x)] \,\mathrm{d}x \,\mathrm{d}t = 0$ for any $\mathbf{C^1}$ map $\varphi\colon]0,+\infty[\times \mathbb{R} \to \mathbb{R}^n$ with compact support.

(Above and in the sequel, we use the fact that any function $u = u(t,x)$ of two variables identifies a map $u = u(t)$ attaining values in a suitable function space.) Note that a distributional solution can be arbitrarily modified at a countable set of times on all $\mathbb{R}$ still remaining a distributional solution. The usual relations between classical and weak solutions apply: any classical solution is also a weak solution while a smooth weak solution is also a classical solution.

The simplest example of a possible non smooth solution to (1.1) is

$$u(t,x) = \begin{cases} u^l & \text{if } x < \Lambda t, \\ u^r & \text{if } x > \Lambda t \end{cases} \tag{2.2}$$

where $u^l, u^r \in \Omega$ and $\Lambda \in \mathbb{R}$. Applying (1.2) or Definition 2.1, we have

Lemma 2.2. *Let $f\colon \Omega \to \mathbb{R}^n$ be smooth. The function u in* (2.2) *is a weak solution to* (1.1) *if and only if*

$$\Lambda \cdot (u^l - u^r) = f(u^l) - f(u^r). \tag{2.3}$$

The vector relation (2.3) is known as the *Rankine-Hugoniot condition*. In general, it does not single out a unique solution to (2.1).

E x a m p l e 2.3. Fix an arbitrary $N \in \mathbb{N}$, $N > 0$ and choose $\omega_0, \dots, \omega_{N+1}$ in $[0,1]$ with $\omega_0 = 0$, $\omega_{N+1} = 1$ and $\omega_{h-1} \leqslant \omega_h$ for $h = 1, \dots, N+1$. Then

$$u(t,x) = \begin{cases} 0 & \text{if } x \in \,]-\infty, \omega_1 t/2[, \\ \omega_n & \text{if } x \in \,](\omega_{h-1} + \omega_h)t/2, (\omega_h + \omega_{h+1})t/2[\quad \text{for } h = 1, \dots, N, \\ 1 & \text{if } x \in \,]\omega_{N+1} t/2, +\infty[\end{cases}$$

is a weak solution to Burgers' equation $\partial_t u + \partial_x(u^2/2) = 0$, since the Rankine-Hugoniot condition (2.3) is satisfied along any jump.

Several criteria have been devised to single out a unique solution to (2.1). One, motivated by physical considerations, is based on the concept of entropy and on the second principle of thermodynamics.

Definition 2.4. A pair of $\mathbf{C^1}$ functions (η, q) with $\eta, q\colon \Omega \to \mathbb{R}$ is an *entropy-entropy flux pair* if $\mathrm{D}\eta(u)\mathrm{D}f(u) = \mathrm{D}q(u)$ for all $u \in \Omega$.

Hence, any smooth solution u to (1.1) satisfies also $\partial_t \eta(u) + \partial_x q(u) = 0$. However, a non smooth solution may violate this latter conservation law, giving a contribution with varying sign along discontinuities.

Definition 2.5. A weak solution u to (1.1) is *entropy admissible* if for any entropy-entropy flux pair (η, q) with η convex, the inequality

$$\partial_t \eta(u) + \partial_x q(u) \leqslant 0 \tag{2.4}$$

holds in distributional sense.

Note that, besides physical systems, entropy-entropy flux pairs certainly exist if $n = 1$ or $n = 2$. Another criterion arises from *viscous* approximations.

Definition 2.6. A weak solution u to (1.1) is *admissible in the sense of viscosity* if there exists a positive sequence ε_n with $\varepsilon_n \to 0$ for $n \to +\infty$ such that the solutions u_n to $\partial_t u_n + \partial_x f(u_n) = \varepsilon_n \partial^2_{xx} u_n$ converge in $\mathbf{L}^{\mathbf{1}}_{\mathrm{loc}}$ to u as $n \to +\infty$.

The next admissibility condition is extremely useful in connection with Wave Front Tracking, but only under the assumption (GNL/LD) below.

Definition 2.7. Let $u\colon [0, +\infty[\times \mathbb{R} \to \Omega$ be a weak solution to (1.1). The solution u satisfies the *Lax entropy inequalities* if for any $(\tau, \xi) \in [0, T] \times \mathbb{R}$ such that there exist states $u^l, u^r \in \Omega$ and a speed $\lambda \in \mathbb{R}$ with

$$\lim_{\varrho \to 0+} \frac{1}{\varrho^2} \int_{\tau-\varrho}^{\tau+\varrho} \int_{\xi-\varrho}^{\xi+\varrho} \|u(t,x) - U(t,x)\| \,\mathrm{d}x \,\mathrm{d}t = 0$$

where

$$U(t,x) = \begin{cases} u^l & \text{if } x - \xi < \lambda \cdot (t - \tau), \\ u^r & \text{if } x - \xi > \lambda \cdot (t - \tau), \end{cases}$$

then, for an $i \in \{1, \ldots, n\}$, $\lambda_i(u^l) \geqslant \lambda \geqslant \lambda_i(u^r)$.

Note that the discontinuities in Example 2.3 do not satisfy Definition 2.7. Conservation laws admit a symmetry group.

Lemma 2.8. *Fix a positive ϱ, constants $\tau, \xi \in \mathbb{R}$, and a function $u\colon [0, +\infty[\times \mathbb{R} \to \Omega$. Define $w(t,x) = u(\tau + \varrho t, \xi + \varrho x)$. Then,*

1. *if u is a weak (or distributional) solution to (1.1), then so is w;*
2. *if u is entropy admissible, then so is w;*
3. *if u is admissible in the sense of viscosity, then so is w.*

The *hyperbolic rescaling* is the transformation $t \to \varrho t$ and $x \to \varrho x$. A function $u\colon [0, +\infty[\times \mathbb{R} \to \mathbb{R}^n$ is *self similar* if for all $\varrho > 0$, $u(\varrho t, \varrho x) = u(t, x)$.

Proposition 2.9. *Fix two functions $u_l, u_r\colon [0, +\infty[\times \mathbb{R} \to \Omega$ and a continuous map $\psi\colon [0, +\infty[\to \mathbb{R}$ such that $\lim_{x \to \psi(t)-} u_l(t,x) = \lim_{x \to \psi(t)+} u_r(t,x)$ for a.e. $t \geqslant 0$ and define*

$$w(t,x) = \begin{cases} u_l(t,x) & \text{if } x < \psi(t), \\ u_r(t,x) & \text{if } x > \psi(t). \end{cases}$$

Then,

1. *if u_l, u_r are weak (distributional) solutions to (1.1) in Ω, then so is w;*
2. *if u_l, u_r are entropy admissible, then so is w.*

Note that 1. above requires also the existence of the limits. If $u_l(t)$, $u_r(t)$ are in $\mathrm{BV}(\mathbb{R})$ for all t, both the limits certainly exist.

Corollary 2.10. *Fix $u_* \in \Omega$, a function $u\colon [0,+\infty[\times \mathbb{R} \to \Omega$ and a continuous map $\psi\colon [0,\infty[\to \mathbb{R}$ such that $\lim_{x\to\psi(t)-} u(t,x) = u_*$ for a.e. $t \geqslant 0$. Define*

$$w(t,x) = \begin{cases} u(t,x) & \text{if } x < \psi(t), \\ u_* & \text{if } x > \psi(t). \end{cases}$$

Then,

1. *if u is a weak (or distributional) solution to (1.1) in Ω, then so is w;*
2. *if u is entropy admissible, then so is w.*

3. The Riemann problem

The Riemann problem for (1.1) is the following particular Cauchy problem:

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, \\ u(0,x) = \begin{cases} u^l & \text{if } x < 0, \\ u^r & \text{if } x > 0. \end{cases} \end{cases} \tag{3.1}$$

Note that this problem is self similar, in the sense that the hyperbolic rescaling leaves it unchanged.

If $n > 1$, a solution to (3.1) is constructed using the eigenvalues $\lambda_1, \dots, \lambda_n$ and eigenvectors $r_1, \dots, r_n$ of $\mathrm{D}f$. The term "*i-characteristic field*" often refers to both the maps $u \mapsto r_i(u)$ and $u \mapsto \lambda_i(u)$. Below, we require that (USH) holds and choose the eigenvectors so that $\|r_i(u)\| = 1$ for all i and u.

Definition 3.1. The i-characteristic field is *genuinely nonlinear* if

$$\nabla\lambda_i(u) \cdot r_i(u) \neq 0$$

for all u. It is *linearly degenerate* if $\nabla\lambda_i(u) \cdot r_i(u) = 0$ for all u.

The following assumption greatly simplifies the necessary techniques:

(GNL/LD) each characteristic field is either *genuinely nonlinear* or *linearly degenerate*.

If the i-characteristic field is genuinely nonlinear, we choose the ith eigenvector oriented so that $\nabla\lambda_i(u) \cdot r_i(u) > 0$ for all $u \in \Omega$. Within the framework of Wave Front

Tracking, the results in [9], [10] allow to relax the above assumption, at the price of heavy technicalities. The recent paper [16], through entirely different and new techniques, proves the well-posedness of (2.1) only under assumption (SH), without even requiring (1.1) being in conservation form.

3.1. Rarefaction waves

Lemma 3.2. *If the i-characteristic field is genuinely nonlinear, then for all $u_o \in \Omega$ there exists a positive σ_o and a smooth curve $\sigma \mapsto R_i(u_o, \sigma)$ defined for $\sigma \in [-\sigma_o, \sigma_o]$ with the properties:*

(R1) $\mathrm{d}/\mathrm{d}\sigma R_i(u_o, \sigma) = r_i(R_i(u_o, \sigma))$;

(R2) $R_i(u_o, 0) = u_o$;

(R3) *σ can be chosen so that $\lambda_i(R_i(u_o, \sigma)) = \lambda_i(u_o) + \sigma$, i.e. $\nabla\lambda_i(u) \cdot r_i(u) = 1$.*

The curve $\sigma \mapsto R_i(u_o, \sigma)$ is the *i-rarefaction* through u_o. It solves the Cauchy problem $\begin{cases} u' = r_i(u), \\ u(0) = u_o. \end{cases}$ The choice of σ to parameterise R_i is arbitrary, (R3) facilitates some estimates.

Proposition 3.3. *Assume that the i-characteristic field is genuinely nonlinear and the i-rarefaction curve is parametrized as in* [(R3), *Lemma* 3.2]. *If there exists a $\sigma_i \in [0, \sigma_o]$ such that $u^r = R_i(u^l, \sigma_i)$, then the function*

$$u(t,x) = \begin{cases} u^l & \text{if } x < \lambda_i(u^l)t, \\ R_i(u^l, \sigma) & \text{if } x = \lambda_i(R_i(u^l, \sigma))t \text{ for } \sigma \in [0, \sigma_i], \\ u^r & \text{if } x > \lambda_i(u^r)t \end{cases} \tag{3.2}$$

1. *is a weak solution to* (3.1);
2. *is continuous and self similar;*
3. *is entropy admissible (see Definition* (2.5)*) with equality in* (2.4), *if* (1.1) *admits an entropy-entropy flux pair;*
4. *is such that for all $t > 0$, the map $x \mapsto u(t,x)$ is Lipschitz with the constant $1/[(\sup_u \|\nabla\lambda_i \cdot r_i\|)t]$.*

The solution (3.2) is a *(centered) rarefaction wave* with *strength* σ_i. Note that if $u^r = R_i(u^l, \sigma)$ with $\sigma < 0$, the above construction (3.2) is not possible. The case $\sigma_i = 0$ of a null rarefaction is considered for completeness.

