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A GENERALIZATION OF THE HOLDITCH THEOREM FOR
THE PLANAR HOMOTHETIC MOTIONS

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Abstract. In this paper, under the one-parameter closed planar homothetic motion, a generalization of Holditch Theorem is obtained by using two different line segments (with fixed lengths) whose endpoints move along two different closed curves.

Keywords: Steiner formula, Holditch Theorem, homothetic motion

MSC 2000: 53A17

1. INTRODUCTION

H. Holditch [4] gave the following noteworthy classical theorem in 1858: if the endpoints A , B of a fixed segment \overline{AB} with length $a + b$ are rotated once along an oval (Eilinie) k in the Euclidean plane, then a given fixed point X ($\overline{AX} = a$, $\overline{XB} = b$) of \overline{AB} describes a closed, not necessarily convex, curve $k(X)$. The area F of the *Holditch-Ring* bounded by the curves k and $k(X)$ is $F = \pi ab$. Later, this classical result was generalized by different methods [1]–[9], [11]–[13].

Let E and E' be the moving and fixed Euclidean planes and $\{O; e_1, e_2\}$ and $\{O'; e'_1, e'_2\}$ their coordinate systems, respectively. By taking $\overline{OO'} = \mathbf{u} = u_1 e_1 + u_2 e_2$ for $u_1, u_2 \in \mathbb{R}$, the motion defined by the transformation

$$(1) \quad \mathbf{x}' = h\mathbf{x} - \mathbf{u}$$

is called a *one-parameter planar homothetic motion* and denoted by E/E' , where h is a homothetic scale of the motion E/E' and \mathbf{x} and \mathbf{x}' are the position vectors with respect to the moving and fixed rectangular coordinate systems of a point $X \in E$, respectively. The homothetic scale h and the vectors \mathbf{x} , \mathbf{x}' and \mathbf{u} are continuously differentiable functions of a real parameter t . Furthermore, at the initial time $t = 0$

the coordinate systems coincide. Taking $\varphi = \varphi(t)$ as the rotation angle between \mathbf{e}_1 and \mathbf{e}'_1 , equations

$$(2) \quad \begin{aligned} \mathbf{e}_1 &= \cos \varphi \mathbf{e}'_1 + \sin \varphi \mathbf{e}'_2, \\ \mathbf{e}_2 &= -\sin \varphi \mathbf{e}'_1 + \cos \varphi \mathbf{e}'_2 \end{aligned}$$

can be written. If

$$(3) \quad \begin{aligned} u_j(t+T) &= u_