3.2. Shock waves

Lemma 3.4. *If the i-characteristic field is genuinely nonlinear, then for all $u_o \in \Omega$ there exists a positive σ_o, a smooth curve $\sigma \mapsto S_i(u_o, \sigma)$ defined for $\sigma \in [-\sigma_o, \sigma_o]$ and a function $\Lambda_i(u_o, \cdot)\colon [-\sigma_o, \sigma_o] \to \mathbb{R}$ with the properties*

(S1) $f(S_i(u_o,\sigma)) - f(u_o) = \Lambda_i(u_o,\sigma)\cdot(S_i(u_o,\sigma) - u_o)$;
(S2) $S_i(u_o,0) = u_o$, $\mathrm{d}/\mathrm{d}\sigma S_i(u_o,0) = r_i(u_o)$ and $\Lambda_i(u_o,0) = \lambda_i(u_o)$;
(S3) *the parameter* σ *can be chosen so that* $\lambda_i(S_i(u_o,\sigma)) = \lambda_i(u_o) + \sigma$.

The curve $\sigma \mapsto S_i(u_o,\sigma)$ is the *i-shock* through u_o. It is the unique curve of solutions to the Rankine-Hugoniot condition (2.3) exiting u_o tangent to $r_i(u_o)$.

Proposition 3.5. *Assume that the i-characteristic field is genuinely nonlinear and the i-shock curve is parametrized as in* [(S3), *Lemma* 3.4]. *If there exists a* $\sigma_i \in [-\sigma_o, \sigma_o]$ *such that* $u^r = S_i(u^l,\sigma_i)$, *then the function*

$$u(t,x) = \begin{cases} u^l & \text{if } x < \Lambda_i(u^l,\sigma_i)t, \\ u^r & \text{if } x > \Lambda_i(u^l,\sigma_i)t \end{cases} \tag{3.3}$$

1. *is a weak solution to* (3.1);
2. *is self similar;*
3. *is entropy admissible (see Definition* 2.5*), provided* $\sigma \leqslant 0$ *and* (1.1) *admits an entropy-entropy flux pair;*
4. *is such that for all* $t > 0$, *the maps* $x \mapsto u(t,x)$ *and* $x \mapsto \lambda_i(u(t,x))$ *have a jump discontinuity at* $\Lambda_i(u^l,\sigma_i)t$.

The solution (3.3) is a *shock*. To stress the admissibility in the case $\sigma \leqslant 0$, "*entropic shock*" is often used. If $\sigma_i > 0$ then, differently from the case of rarefactions, the solution (3.3) is still well defined and it is a weak solution to (3.1), see Example 2.3. However, it is not entropic. Furthermore, it leads to a construction which is not *consistent*, according to Definition 3.12.

Proposition 3.6. *Assume the i-characteristic field is genuinely nonlinear and the i-shock curve is parametrized as in* [(S3), *Lemma* 3.4]. *The following statements are equivalent:*

1. *the weak solution* (3.3) *is entropy admissible, if* (1.1) *admits an entropy-entropy flux pair;*
2. *the weak solution* (3.3) *is admissible in the sense of viscosity;*
3. $\lambda_i(u^l) > \Lambda_i(u^l,\sigma_i) > \lambda_i(u^r)$;
4. $\sigma_i < 0$.

Remark that the last statement above depends on the choice of the orientation of r_i. Condition 2 is far from immediate, see [67, Chapter 7]. The inequalities at 3 are known as *Lax inequalities*, see Definition 2.7.

3.3. Contact discontinuities

Lemma 3.7. *If the i-characteristic field is linearly degenerate, then for all $u_o \in \Omega$ there exists a positive σ_o and a smooth curve $\sigma \mapsto \mathcal{L}_i(u_o, \sigma)$ defined for $\sigma \in [-\sigma_o, \sigma_o]$ with the properties*

(CD1) $\mathrm{d}/\mathrm{d}\sigma \mathcal{L}_i(u_o, \sigma) = r_i(\mathcal{L}_i(u_o, \sigma))$;
(CD2) $f(\mathcal{L}_i(u_o, \sigma)) - f(u_o) = \lambda_i(u_o) \cdot (\mathcal{L}_i(u_o, \sigma) - u_o)$;
(CD3) $\mathcal{L}_i(u_o, 0) = u_o$;
(CD4) $\lambda_i(\mathcal{L}_i(u_o, \sigma)) = \lambda_i(u_o)$, *for all* $\sigma \in [-\sigma_o, \sigma_o]$;
(CD5) *the arc-length can be chosen as the parameter* σ.

In the linearly degenerate case, shock and rarefaction curves coincide.

Proposition 3.8. *Assume the i-characteristic field is linearly degenerate and the i-shock curve is parametrized as in Lemma 3.7. If there exists a $\sigma_i \in [-\sigma_o, \sigma_o]$ such that $u^r = \mathcal{L}_i(u^l, \sigma_i)$, then the function*

$$u(t,x) = \begin{cases} u^l & \text{if } x < \lambda_i(u^l)t, \\ u^r & \text{if } x > \lambda_i(u^l)t \end{cases} \tag{3.4}$$

1. *is a weak solution to* (3.1);
2. *is self similar;*
3. *is entropy admissible (see Definition* 2.5*) with equality in* (2.4)*, if* (1.1) *admits an entropy-entropy flux pair;*
4. *satisfies Lax inequalities (see Definition* 2.7*), with equality signs;*
5. *is such that for all $t > 0$, the map $x \mapsto u(t,x)$ has a jump discontinuity at $\lambda_i(u^l)t$, while the map $x \mapsto \lambda_i(u(t,x))$ is constant.*

3.4. Lax curves

Lemma 3.9. *If the i-characteristic field is genuinely nonlinear, let us parametrize shock and rarefaction curves so that*

$$\begin{aligned} \lambda_i(R_i(u_o,\sigma)) &= \lambda_i(u_o) + \sigma \qquad \partial_\sigma \lambda_i(R_i(u_o,\sigma)) = 1, \\ \lambda_i(S_i(u_o,\sigma)) &= \lambda_i(u_o) + \sigma \qquad \partial_\sigma \lambda_i(S_i(u_o,\sigma)) = 1. \end{aligned} \tag{3.5}$$

For all $u_o \in \Omega$, there exists a constant C such that for all $\sigma \in [-\sigma_o, \sigma_o]$,

$$\|R_i(u_o,\sigma) - S_i(u_o,\sigma)\| \leqslant C \cdot |\sigma|^3. \tag{3.6}$$

Definition 3.10. If the i-characteristic field is either genuinely nonlinear or linearly degenerate, through each $u_o \in \Omega$ let us define the *i-Lax curve*

$$\mathcal{L}_i(u_o, \sigma) = \begin{cases} S_i(u_o, \sigma) & \text{if } \sigma < 0, \\ R_i(u_o, \sigma) & \text{if } \sigma \geqslant 0. \end{cases} \tag{3.7}$$

If the i-field is linearly degenerate, then the i-Lax curve is defined in Lemma 3.7.

Theorem 3.11. *Under assumptions* (SH) *and* (GNL/LD), *any u_o in Ω has a neighbourhood $\mathcal{U}$ such that*

1. (USH) *holds in $\mathcal{U}$;*
2. *for any $\overline{u}$ in $\mathcal{U}$ there exists $\overline{\sigma} > 0$ such that for $i = 1, \dots, n$ the i-Lax curve $\mathcal{L}_i$ through $\overline{u}$ is defined on $[-\overline{\sigma}, \overline{\sigma}]$;*
3. *for any two points u^l, u^r in $\mathcal{U}$ there exists a unique n-tuple $(\sigma_1, \dots, \sigma_n)$ such that there are $n+1$ states $u_0, \dots, u_n$ in Ω satisfying $u_0 = u^l, \dots, u_1 = \mathcal{L}_1(u_0, \sigma_1), \dots, u_i = \mathcal{L}_i(u_{i-1}, \sigma_i), \dots, u_n = u^r$;*
4. *for any two points u^l, u^r in $\mathcal{U}$, the Riemann problem* (3.1) *admits a weak solution obtained as the juxtaposition (for $i = 1, \dots, n$) of*
   4.1. *an i-rarefaction* (3.2), *if the i-field is genuinely nonlinear and $\sigma_i \geqslant 0$;*
   4.2. *an i-shock* (3.3), *if the i-field is genuinely nonlinear and $\sigma_i < 0$;*
   4.3. *an i-contact discontinuity* (3.4), *if the i-field is linearly degenerate;*
5. *the solution so constructed satisfies Lax inequalities, Definition* 2.7 *and, if* (1.1) *admits an entropy-entropy flux pair, is entropy admissible;*
6. *any two weak solutions to* (2.1) *valued in $\mathcal{U}$ and consisting of the juxtaposition of rarefactions, entropic shocks or contact discontinuities, coincide.*

A *Riemann Solver* is a map that with the initial data u^l, u^r in (3.1) associates a self similar weak solution to (3.1), computed at, say, time $t = 1$. The Riemann solver defined in Theorem 3.11 is the *Lax* Riemann solver.

The ordering in the wave speeds induced by (SH) and the Lax inequalities lead to introduce the following property, enjoyed by the Lax Riemann solver.

Definition 3.12. The Riemann solver $\mathcal{R}$ is *consistent* if the following holds:

$$\left.\begin{aligned} \mathcal{R}(u^l, u^m)(\overline{x}) &= u^m \\ \mathcal{R}(u^m, u^r)(\overline{x}) &= u^m \end{aligned}\right\} \Rightarrow \mathcal{R}(u^l, u^r) = \begin{cases} \mathcal{R}(u^l, u^m) & \text{if } x < \overline{x}, \\ \mathcal{R}(u^m, u^r) & \text{if } x \geqslant \overline{x}, \end{cases} \tag{C1}$$

$$\mathcal{R}(u^l, u^r)(\overline{x}) = u^m \Rightarrow \begin{cases} \mathcal{R}(u^l, u^m) = \begin{cases} \mathcal{R}(u^l, u^r) & \text{if } x \leqslant \overline{x}, \\ u^m & \text{if } x > \overline{x}, \end{cases} \\ \mathcal{R}(u^m, u^r) = \begin{cases} u^m & \text{if } x < \overline{x}, \\ \mathcal{R}(u^l, u^r) & \text{if } x \geqslant \overline{x} \end{cases} \end{cases} \tag{C2}$$

Both these properties are enjoyed by the Lax Riemann solver. Essentially, (C1) states that whenever two solutions to two Riemann problems can be placed side by side, then their juxtaposition is again a solution to a Riemann problem, see Fig. 1. Condition (C2) is the vice-versa.

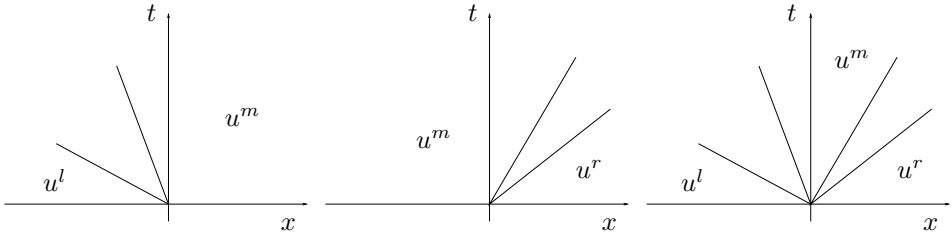

Figure 1. Consistency of a Riemann solver.

4. Existence of solutions to the Cauchy problem

Proposition 2.9 and Definition 3.12 show that solutions to conservation laws can be obtained through a suitable "*gluing*" of other known solutions. The previous section allows to solve Riemann problems. It is thus natural to try to construct solutions to (2.1) through the juxtaposition of solutions to Riemann problems.

Approximate $\overline{u}$ in (2.1) through a piecewise constant function $\overline{u}^\varepsilon$ such that

$$\lim_{\varepsilon \to 0} \|\overline{u} - \overline{u}^\varepsilon\|_{\mathbf{L^1}} = 0. \tag{4.1}$$

It is now natural to proceed by solving the Riemann problems at the points of jump of $\overline{u}$ and gluing the various solutions. However, as soon as a Riemann problem is solved by means of a rarefaction, the approximate solution $u^\varepsilon(t,\cdot)$ to (2.1) so constructed ceases to be piecewise constant at $t = 0+$. Therefore, it is useful to define an ε-approximate Riemann solver $\mathcal{R}^\varepsilon$ so that $\mathcal{R}^\varepsilon(u^l, u^r)$ is piecewise constant. Contact discontinuities and (entropic) shocks are not approximated. Concerning rarefactions, fix $\varepsilon > 0$ and assume that $u^r = \mathcal{L}_i(u^l, \sigma_i)$ with $\sigma_i > 0$. Then, split the rarefaction into $p_i = \lfloor \sigma_i/\varepsilon \rfloor + 1$ waves (here, $\lfloor \xi \rfloor$ is the integer part of ξ) and define the intermediate states $w_j = \mathcal{L}_i(u_o, j\sigma_i/p_i)$ for $j = 0, \dots, p_i$. Finally, define the ε-approximate rarefaction as

$$u^\varepsilon(t,x) = \begin{cases} u^l & \text{if } x < \lambda_i(w_1)t, \\ w_j & \text{if } x \in]\lambda_i(w_j)t, \lambda_i(w_{j+1})t[\text{ and } j = 1, \dots, p_i - 1, \\ u^r & \text{if } x > \lambda_i(u^r)t. \end{cases} \tag{4.2}$$

The ε-approximate Riemann solver $\mathcal{R}^\varepsilon$ can now be defined as the Lax Riemann solver, substituting the exact rarefactions (3.2) by the ε-approximate ones (4.2). By

means of $\mathcal{R}^\varepsilon$, the Riemann problems at the points of jumps of $\overline{u}^\varepsilon$ are approximately solved and an ε-approximate solution $t \mapsto u^\varepsilon(t,\cdot)$ to (2.1) is defined up to the first time, say t_1, at which two discontinuities collide, see Fig. 2. At time t_1, a new Riemann problem arises where the discontinuities meet. Thus, $\mathcal{R}^\varepsilon$ can again be applied and the ε-approximate solution u^ε can be extended up to the next interaction point. Note that this procedure can be applied also if more than 2 waves collide. Two difficulties may stop this construction.

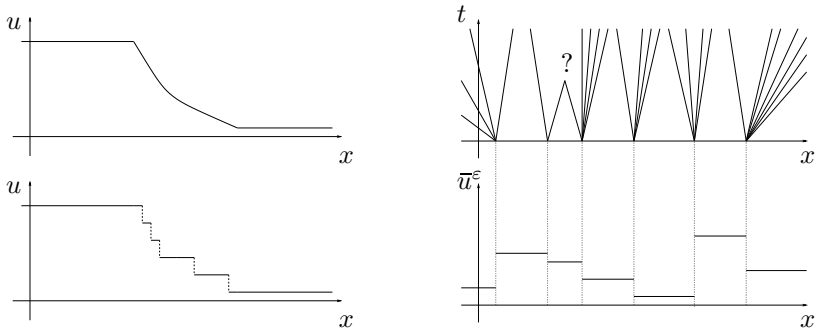

Figure 2. Left, an exact (above) and an approximated (below) rarefaction. Right, the beginning of wave front tracking.

(D1) The solution to a Riemann problem with data in $\mathcal{U}$ need not attain values only in $\mathcal{U}$. Hence the Riemann problems arising in the iteration of the above procedure may well be unsolvable, see Fig. 3, left.

(D2) A Riemann solver may not prolong the ε-solution u^ε, as for instance in case there exists a point (t_*, x_*) such that u^ε suffers a discontinuity at each (t_*, x_n) with $\lim_{n\to+\infty} x_n = x_*$ and $x_n \neq x_*$, see Fig. 3, right.

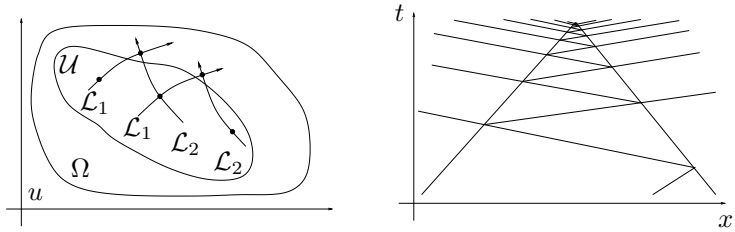

Figure 3. Left, the recursive solution to Riemann problems may lead to exiting $\mathcal{U}$. Right, a cluster point of interaction points.

Suitable *a priori* estimates on the approximate solution allow to ensure that the range of u^ε remains where Riemann problems can be solved. More precisely, an upper bound uniform in ε for the total variation $\mathrm{TV}(u^\varepsilon(t))$ will be obtained through Glimm functionals. This bound, since $u^\varepsilon(t) \in \mathbf{L^1}$ for all t, gives an estimate for the

diameter of $u^\varepsilon(t,\mathbb{R})$. Note also that these bounds also allow to use Helly's theorem to obtain a convergent (sub)sequence of approximate solutions.

On the contrary, (D2) requires technical modifications in the algorithm above. Indeed, a cluster point of interaction points as shown in Fig. 3, right, can indeed arise, see [14], [54] or [48, §13.9].

4.1. Glimm functionals

Write the ε-solution at time t constructed following the above procedure as

$$u^\varepsilon(x) = \sum_\alpha u^\alpha \chi_{]x_\alpha, x_{\alpha+1}]}(x) \quad \text{with} \tag{4.3}$$
$$u^{\alpha+1} = \mathcal{L}^n(\dots \mathcal{L}^i(\dots \mathcal{L}^1(u^\alpha, \sigma_{1,\alpha}), \dots \sigma_{i,\alpha}), \dots \sigma_{n,\alpha}),$$

i.e. $\sigma_{i,\alpha}$ is the (total) size of the i-wave in the solution to the Riemann problem at x_α. For any fixed initial datum $\overline{u}^\varepsilon$, a somewhat "*intrinsic*" measure of $\mathrm{TV}(u^\varepsilon)$ is given by the total strength of waves

$$V = \sum_\alpha \sum_{i=1}^n |\sigma_{i,\alpha}|. \tag{4.4}$$

Note that V is a functional defined on all piecewise constant functions attaining values in the set $\mathcal{U}$ where Riemann problems can be solved.

Let an interaction take place at time t_*. To estimate the variation $\Delta V(t_*) = V(u^\varepsilon(t_*+)) - V(u^\varepsilon(t_*-))$ of V, the following interaction estimates are essential. Remark that interactions take place also in the linear case. But there, waves simply cross each other. In the nonlinear case, any interaction may cause the birth of new waves, see the remarks about Temple systems in Paragraph 4.2.2.

Lemma 4.1. *Assume as in* Fig. 4 *that two waves of different (right) or of the same (left) family interact and that all states on the sides of the discontinuities are in $\mathcal{U}$. Then there exists a constant C such that, in both the cases, the following estimates hold:*

$$|\sigma_i^+ - \sigma_i^-| + |\sigma_j^+ - \sigma_j^-| + \sum_{l\neq i,j} |\sigma_l^+| \leqslant C \cdot |\sigma_i^- \sigma_j^-| \tag{4.5}$$
$$|\sigma_i^+ - (\sigma_i' + \sigma_i'')| + \sum_{l\neq i} |\sigma_l^+| \leqslant C \cdot |\sigma_i' \sigma_i''| \cdot (|\sigma_i'| + |\sigma_i''|).$$

The constant C in (4.5) depends on a compact set containing all states on the sides of the interacting waves. Moreover, if the interacting waves are sufficiently

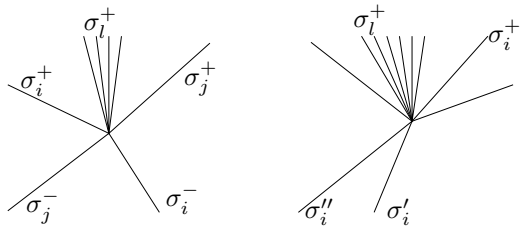

Figure 4. Interactions between waves of different families, left, and of the same family, right.

small, these estimates show that waves do not change sign at interactions. A further consequence of (4.5) is that, in both cases, the function V may well increase at any interaction. In an interaction between waves of different families

$$\Delta V(t_*) = |\sigma_i^+| + |\sigma_j^+| + \sum_{l\neq i,j} |\sigma_l^+| - (|\sigma_i^-| + |\sigma_j^-|) \leqslant C \cdot |\sigma_i^- \sigma_j^-| \tag{4.6}$$

holds, while for interacting waves of the same family we have

$$\Delta V(t_*) = |\sigma_i^+| + \sum_{l\neq i} |\sigma_l^+| - (|\sigma_i'| + |\sigma_i''|) \leqslant C \cdot |\sigma_i' \sigma_i''| \cdot (|\sigma_i'| + |\sigma_i''|).$$

The increase in V is at most *quadratic* in the sizes of the interacting waves.

On the other hand, (SH) ensures that any two waves "*may interact at most once*". Following Glimm [52], we thus introduce the *Glimm interaction potential*

$$Q = \sum_{(\sigma_{i,\alpha},\sigma_{j,\beta})\in\mathcal{A}} |\sigma_{i,\alpha}\sigma_{j,\beta}|, \tag{4.7}$$

$\mathcal{A}$ being the set of the approaching waves, i.e. waves that "*may potentially interact*".

Definition 4.2. The waves $\sigma_{i,\alpha}$ and $\sigma_{j,\beta}$ are *approaching* either if $x_\alpha < x_\beta$ and $i > j$, or if $i = j$, the i-field is genuinely nonlinear and $\min\{\sigma_{i,\alpha}, \sigma_{j,\beta}\} < 0$.

The above definition is motivated by the fact that two adjacent rarefactions or contact discontinuities may not interact.

If the interacting waves belong to different families (Fig. 4, left), then

$$\begin{aligned}\Delta Q(t_*) &= -|\sigma_i^-\sigma_j^-| + (|\sigma_i^+| - |\sigma_i^-|) \sum_{\sigma_{k,\beta}\,:\,(\sigma_i^+,\sigma_{k,\beta})\in\mathcal{A}} |\sigma_{k,\beta}| \\ &\quad + (|\sigma_j^+| - |\sigma_j^-|) \sum_{\sigma_{k,\beta}\,:\,(\sigma_j^+,\sigma_{k,\beta})\in\mathcal{A}} nit|\sigma_{k,\beta}| + \sum_{l\neq i,j} |\sigma_l^+| \sum_{\sigma_{k,\beta}\,:\,(\sigma_l^+,\sigma_{k,\beta})\in\mathcal{A}} |\sigma_{k,\beta}| \\ &\leqslant (-1 + C\cdot V(t_*-)) \cdot |\sigma_i^-\sigma_j^-|\end{aligned} \tag{4.8}$$

and a similar result holds for interacting waves of the same family.

Theorem 4.3. *There exists a positive constant δ such that $B(0,\delta) \subseteq \mathcal{U}$ and*

(Υ1) *any Riemann problem with data in $B(0,\delta)$ admits a unique ε-solution;*

(Υ2) *there exists a constant C such that the estimates* (4.5) *hold whenever the interacting waves separate states in $B(0,\delta)$;*

(Υ3) *the functional $\Upsilon = V + 3C \cdot Q$ with V as in* (4.4) *and Q as in* (4.7) *is such that for any initial datum $\overline{u}^\varepsilon$ satisfying $\Upsilon(\overline{u}^\varepsilon) < \delta$, the functional $t \mapsto \Upsilon(u^\varepsilon(t))$ is non-increasing along the ε-solution constructed above;*

(Υ4) *Υ is uniformly equivalent to the total variation, i.e. there exists a constant K such that for any piecewise constant function with values in $B(0,\delta)$ we have*

$$\frac{1}{K} \cdot \Upsilon(u) \leqslant \mathrm{TV}(u) \leqslant K \cdot \Upsilon(u).$$

Above, $B(0,\delta)$ is the open sphere centered at u with radius δ.

Remark 4.4. The proof of the above result is here described in the case that at most two waves may interact at a single interaction point (t_*, x_*). In general, this is not true and different ways to bypass this difficulty have been devised. One possibility is to change by an "*arbitrarily small*" quantity the speed of waves so that no more than two waves may interact at a single point, see [22]. This allows to prove the above result, but the approximations so obtained fail to depend Lipschitz continuously upon the initial data, see [25, Example 1].

We have assumed above that at most one interaction takes place at any t_*. The case of more interactions at the same t_* does not require a specific treatment, due to the *finite propagation speed* displayed by (1.1) and by the present algorithm.

4.2. Control on the number of interactions

The usual way to prevent the formation of cluster points of discontinuities (difficulty (D2) above) is to bound the number of interaction points. More precisely, interaction points are proved to be finite on any compact subset of $[0, +\infty[\times \mathbb{R}$.

The bound on the total variation [(Υ3), Theorem 4.3] shows that the number of interaction points can be bounded once the number of *small* waves can be controlled. To this aim, several techniques have been considered. We consider first the "*general case*", i.e. a construction that works under the assumptions

$$\text{(SH)}, \quad \text{(GNL/LD)}, \quad n \geqslant 1, \quad \mathrm{TV}(\overline{u}) \text{ small}.$$

Separately, we consider the case of Temple systems, where a geometrical assumption is required, but neither (GNL/LD) nor $\mathrm{TV}(\overline{u})$ small are necessary. Finally, we present a construction that works only in the case $n = 2$ but can be extended to various other situations, such as systems with phase transitions [36].

4.2.1. The general case

In the case $n \geqslant 1$ with initial data having small total variation, various techniques in literature [13], [22], [53], [64], [65] lead to the suppression of sufficiently small waves. Here, we follow the construction from [22, Chapter 7], see also [23].

The first simplification is achieved avoiding further splitting of rarefactions. When a rarefaction hits a wave of another family, its size slightly varies, see the first estimate in (4.5). Strict application of the above procedure would lead to the splitting of this rarefaction into waves having size at most ε. To reduce the number of waves (and, hence, of interactions) we convene not to split any rarefaction after its birth. Indeed, it holds that the size $\sigma(t)$ of a rarefaction born at time t_o is uniformly bounded, i.e. $|\sigma(t)| \leqslant K\varepsilon$ for all times $t \geqslant t_o$, the constant K depending only on the total variation of the initial data.

Let $\hat{\lambda}$ be an upper bound for all characteristic speeds and let us fix a threshold ϱ with, say, $\varrho \ll \varepsilon$. Use the *Accurate Riemann solver* at time $t = 0$ and whenever the product of the interacting waves is in absolute value greater than ϱ.

When two waves σ_i, σ_j with $|\sigma_i \sigma_j| \leqslant \varrho$ interact, use the following *Simplified Riemann solver*: prolong the incoming waves with waves of the same family and size. Then, introduce a further *non-physical* wave to adjust the states on the right, see Fig. 5, left. More precisely, we have

$$\begin{aligned} &\text{Before:} \qquad u_-^m = \mathcal{L}_i(u^l, \sigma_i), \qquad u^r = \mathcal{L}_j(u_-^m, \sigma_j);\\ &\text{After:} \qquad u_+^m = \mathcal{L}_j(u^l, \sigma_j), \qquad u_+^r = \mathcal{L}_i(u_+^m, \sigma_i). \end{aligned}$$

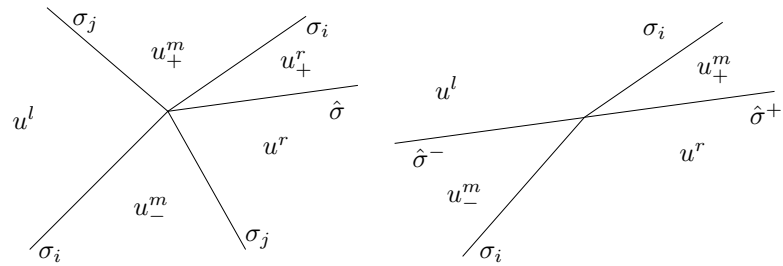

Figure 5. Left, the simplified Riemann solver and a non-physical wave. Right, a non-physical wave hits an i-wave.

The non-physical wave $\hat{\sigma}$ separates the states u_+^r and u^r, is assigned the size $\hat{\sigma} = \|u_+^r - u^r\|$ and is considered to belong to a fictitious linearly degenerate $(n+1)$st family. Moreover, whenever a non-physical wave $\hat{\sigma}$ hits a physical one σ_i, the former proceeds with unchanged size, while the size of the latter needs to be slightly adjusted,

see Fig. 5, right:

$$\begin{aligned} &\text{Before:} \qquad \|u_-^m - u^l\| = \hat\sigma^-, \quad u^r = \mathcal{L}_i(u_-^m, \sigma_i);\\ &\text{After:} \qquad u_+^m = \mathcal{L}_i(u^l, \sigma_i), \quad \|u^r - u_+^m\| = \hat\sigma^+. \end{aligned}$$

This modification makes the derivation of the bounds on the total variation much more intricate: at any step, various cases need to be considered depending on the nature (physical or non-physical) of the waves considered. A key estimate in this whole procedure is that if ϱ is sufficiently smaller than ε, then the total strength of all non-physical waves is bounded by ε:

$$\sum_{\sigma_\alpha \text{ non-physical}} \|\sigma_\alpha\| = \mathcal{O}(1) \cdot \varepsilon. \tag{4.9}$$

4.2.2. Temple systems

Definition 4.5. System (1.1) is a *Temple system* if the following holds:

(T1) assumption (USH);
(T2) shock and rarefaction curves coincide;
(T3) there exist coordinates w such that $\partial u/\partial w_i$ is the ith right eigenvector of $\mathrm{D}f$.

(In literature, this definition is subject to *variazioni*, see [12], [15], [66], [67], 69]).

A typical property of Temple systems is a sort of decoupling, in the sense that solving any Riemann problem is equivalent to solving n scalar Riemann problems. Besides, in Temple systems interactions have a "*linear*" behaviour. Indeed, if the waves $\sigma_i^1, \dots, \sigma_i^{m_i}$ of the i-family interact, for $i = 1, \dots, n$, the i-waves exiting the interaction have total size $\sigma_i^+ = \sum_{l=1}^{m_i} \sigma_i^l$, using the parametrization defined in Lemmas 3.2, 3.4 and 3.7. Note however that in an interaction involving only 2 waves of different families, the waves that exit the interaction have the same size as those entering it, but possibly different speed. Moreover, if no i-wave enters the interaction point, then no i-wave exits it.

As a consequence, the Wave Front Tracking algorithm in [12], [15] defines solutions attaining values on a *fixed* grid with mesh size ε. Hence, no wave can have size smaller than ε. This, together with a careful use of the decrease of Q, allows to prove that there is a *finite* number of waves on all $[0, +\infty[\times \mathbb{R}$. Another consequence of the introduction of the ε-grid is that all the functionals V, Q and Υ formally depend also on ε.

The results in [15] allow to remove (GNL/LD) in Temple systems. Morever, the total variation of the initial data is not required to be small.

4.2.3. 2×2 **systems**

In the case $n = 2$, a careful definition of the approximate solution implies that all adjacent small waves of the same family are parallel. Indeed, when small waves are involved, the flow f is approximated essentially through a Temple system and adjacent waves of the same family having size σ with $|\sigma| < \varepsilon$ may be parallel. In this approximation, it is essential to substitute the Lax curves (3.7) by the approximations (see [25])

$$\mathcal{L}_i^\varepsilon(u_o, \sigma) = \begin{cases} S_i(u_o, \sigma) & \text{if } \sigma < -2\sqrt{\varepsilon}, \\ \varphi\Big(\dfrac{\sigma}{\sqrt{\varepsilon}}\Big) S_i(u_o, \sigma) + \Big(1 - \varphi\Big(\dfrac{\sigma}{\sqrt{\varepsilon}}\Big)\Big) R_i(u_o, \sigma), & \\ & \text{if } \sigma \in [-2\sqrt{\varepsilon}, -\sqrt{\varepsilon}], \\ R_i(u_o, \sigma) & \text{if } \sigma > -\sqrt{\varepsilon} \end{cases} \tag{4.10}$$

with φ being a $\mathbf{C}^\infty$ function such that

$$\begin{cases} \varphi(\sigma) = 1 & \text{if } \sigma \in \,]-\infty, -2], \\ \varphi'(\sigma) \in [-2, 0] & \text{if } \sigma \in \,]-2, -1[, \\ \varphi(\sigma) = 0 & \text{if } \sigma \in [-1, +\infty[. \end{cases}$$

The ε-approximate rarefactions are then obtained cutting the rarefactions along a fixed grid of size ε. Note that also all shocks with size $|\sigma| < 2\sqrt{\varepsilon}$ are approximated, due to the interpolation (4.10), and (3.6) provides bounds for this error.

The speeds of these waves are assigned so that the coordinates along the approximate rarefactions (4.10) are *exact* solutions to scalar conservation laws with piecewise linear flux function, see [25, Section 2] for the explicit formulæ. Hence, adjacent small waves propagate parallelly and less interactions take place.

With this construction, it is then possible to show directly that no compact set may contain a cluster point of interaction points. This step depends essentially on the assumption $n = 2$.

An inductive procedure allows to extend estimates of the type (4.5) to the case of multiple interactions. Hence, the "*arbitrarily small*" change of the wave speeds to avoid multiple interaction is not necessary and the approximate solutions depend Lipschitz continuously on the initial data.

Note that when the initial data is a perturbation of a *large* jump, then the number of interactions may well be infinite over all $[0, +\infty[\times \mathbb{R}$, see [26].

If (1.1) is a Temple system, the present algorithm provides the same solutions as those specifically constructed for Temple systems, the only difference being that here approximate solutions do not need take values in a fixed ε-grid.

4.3. The limit

Before passing to the limit for $\varepsilon \to 0$, we estimate how far the ε-approximate solution is from being an exact solution to (2.1). First, there is an error due to the initial datum, which vanishes as $\varepsilon \to 0$ thanks to (4.1).

Then, another error is due to the fact that not all discontinuities in u^ε satisfy the Rankine-Hugoniot conditions [(S1), Lemma 3.4]. In the general case (Paragraph 4.2.1), rarefaction waves and non-physical waves violate it. Let $\Delta u^\varepsilon(t, x_\alpha) = u^\varepsilon(t, x_\alpha+) - u^\varepsilon(t, x_\alpha-)$ and define $\Delta f(u^\varepsilon(t, x_\alpha))$ similarly. The Rankine-Hugoniot conditions along a rarefaction wave supported at x_α with propagation speed $\dot{x}_\alpha$ are missed with an error

$$|\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon \cdot \sigma_\alpha,$$

σ_α being the rarefaction at x_α. This second order estimate is a consequence of (3.6). Summing over all rarefactions, if $\Upsilon(\bar{u}) < \delta$, the total error is

$$\sum_{\sigma_\alpha \text{ rarefaction}} |\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon \cdot \delta$$

and converges to 0 as $\varepsilon \to 0$. If a non-physical wave is supported at x_α, then

$$|\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot |\sigma_\alpha|$$

and the bound is now only of the first order. On the other hand, when summing over all non-physical waves, by (4.9) we obtain

$$\sum_{\sigma_\alpha \text{ non-physical}} |\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon.$$

The entropy inequalities are yet another source of error: neither approximate rarefactions nor non-physical waves satisfy them. The corresponding bounds are similar to those for Rankine-Hugoniot conditions, see [4] and [5, Theorem 5.1].

Before the final limit, note that ε-solutions are Lipschitz with respect to t.

Lemma 4.6. *With the above definitions of ε-solution, there exist constants δ, L such that for all piecewise constant initial data $u \in \mathbf{L^1}$ with $\Upsilon(u) < \delta$, the ε-solution u^ε satisfies $\|u^\varepsilon(t'') - u^\varepsilon(t')\|_{\mathbf{L^1}} \leqslant L \cdot |t'' - t'|$ uniformly in ε.*

Above, L depends on δ and on the maximal characteristic speed.

The existence of solutions to (2.1) is now at hand. Theorem 4.3 provides a bound uniform in ε on the total variation of the ε-approximate solution $u^\varepsilon(t, \cdot)$ at any time

$t \geqslant 0$. Fix a positive sequence ε_n with $\lim_{n\to+\infty} \varepsilon_n = 0$. A slight modification of the classical Helly's Compactness Theorem [22, Theorem 2.4] allows to extract from ε_n a subsequence ε_{n_k} such that there exists a function $u\colon [0,+\infty[\to \mathbf{L^1}(\mathbb{R})$ with the properties

1. $u^{\varepsilon_{n_k}} \to u$ in $\mathbf{L^1_{loc}}([0,+\infty[\times \mathbb{R})$ as $k \to +\infty$;
2. $\|u(t'') - u(t')\|_{\mathbf{L^1}} \leqslant L \cdot |t'' - t'|$;
3. $\mathrm{TV}(u(t)) \leqslant K\delta$ with K as in [(Υ4), Theorem 4.3].

The above properties lead to the proof that u is a weak entropic solution.

Remark 4.7. The existence of solutions was obtained by means of a *compactness* argument. Hence, neither uniqueness nor continuous dependence are directly available through the same method.

In Temple systems, the limiting procedure is somewhat simpler, since non-physical waves are absent. In the 2×2 case, also small shocks cause some error, due to the interpolation (4.10), and need to be considered separately.

5. Stability

This section is devoted to the proof of continuous dependence on the initial data. In the case of conservation laws, this proof usually preceeds that of uniqueness. For the stability with respect to the flow f, see [17].

The results in the previous sections prove the following theorem.

Theorem 5.1. *Let f be smooth and satisfy* (SH), (GNL/LD). *Then there exist positive δ, K, L and, for all small ε, a map $S^\varepsilon\colon [0,+\infty[\times \mathcal{D}^\varepsilon \to \mathcal{D}^\varepsilon$ satisfying*

1. *the domain $\mathcal{D}^\varepsilon = \{u \in \mathbf{L^1}\colon u$ piecewise constant and $\Upsilon(u) < \delta\}$ is invariant with respect to S^ε;*
2. *$\mathcal{D}^\varepsilon \supseteq \{u \in \mathbf{L^1}\colon u$ piecewise constant and $\mathrm{TV}(u) < K\delta\}$;*
3. *if $n = 2$ or if* (1.1) *is a Temple system, then S^ε is a semigroup;*
4. *along a physical discontinuity at, say, $x_\alpha(t)$, the Rankine-Hugoniot condition* [(S1), *Lemma* 3.4] *is approximately satisfied:*

$$|\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon \cdot |\sigma_\alpha|,$$

while along a non-physical discontinuity we have

$$|\dot{x}_\alpha \cdot \Delta u^\varepsilon(t, x_\alpha) - \Delta f(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot |\sigma_\alpha|;$$

5. *if an entropy-entropy flux pair exists along a physical discontinuity at* $x_\alpha(t)$*, the error in the entropy condition* (2.4) *is bounded by*

$$|\dot{x}_\alpha \cdot \Delta\eta(u^\varepsilon(t, x_\alpha)) - \Delta q(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot \varepsilon \cdot |\sigma_\alpha|,$$

while in the case of a nonphysical discontinuity we have

$$|\dot{x}_\alpha \cdot \Delta\eta(u^\varepsilon(t, x_\alpha)) - \Delta q(u^\varepsilon(t, x_\alpha))| = \mathcal{O}(1) \cdot |\sigma_\alpha|;$$

6. *the total size of non-physical waves is bounded as in* (4.9)*;*
7. S^ε *is* $\mathbf{L^1}$*-Lipschitz with respect to time:* $\|S^\varepsilon_{t_1} u - S^\varepsilon_{t_2} u\|_{\mathbf{L^1}} \leqslant L \cdot \|t_2 - t_1\|$.

Note that in the general case, due to the presence of the Simplified Riemann solver, S^ε is *not* a semigroup. Indeed, assume that an interaction takes place at time t_*. The definition of S^ε after t_* depends on the *history* before t_*. In the general case, the semigroup property is recovered in the limit as a consequence of the uniqueness of the limit of the approximate semigroups.

Our next target is the regularity of S^ε as a function of the initial data. Two entirely different techniques are available, the former in the general case, while the latter works for Temple and 2×2 systems.

5.1. The functional

This section follows the ideas introduced in [34], [60], [61], [62].

The limit semigroup is proved Lipschitz once a functional $\Phi\colon\ \mathcal{D}^\varepsilon \times \mathcal{D}^\varepsilon \to [0, +\infty[$ with the following properties, uniform in ε, is introduced:

(Φ1) for a suitable positive κ, $1/\kappa \cdot \|u - w\|_{\mathbf{L^1}} \leqslant \Phi(u, w) \leqslant \kappa \cdot \|u - w\|_{\mathbf{L^1}}$;

(Φ2) for any two ε-solutions u and w, $\Phi(u(t), w(t)) \leqslant \Phi(u(0), w(0)) + \mathcal{O}(1) \cdot \varepsilon \cdot t$.

To define Φ, introduce first a sort of distance in $\mathcal{U}$ by means of (1.1). In the next lemma and in all this paragraph, we use the term "*i-shock*" also for the i-Lax curves of linearly degenerate families. Moreover, also the non entropic parts of shock curves are used due to the interpolation (4.10).

Lemma 5.2. *Any* $\overline{u} \in \Omega$ *admits a neighborhood* $\mathcal{U}$ *such that*

1. *the conclusions of Theorem* 3.11 *hold in* $\mathcal{U}$*;*
2. *for any two points* u, w *in* $\mathcal{U}$*, there exists a unique* n*-tuple* $(q_1, \ldots, q_n)$ *such that there are* $n+1$ *states* $u_0, \ldots, u_n$ *in* Ω *satisfying* $u_0 = u$, $u_1 = \mathcal{L}_1(u_0, \sigma_1), \ldots, u_i = S_i(u_{i-1}, q_i), \ldots, u_n = w$*;*
3. *there exists a positive* c *such that* $1/c \cdot \|w - u\| \leqslant \sum_{i=1}^{n} |q_i| \leqslant c \cdot \|w - u\|$.

In other words, the q_is are the sizes of the i-shocks that solve the Riemann problem (3.1) with $u^l = u$ and $u^r = w$ having the minimum number of (possibly non

entropic) shocks. The last statement above amounts to say that the q_is provide a measure of the distance in $\mathcal{U}$ equivalent to the Euclidean one. However, setting $d(u,w) = \sum_i |q_i|$, d is not a distance for, in general, since neither the triangular inequality nor the symmetric property $d(u,w) = d(w,u)$ hold.

Theorem 5.3. *Let f satisfy* (SH) *and* (GNL/LD). *Let $q_1(x), \dots, q_n(x)$ be the shock sizes defined in Lemma* 5.2 *with reference to $u(x)$ and $w(x)$, for $u, w \in \mathcal{D}^\varepsilon$. Reducing the δ in* [1, *Theorem* 5.1], *if necessary, it is possible to define weights $W_1, \dots, W_n \colon \mathbb{R} \to [0, +\infty[$ so that*

$$\Phi(u,w) = \int_{\mathbb{R}} \sum_{i=1}^{n} |q_i(x)| W_i(x)\, \mathrm{d}x$$

satisfies (Φ1) *and* (Φ2).

Note that (Φ1) holds as soon as the weights W_i are uniformly bounded, say $W_i(x) \in [1,2]$ for all x. All difficulties in the proof of the well-posedness of (1.1) are thus reduced to the search for these weights, whose explicit definition is far from immediate. For any u in $\mathcal{D}^\varepsilon$, written as in (4.3), let $\mathcal{J}(u)$ be the set of jumps in u. Define, for $i = 1, \dots, n$,

$$W_i(x) = 1 + \kappa_1 A_i(x) + \kappa_1 \kappa_2 (Q(u) + Q(w)) \tag{5.1}$$

and, if the i-field is linearly degenerate, set

$$\begin{aligned} A_i(x) = {} & \sum \{ |\sigma_{j,\alpha}| \colon \alpha \in \mathcal{J}(u) \cup \mathcal{J}(w),\ x_\alpha < x,\ j \in \{i+1, \dots, n\} \} \\ & + \sum \{ |\sigma_{j,\alpha}| \colon \alpha \in \mathcal{J}(u) \cup \mathcal{J}(w),\ x_\alpha > x,\ j \in \{1, \dots, i-1\} \} \end{aligned}$$

where $\sum \{\sigma \colon \sigma \in A\} = \sum_{\sigma \in A} \sigma$. If the i-field is genuinely nonlinear, let

$$\begin{aligned} A_i(x) = {} & \sum \{ |\sigma_{j,\alpha}| \colon \alpha \in \mathcal{J}(u) \cup \mathcal{J}(w),\ x_\alpha < x,\ j \in \{i+1, \dots, n\} \} \\ & + \sum \{ |\sigma_{j,\alpha}| \colon \alpha \in \mathcal{J}(u) \cup \mathcal{J}(w),\ x_\alpha > x,\ j \in \{1, \dots, i-1\} \} \\ & + \begin{cases} \sum \{ |\sigma_{i,\alpha}| \colon \alpha \in \mathcal{J}(u),\ x_\alpha < x \} + \sum \{ |\sigma_{i,\alpha}| \colon \alpha \in \mathcal{J}(w),\ x_\alpha > x \}, \\ \qquad\qquad \text{if } q_i(x) < 0, \\ \sum \{ |\sigma_{i,\alpha}| \colon \alpha \in \mathcal{J}(u),\ x_\alpha > x \} + \sum \{ |\sigma_{i,\alpha}| \colon \alpha \in \mathcal{J}(w),\ x_\alpha < x \} \\ \qquad\qquad \text{if } q_i(x) > 0. \end{cases} \end{aligned}$$

Note that non-physical waves have no role in the construction of the weights.

With this definition, Theorem 5.3 can be proved. The particular above choice of the weights is partly motivated as follows. Rewrite Φ as

$$\begin{aligned}\Phi(u(t),w(t)) = &\int_{\mathbb{R}} \left(\sum_{j=1}^{n} |q_j(x)|\right) \mathrm{d}x \\ &+ \sum_{\sigma_{i,\alpha}\in\mathcal{J}(u)\cup\mathcal{J}(w)} \sum_{j=1}^{n} |\sigma_{i,\alpha}| \int_{\mathcal{I}^j_{i,\alpha}} |q_j(x)|\,\mathrm{d}x \\ &+ \kappa_1\kappa_2(Q(u)+Q(w)) \int_{\mathbb{R}} \left(\sum_{j=1}^{n} |q_j(x)|\right) \mathrm{d}x\end{aligned}$$

where $\mathcal{I}^j_{i,\alpha}$ is the subset of $\mathbb{R}$ where $q_j(x)$ approaches $\sigma_{i,\alpha}$ in the sense of Definition 4.2. The first term is essentially the $\mathbf{L^1}$ norm, see [3, Lemma 5.2]. The last term exploits the fact that the interaction potential decreases and is used to ensure the decrease of Φ at interaction times. The second term is the key point. Call u_j and w_j the coordinate of u and w along the j-Lax curve. Each summand in this sum is the size of a wave $\sigma_{i,\alpha}$ in u or in w, multiplied by the area selected by u_j and w_j over $\mathcal{I}^j_{i,\alpha}$. This set, a union of intervals, is defined as the place where $q_j(x)$ approaches $\sigma_{i,\alpha}$ in the sense of Definition 4.2.

Consider for example the jth summand in the first sum defining A_i, in the case $i > j$ with the i-field genuinely nonlinear. Then $\int_{\mathcal{I}^j_{i,\alpha}} |q_j(x)|\,\mathrm{d}x$ is a measure of the area between u_j and w_j to the right of x_α. Due to (SH), this area decreases in time, see Fig. 6, since $\lambda_i > \lambda_j$. For analogous geometric interpretations of the other summands in A_i, see [23, Chapter 5].

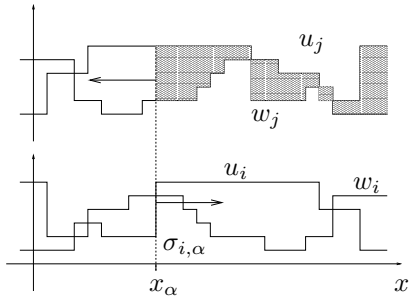

Figure 6. The dashed area between u_j and w_j to the right of x_α diminishes.

Technically, to achieve the proof of the key estimate (Φ2), the first step is taking the derivative of $\Phi(u(t),w(t))$. Denote $q_i^{\alpha\pm} = q_i(x_\alpha\pm)$ and similarly for W_i, while $\dot{x}_\alpha$ denotes the speed of the wave at x_α (at each x_α there is a unique such wave outside

interaction times). Then

$$\begin{aligned}\frac{\mathrm{d}\Phi(u(t),w(t))}{\mathrm{d}t} &= \sum_{\alpha}\sum_{i=1}^{n}\big(|q_i^{\alpha-}|W_i^{\alpha-} - |q_i^{\alpha+}|W_i^{\alpha+}\big)\cdot\dot{x}_\alpha \\ &= \sum_{i,\alpha}(|q_i^{\alpha+}|W_i^{\alpha+}(\lambda_i^{\alpha+}-\dot{x}_\alpha) - |q_i^{\alpha-}|W_i^{\alpha-}(\lambda_i^{\alpha-}-\dot{x}_\alpha))\end{aligned}$$

where $\lambda_i^{\alpha\pm} = \lambda_i(u(x_\alpha\pm))$. In the last line above, the equality $|q_i^{\alpha-}|W_i^{\alpha-}\lambda_i^{\alpha-} = \big|q_i^{(\alpha-1)+}\big|W_i^{(\alpha-1)+}\lambda_i^{(\alpha-1)+}$ was used. Then, upper bounds on this quantity are derived through suitable interaction estimates. Indeed, by considering various cases, one shows that for any physical wave $\sigma_{i,\alpha}$, for any j and α,

$$|q_j^{\alpha+}|W_j^{\alpha+}(\lambda_j^{\alpha+}-\dot{x}_\alpha) - |q_j^{\alpha-}|W_j^{\alpha-}(\lambda_j^{\alpha-}-\dot{x}_\alpha) = \mathcal{O}(1)\cdot\varepsilon\cdot|\sigma_{i,\alpha}|$$

while if $\sigma_{i,\alpha}$ is a non-physical wave

$$|q_j^{\alpha+}|W_j^{\alpha+}(\lambda_j^{\alpha+}-\dot{x}_\alpha) - |q_j^{\alpha-}|W_j^{\alpha-}(\lambda_j^{\alpha-}-\dot{x}_\alpha) = \mathcal{O}(1)\cdot|\sigma_{i,\alpha}|.$$

Summing over all i and α, we get $\mathrm{d}\Phi(u(t),w(t))/\mathrm{d}t \leqslant c\cdot\varepsilon\cdot\delta$ for a $c>0$ and for all times when no interaction occur. A suitable choice of κ_2 in (5.1) implies that $\Delta\Phi(u(t),w(t)) \leqslant 0$ at any interaction time, completing the proof of (Φ2).

Concerning the final limiting procedure, choose a positive sequence ε_ν with $\varepsilon_\nu\to 0$ as $\nu\to+\infty$ and approximate the initial data $\overline{u}$ by means of a sequence $\overline{u}_\nu$ with $\lim\limits_{\nu\to+\infty}\|\overline{u}-\overline{u}_\nu\|_{\mathbf{L^1}} = 0$. Then, by ($\Phi$1) and ($\Phi$2),

$$\begin{aligned}\|S_t^{\varepsilon_{\nu_1}}\overline{u}_{\nu_1} - S_t^{\varepsilon_{\nu_2}}\overline{u}_{\nu_2}\|_{\mathbf{L^1}} &\leqslant \kappa\cdot\Phi(S_t^{\varepsilon_{\nu_1}}\overline{u}_{\nu_1}, S_t^{\varepsilon_{\nu_2}}\overline{u}_{\nu_2}) \\ &\leqslant \kappa\cdot\Phi(\overline{u}_{\nu_1},\overline{u}_{\nu_2}) + \mathcal{O}(1)\cdot\max\{\varepsilon_{\nu_1},\varepsilon_{\nu_2}\}\cdot t \\ &\leqslant \kappa^2\cdot\|\overline{u}_{\nu_1}-\overline{u}_{\nu_2}|_{\mathbf{L^1}} + \mathcal{O}(1)\cdot\max\{\varepsilon_{\nu_1},\varepsilon_{\nu_2}\}\cdot t,\end{aligned}$$

which shows the existence of the limit by a completeness argument.

R e m a r k 5.4. The proof of the Lipschitz dependence of the approximate solution allows also to obtain an existence proof entirely independent of the previous one. Moreover, the compactness argument is now substituted by a completeness argument. These observations apply also to the other technique exposed below.

5.2. Pseudopolygons

In the present section, we follow [25].

In Temple and 2×2 systems, the ε-approximate semigroups are $\mathbf{L^1}$-Lipschitz with a Lipschitz constant bounded uniformly in ε. To prove it, any two initial data u,

w in $\mathcal{D}^\varepsilon$ are interpolated through a curve γ and the whole curve evolves with S^ε, leading to $S^\varepsilon_t \circ \gamma$. An estimate of $\|S^\varepsilon_t u - S^\varepsilon_t w\|_{\mathbf{L^1}}$ is obtained through the length of $S^\varepsilon_t \circ \gamma$. In turn, this is achieved by means of an estimate of how the vector tangent to $S^\varepsilon_t \circ \gamma$ varies with time.

The key idea is to define γ shifting the locations of the jumps in the initial data u at constant rates, and then studying the rates at which the jumps in the corresponding solution $S^\varepsilon_t u$ are shifted, for any fixed $t > 0$.

Definition 5.5. Let $]a, b[$ be an open interval. An *elementary path* is a map $\gamma\colon\,]a, b[\to \mathcal{D}^\varepsilon$ of the form $\gamma(\theta) = \sum_{\alpha=1}^{N} \omega_\alpha \cdot \chi_{]x^\theta_{\alpha-1}, x^\theta_\alpha]}$, where $x^\theta_\alpha = \overline{x}_\alpha + \xi_\alpha \theta$ with $x^\theta_{\alpha-1} < x^\theta_\alpha$ for all $\theta \in]a, b[$ and $\alpha = 1, \dots, N$.

For each θ, $\gamma(\theta)$ is piecewise constant with bounded support. As θ varies, the values ω_α remain constant while the locations of the jumps x^θ_α shift with constant speeds ξ_α leaving the ordering of the x^θ_α unchanged.

Definition 5.6. A continuous map $\gamma\colon\, [a, b] \to \mathcal{D}^\varepsilon$ is a *pseudopolygon* if there exist countably many disjoint open intervals $J_h \subset [a, b]$ such that the restriction of γ to each J_h is an elementary path and the set $[a, b] \setminus \bigcup_{h \geqslant 1} J_h$ is countable.

Every couple of initial conditions u, w in $\mathcal{D}^\varepsilon$ can be joined by a pseudopolygon. A remarkable property of the algorithm described in Paragraphs 4.2.2 and 4.2.3 is that it preserves pseudopolygons.

Proposition 5.7. *Let γ be a pseudopolygon. Then, for all $t > 0$, the path $S^\varepsilon_t \circ \gamma$ is also a pseudopolygon. Furthermore, there exist countably many open intervals J_h such that $[a, b] \setminus \left(\bigcup_h J_h\right)$ is countable and for any h, the functions $(S^\varepsilon_t \circ \gamma)(\theta)$, $\theta \in J_h$, all have the same number of waves, interacting at the same number of points in $[0, t] \times \mathbb{R}$. As θ varies, these waves and the interaction points are shifted with constant speeds in the (t, x)-plane.*

The $\mathbf{L^1}$-length of a pseudopolygon γ is the sum of the lengths of the elementary paths obtained by restricting γ to each subinterval J_h. It is thus sufficient to study the case where γ is an elementary path and the wave configuration of the solution $S^\varepsilon_t \circ \gamma$ on the strip $[0, T] \times \mathbb{R}$ is fixed for all $\theta \in]a, b[$. For a fixed $t \in [0, T]$, let $u^\theta(t) = (S^\varepsilon_t \circ \gamma)(\theta)$ have the form $u^\theta(t, x) = \sum_{\alpha=1}^{N} u^\alpha \chi_{]x^\theta_{\alpha-1}(t), x^\theta_\alpha(t)]}(x)$ where $x^\theta_\alpha(t) = \overline{x}_\alpha(t) + \xi_\alpha \theta$. The $\mathbf{L^1}$-length of the path $S^\varepsilon_t \circ \gamma$ is then measured by $\|S^\varepsilon_t \circ \gamma\| = \int_a^b \sum_\alpha |\partial_\theta x^\theta_\alpha| \, \|\Delta u^\theta(x_\alpha)\| \, \mathrm{d}\theta$. In order to relate the length of $S^\varepsilon_t \circ \gamma$ with the length of the path γ interpolating the initial conditions, for any given $\theta \in]a, b[$ we study how

the sum $\sum_\alpha |\partial_\theta x^\theta_\alpha| \, \|\Delta u^\theta(x_\alpha)\|$ varies in time. Clearly, this sum can change only at those times where an interaction takes place.

Consider, for instance, an interaction as in Fig. 4, left, between two waves σ_j^-, σ_i^- located on the lines $x_j^{\theta-}(t) = x_j^-(t) + \xi_j^-\theta$ and $x_i^{\theta-}(t) = x_i^-(t) + \xi_i^-\theta$. As θ varies, the waves before the interaction shift at the rates $\xi_j^- = \partial_\theta x_j^{\theta-}(t)$, $\xi_i^- = \partial_\theta x_i^{\theta-}(t)$. Assume that the interaction produces n_k waves of the k-family for $k = 1, \dots, n$, having sizes $\sigma^+_{k,1}, \dots, \sigma^+_{k,n_k}$. The interaction point $P^\theta = (\bar t^\theta, \bar x^\theta)$ shifts at a constant rate, as do the locations of the outgoing waves, say $x^{\theta+}_{k,\ell}(t) = x^+_{k,\ell}(t) + \xi^+_{k,\ell}\theta$ for $\ell = 1, \dots, n_k$ and $k = 1, \dots, n$. The next lemma provides the basic estimate on strengths and shift rates of waves before and after an interaction, generalizing (4.5) to the case of shifting interactions.

Lemma 5.8. *There exists a constant C independent of ε such that, whenever two waves interact, the quantities introduced above satisfy*

$$\sum_{k=1}^{n} \sum_{\ell=1}^{n_k} |\sigma^+_{k,\ell} \xi^+_{i,\ell}| \leqslant |\sigma_i^- \xi_i^-| + |\sigma_j^- \xi_j^-| + C \cdot |\sigma_i^- \sigma_j^-| (|\xi_i^-| + |\xi_j^-|).$$

The $\mathbf{L^1}$-length of $S^\varepsilon_t \circ \gamma$ may well increase in time. Lemma 5.8 allows to control this increase by means of an interaction potential. Indeed, we will introduce on $\mathcal{D}^\varepsilon$ a metric equivalent to the $\mathbf{L^1}$-distance and such that with respect to it, the semigroup S^ε turns out to be contractive.

We define the *weighted length* of an elementary path $\gamma\colon\ ]a, b[\to \mathcal{D}^\varepsilon$ by

$$\|\gamma\|_\varepsilon = \sum_\alpha \sum_{i=1}^{n} (b-a) \cdot \|\sigma_{i,\alpha} \xi_{i,\alpha}| \cdot W_{i,\alpha}$$

for suitable weights $W_{i,\alpha}$. In the more general case where γ is a pseudopolygon, we define its *weighted length* $\|\gamma\|_\varepsilon$ as the sum of the weighted lengths of its elementary paths. As soon as the weigths $W_{i,\alpha}$ satisfy uniform bounds

$$1 \leqslant W_{i,\alpha} \leqslant c, \tag{5.2}$$

it is immediate to prove the equivalence between the $\mathbf{L^1}$-length and the weighted length, i.e. there exists a constant $\tilde c$ such that $1/\tilde c \cdot \|\gamma\|_{\mathbf{L^1}} \leqslant \|\gamma\|_\varepsilon \leqslant \tilde c \cdot \|\gamma\|_{\mathbf{L^1}}$.

The next proposition contains the key difficulties of the present approach, namely the definition of the weights $W_{i,\alpha}$.

Proposition 5.9. *There exist constants δ and c such that for all small ε, for any elementary path $\gamma\colon [a,b] \to \mathcal{D}^\varepsilon$ with $S^\varepsilon_t \circ \gamma$ having a fixed wave configuration on $[0,T]$, there exist weights $W_{i,\alpha}$ satisfying* (5.2) *uniformly in ε and such that the map $t \mapsto \|S^\varepsilon_t \circ \gamma\|_\varepsilon$ is non-increasing for $t \in [0,T]$.*

Remark that in the case $n = 2$, the weights $W_{i,\alpha}$ can be explicitely defined, by quite different expressions depending on the specific situation, see [25], [26], [36], [38], [44]. On the contrary, in the case of Temple systems with $n > 2$, only an implicit (backward) recursive construction is currently available [12], [15].

On $\mathcal{D}^\varepsilon$ define now the metric

$$d_\varepsilon(u,w) = \inf\{\|\gamma\|_\varepsilon\colon\ \gamma \text{ a pseudopolygon in } \mathcal{D}^\varepsilon \text{ joining } u \text{ with } w\}. \tag{5.3}$$

Note that d_ε does not fit in the set of metrics considered in [70]. Apply now Proposition 5.9 to obtain the following lemma.

Lemma 5.10. *There exists a $\delta > 0$ such that, restricted to $\mathcal{D}^\varepsilon$, the distance d_ε in* (5.3) *is uniformly equivalent to the $\mathbf{L^1}$-distance. Moreover, S^ε is contractive with respect to d_ε: $d_\varepsilon(S^\varepsilon_t u, S^\varepsilon_t w) \leqslant d_\varepsilon(u,w)$ for all $t \geqslant 0$ and all u, w in $\mathcal{D}^\varepsilon$.*

In terms of the $\mathbf{L^1}$-metric, we obtain $\|S^\varepsilon_t u - S^\varepsilon_t w\|_{\mathbf{L^1}} \leqslant L \cdot \|u - w\|_{\mathbf{L^1}}$ for all $t \geqslant 0$, $u, w \in \mathcal{D}^\varepsilon$ and for some L independent of ε. To complete the proof of the $\mathbf{L^1}$-Lipschitz dependence, choose $\varepsilon_\nu = 2^{-\nu}$, $\nu \in \mathbb{N}$, and define S as $S_t u = \lim\limits_{\nu \to +\infty} S^{\varepsilon_\nu}_t u_\nu$, with $u_\nu \in \mathcal{D}^{\varepsilon_\nu}$ and $\|u_\nu - u\|_{\mathbf{L^1}} \leqslant \varepsilon_\nu$. The existence of the previous limit follows from a *completeness* argument based on Lemma 6.2. The domain of S is $\mathcal{D} = \{u \in \mathbf{L^1}\colon\ \exists u_\nu \in \mathcal{D}^{\varepsilon_\nu} \text{ and } \|u_\nu - u\|_{\mathbf{L^1}} \leqslant \varepsilon_\nu\}$.

6. Uniqueness

A uniqueness result consists in the selection of a class such that

(U1) an existence result provides solutions in this class;

(U2) two solutions in this class coincide.

In the case of conservation laws, two different kinds of uniqueness results can be considered: the one referring to the *semigroup* constructed above and another one referring to the *single solution* to (2.1). The first result in this direction, in the case $n = 2$, was obtained in [26]. In the general case, see [29], [31].

6.1. Uniqueness of the semigroup

The key difficulty in proving the uniqueness of the semigroup S constructed above consists in the selection of the properties of S that define the class where S is unique. This class was selected by Bressan in [19] through the following definition.

Definition 6.1. The system (1.1) generates a *Standard Riemann Semigroup* $S\colon [0,+\infty[\times \mathcal{D} \to \mathcal{D}$ if there exist positive constants δ and L such that

(SRS1) $\mathcal{D} \supseteq \{u \in \mathbf{L^1}\colon\ u(\mathbb{R}) \subseteq \Omega \text{ and } \mathrm{TV}(u) \leqslant \delta\}$;

(SRS2) S is a semigroup: $S_0 = \mathrm{Id}$ and $S_{t_1} \circ S_{t_2} = S_{t_1+t_2}$;

(SRS3) S is Lipschitz: $\|S_{t_2}u_2 - S_{t_1}u_1\|_{\mathbf{L^1}} \leqslant L(|t_2 - t_1| + \|u_2 - u_1\|_{\mathbf{L^1}})$;

(SRS4) if $u \in \mathcal{D}$ is piecewise constant, then for t small, $S_t u$ is the gluing of the Lax solutions to Riemann problems at the points of jump in u.

Note that no reference is made to the fact that the orbits of S yield solutions to (1.1). Indeed, the semigroups constructed in Section 5 enjoy all the above properties and, moreover, the map $t \mapsto S_t\overline{u}$ is a weak entropic solution to (2.1). Hence, a further byproduct of Theorem 6.3 is that any semigroup satisfying (SRS1), ..., (SRS4) also yields weak entropic solutions to (1.1).

The admissibility conditions considered in Section 2 appear in Definition 6.1 only through the Lax [55] solution to Riemann problems. In other words, the choice of the Lax Riemann solver uniquely determines also the solutions to Cauchy problems, once Lipschitz continuous dependence on the initial data is required.

The starting point for the uniqueness of the semigroup is the following abstract result, presented here at the level of metric spaces, see [22], [39].

Lemma 6.2. *Let $(\mathcal{D}, d)$ be a complete metric space, $S\colon [0,+\infty[\times \mathcal{D} \to \mathcal{D}$ a Lipschitz semigroup with Lipschitz constant L and $w\colon [0,T] \to \mathcal{D}$ a Lipschitz map. Then $d(w(T), S_T w(0)) \leqslant L \cdot \int_0^T \liminf_{h\to 0+} d(w(t+h), S_h w(t))/h \,\mathrm{d}t$.*

The above lemma reduces the problem of computing the distance between the orbits of different semigroups to that of computing the difference between the "*tangent vectors*" [30], [33]. This fits particularly well with conservation laws, since the tangent vector to solutions is essentially characterized by the Lax Riemann solver, at least in the case of a piecewise constant initial datum.

Theorem 6.3. *Let $S\colon [0,+\infty[\times \mathcal{D} \to \mathcal{D}$ be the semigroup constructed above through Wave Front Tracking. Let $\tilde{S}\colon [0,+\infty[\times \tilde{\mathcal{D}} \to \tilde{\mathcal{D}}$ be another* SRS *generated by* (1.1) *with $\tilde{\mathcal{D}} \supseteq \mathcal{D}$. Then for all $u \in \mathcal{D}$ and all $t \geqslant 0$, $S_t u = \tilde{S}_t u$.*

In the proof of Theorem 6.3, Wave Front Tracking approximations have a key role, thanks to their relation with Lax solutions to Riemann problems.

6.2. Uniqueness of the single solution

To meet the requirement (U1) above, it is now necessary to find such properties of the solutions yielded by the SRS such that single solutions to Cauchy problems

could be fully characterized. As a first intermediate step, assumed the existence of a SRS, we seek those conditions on a solution u implying that $u(t) = S_t u(0)$.

Let $u_*\colon [0,T] \to \mathcal{D}$ be a weak solution to (2.1). Define $U^\sharp_{(u_*,\tau,\xi)}$ as the solution to the Riemann problem

$$\begin{cases} \partial_t u + \partial_x f(u) = 0, & t \geqslant \tau,\ \ x \in \mathbb{R}, \\ u(\tau,x) = \begin{cases} \lim\limits_{x\to\xi-} u_*(\tau,x) & \text{if } x < \xi, \\ \lim\limits_{x\to\xi+} u_*(\tau,x) & \text{if } x > \xi. \end{cases} \end{cases}$$

Let $U^\flat_{(u_*,\tau,\xi)}$ be the solution to the linear Cauchy problem

$$\begin{cases} \partial_t u + \mathrm{D}f(u(\tau,\xi))\partial_x u = 0, & t \geqslant \tau,\ \ x \in \mathbb{R}, \\ u(\tau,x) = u_*(\tau,x). \end{cases}$$

By means of $U^\flat$ and $U^\sharp$, the first property of the semigroup is singled out.

Proposition 6.4. *Let $S\colon [0,+\infty[\times \mathcal{D} \to \mathcal{D}$ be the* SRS *constructed above by means of Wave Front Tracking. Let $\hat\lambda$ be an upper bound for the moduli of all characteristic speeds. Then for all $u \in \mathcal{D}$ and for all $\tau \geqslant 0$*

1. *for all $\xi \in \mathbb{R}$, $\lim\limits_{h\to 0+} 1/h \int_{\xi-h\hat\lambda}^{\xi+h\hat\lambda} \|(S_{\tau+h}u)(x) - U^\sharp_{(u,\tau,\xi)}(h,x)\|\,\mathrm{d}x = 0$;*
2. *there exists a $C > 0$ such that for all $\xi \in]a,b[$ and $h \in]0,(b-a)/(2\hat\lambda)[$, $1/h \int_{a+h\hat\lambda}^{b-h\hat\lambda} \|(S_{\tau+h}u)(x) - U^\flat_{(u,\tau,\xi)}(h,x)\|\,\mathrm{d}x \leqslant C \cdot \big[\mathrm{TV}\big(u(\tau)|_{]a,b[}\big)\big]^2$.*

We now obtain a uniqueness result suited to the above properties.

Theorem 6.5. *Let $S\colon [0,+\infty[\times \mathcal{D} \to \mathcal{D}$ be the* SRS *constructed above. Let $\hat\lambda$ be a finite upper bound for the moduli of all characteristic speeds. If $u_*\colon [0,T] \to \mathcal{D}$ be an $\mathbf{L^1}$-Lipschitz continuous weak solution to* (2.1) *such that*

1. *for all $\xi \in \mathbb{R}$, $\lim\limits_{h\to 0+} 1/h \int_{\xi-h\hat\lambda}^{\xi+h\hat\lambda} \|u_*(\tau+h,x) - U^\sharp_{(u_*,\tau,\xi)}(h,x)\,\mathrm{d}x = 0$,*
2. *there exists a $C > 0$ such that for all $\xi \in]a,b[$ and $h \in]0,(b-a)/(2\hat\lambda)[$, $1/h \int_{a+h\hat\lambda}^{b-h\hat\lambda} \|u_*(\tau+h,x) - U^\flat_{(u_*,\tau,\xi)}(h,x)\|\,\mathrm{d}x \leqslant C \cdot \big[\mathrm{TV}\big(u(\tau)|_{]a,b]}\big)\big]^2$.*

Then u coincides with a semigroup trajectory: $u(t) = S_t(u(0))$.

Note that the existence of the SRS S and the fact that u_* attains values in the domain $\mathcal{D}$ of S are essential. Indeed, they are unavoidable also in the next result, where the integral bounds provided by $U^\sharp$ and $U^\flat$ are substituted by the Lax entropy inequalities and a bound on the total variation.

Theorem 6.6. *Let system* (1.1) *satisfy* (SH), (GNL/LD) *and generate an* SRS $S\colon [0,+\infty[\times\mathcal{D}\to\mathcal{D}$. *Let* $u\colon [0,T]\to\mathcal{D}$ *be such that*

1. u *is a weak solution of* (2.1);
2. u *is* $\mathbf{L^1}$*-continuous;*
3. u *satisfies the Lax entropy condition, see Definition* 2.7;
4. *there exists a* $\delta>0$ *such that for every Lipschitz curve* $\gamma\colon [a,b]\to[0,+\infty[$ *with Lipschitz constant* δ, *the map* $t\mapsto u(\gamma(x),x)$ *has bounded variation.*

Then for all $t\in[0,T]$, $u(t)=S_t\overline{u}$.

7. Other problems

Below we briefly consider (1.1) with boundary and models displaying phase transitions. Other areas in which Wave Front Tracking has been successful are: balance laws, see [5], [6], [45]; control problems, see [7], [8], [24], [40], [42]; the structural stability of (1.1), see [32] and the dependence of the solutions to (1.1) on f, see [17].

7.1. The Initial Boundary Value Problem

Different approaches to the Initial Boundary Value Problem (IBVP) for (1.1) are present in the current literature. First, if the range of the speed $\dot\Psi$ of the boundary is separated from that of the eigenvalues, i.e.

$$\exists\, i_*\colon \sup_{u\in\Omega}\lambda_{i_*}(u)<\inf_{t\in[0,+\infty[}\dot\Psi(t)\quad\text{and}\quad\sup_{t\in[0,+\infty[}\dot\Psi(t)<\inf_{u\in\Omega}\lambda_{i_*+1}(u),\tag{7.1}$$

then $n-i$ scalar conditions can be imposed along the boundary, leading to a *non-characteristic* problem

$$\begin{cases}\partial_t u+\partial_x[f(u)]=0, & (t,x)\in[0,+\infty[\times[\Psi(t),+\infty[,\\ u(0,x)=\overline{u}(x), & x\in[\Psi(0),+\infty[,\\ b(u(t,\Psi(t)))=g(t), & t\in[0,+\infty[\end{cases}\tag{7.2}$$

where the boundary profile $\Psi\colon[0,+\infty[\to\mathbb{R}$ is assumed continuous. Denote $E=\{(t,x)\in[0,+\infty[\times\mathbb{R}\colon\ x\in[\Psi(t),+\infty[\}$. In the case (7.1), $b\colon\Omega\to\mathbb{R}^{n-i_*}$ and $g\colon[0,+\infty[\to\mathbb{R}^{n-i_*}$ essentially fix $n-i_*$ components of u. As the *solution* to (7.2) we choose a solution to (1.1) for (t,x) in the interior of E satisfying the boundary condition in the sense of the trace, i.e. $\lim\limits_{x\to\Psi(t)+}b(u(t,\Psi(t)))=g(t)$. The non-characteristic problem (7.2) is considered in [2], [3], [4], [7].

In the case of a *characteristic* boundary, i.e. when (7.1) is violated, a more general definition of solution was proposed in [51]. This definition is not suitable for

applications to gas dynamics, since it does not consider the boundary layer effects due to viscosity. Nevertheless, it is fully *intrinsic*, for it is based on the solution to Riemann problems. Indeed, consider the IBVP

$$\text{(7.3)} \qquad \begin{cases} \partial_t u + \partial_x[f(u)] = 0, & (t,x) \in [0,+\infty[\times [\Psi(t),+\infty[, \\ u(0,x) = \overline{u}(x), & x \in [\Psi(0),+\infty[, \\ u(t,\Psi(t)) = \tilde{u}(t), & t \in [0,+\infty[. \end{cases}$$

As in [51], we introduce the following definitions.

Definition 7.1. The *characteristic Riemann problem with boundary* is (7.3) with $\Psi(t) = mt$ and $\overline{u}$, $\tilde{u}$ constant. Its solution is the restriction to E of the Lax solution to the standard Riemann problem

$$\text{(7.4)} \qquad \begin{cases} \partial_t u + \partial_x[f(u)] = 0, \\ u(x,0) = \begin{cases} \tilde{u} & \text{if } x < 0, \\ \overline{u} & \text{if } x > 0. \end{cases} \end{cases}$$

Definition 7.2. Let $u\colon E \to \mathcal{U}$ be such that $x \mapsto u(t,x)$ is in BV for a.e. t. Let w be the Lax solution to the Riemann problem (with a jump at $(\tau, \Psi(\tau))$)

$$\begin{cases} \partial_t w + \partial_x[f(w)] = 0, \\ w(\tau,x) = \begin{cases} \tilde{u}(\tau) & \text{if } x < \Psi(\tau), \\ u(\tau,\Psi(\tau)+) & \text{if } x > \Psi(\tau). \end{cases} \end{cases}$$

u is defined to be a solution of (7.3) if

(i) it is a weak entropic solution to (7.3) for $t > 0$ and $x > \Psi(\tau)$,

(ii) it coincides with $\overline{u}$ at time $t = 0$,

(iii) it satisfies the boundary condition, i.e. for all but countably many $\tau \geqslant 0$, $u(\tau,\Psi(\tau)+) = w(t,x)$ for all $(t,x) \in E$ such that $x - \Psi(\tau) > D_-\Psi(\tau) \cdot (t-\tau)$ with $t > \tau$.

Above, $D_-\Psi(t) = \liminf_{s \to t^-} (\Psi(s) - \Psi(t))/(s-t)$. According to the above definition, and differently from the non-characteristic case (7.2), a solution u to (7.3) may well have a trace $u(t,\Psi(t)+)$ that differs from the boundary data $\tilde{u}(t)$, but only if the Riemann problem defined by $u(t,\Psi(t)+)$ and $\tilde{u}(t)$ is solved by waves supported outside E, i.e. by waves slower than the boundary.

In the two cases

- (SH), (GNL/LD), small total variation, $n = 2$;
- (SH), Temple system, $n \geqslant 1$,

both the problems (7.2) and (7.3) are well posed in $\mathbf{L^1}$, in the sense that they define a unique $\mathbf{L^1}$-Lipschitz solution operator. Moreover, characterizations similar to those provided by Theorems 6.3, 6.5 and 6.6 apply, see [2], [3], [4], [41].

7.2. Phase transitions and combustion

The analytical techniques described above seem particularly useful when dealing with phase transitions. In that case, Ω is the disjoint union of 2 (or more) sets, to be referred as *phases*, say $\Omega = \Omega_1 \cup \Omega_2$. At time t, the system described by (1.1) is in phase i at x if and only if $u(t,x) \in \Omega_i$. A *phase transition* is a jump discontinuity in a solution to (1.1) between states belonging to different phases. The *phase boundary* is its support.

Extensions of the classical Lax solution to (3.1) applicable in presence of phase transitions to specific physical models have been considered, see [1], [11], [68], [71], [73] and [36] for a more abstract approach. The usual difficulty that arises when phase transitions are present is the lack of uniqueness, bypassed by introducing further physical information, either through an admissibility function or through constraints on the structure of the solution.

As a first example, consider the p-system (1.4) closed through the pressure law $p = p(\tau)$ where $p\colon\ \Omega_1 \cup \Omega_2 \to]0, +\infty[$. Ω_1 and Ω_2 are disjoint real intervals representing the liquid and the vapor phase, respectively. Reasonable qualitative properties of this function p are in Fig. 7, left. Assume that u^l and u^r in (3.1) are in different phases.

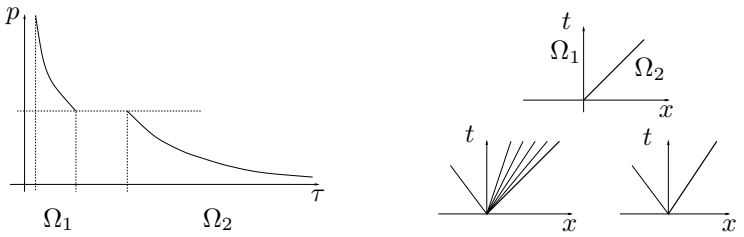

Figure 7. Left, the pressure p as a function of the specific volume τ. Right, configurations in the solution to a Riemann problem for Chapman-Jouguet detonations: above the *unperturbed* one.

In the *subsonic* case (3.1) is well known to be underdetermined. In the case of the p-system, by subsonic we mean that the modulus of the Rankine-Hugoniot speed of the phase boundary is lower than the modulus of the characteristic speeds on the sides of the discontinuity. Two usual criteria used to single out a unique solution are based on the entropy rate dissipation or on visco-capillarity approximation. From a more abstract point of view, it suffices to require that the admissibility condition

$$\Psi(u^l, u^r) = 0 \quad \text{with} \quad \Psi\colon\ (\Omega_1 \times \Omega_2) \cup (\Omega_2 \times \Omega_1) \to \mathbb{R} \tag{7.5}$$

be satisfied by any pair u^l, u^r on the sides of a phase boundary. Within this framework, various analytical results can be obtained [36], [37], [43].

Note that the same mathematical structure (1.3)–(7.5) applies also to models of phase transitions in solids [1], [73].

The structure provided by a system of the form (1.1) on a disconnected domain Ω provides a good model also for Chapman-Jouguet detonations, see [71]. This well known combustion model consists in unburnt gas filling the half line $x > 0$ and burnt gas filling $x < 0$ at time $t = 0$. The chemical reaction is instantaneous. A mathematical description is provided by two Euler systems, one for the burnt and one for the unburnt gas, coupled through a free boundary which models the location of the reaction:

$$\begin{array}{ll} \text{Burnt gas (1)} & \text{Unburnt gas (2)} \\ \begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t \Big(e + \dfrac{v^2}{2} \Big) + \partial_x (pv) = 0, \end{cases} & \begin{cases} \partial_t \tau - \partial_x v = 0, \\ \partial_t v + \partial_x p = 0, \\ \partial_t \Big(e + \dfrac{v^2}{2} \Big) + \partial_x (pv) = 0, \end{cases} \\ S_1 = c_v^1 \ln e + \dfrac{R}{\mu_1} \ln \tau & S_2 = c_v^2 \ln(e - Q) + \dfrac{R}{\mu_2} \ln \tau. \end{array}$$

S is the entropy, e the internal energy, Q the energy to be dissipated through combustion, c_v^i and μ^i are respectively the specific heat and the molar weight of the ith gas.

Chapman-Jouguet strong detonations are characterized by the flame having approximately the sound speed of the burnt gas next to it. We are thus driven to consider the *sonic* situation. As above, we need to introduce a criterion that singles out a unique solution to Riemann problems. Differently from above, here we do not introduce any admissibility condition (7.5) but rather impose restrictions on the *structure* of the solution. The solution to the Riemann problem with the (unperturbed) Chapman-Jouguet data consists of a single sonic phase transition (alias the combustion flame), see Fig. 7, right. Perturbing this data, the solution may contain either a 1-wave, a 2-contact discontinuity and a subsonic phase transition; or a 1-wave, a 2-contact discontinuity and a 3-rarefaction adjacent to the phase transition. With this choice, the Cauchy problem obtained by adding a BV small perturbation to a Chapman-Jouguet Riemann data still admits a solution, see [37] for details. For the sake of completeness, we mention here that in the case of Chapman-Jouguet deflagrations, a unique solution to the Riemann problem is chosen through the introduction of both an admissibility condition (7.5) and a constraint on the structure of the solution, see [38].

A model describing phase transitions in traffic flow is considered in [35].

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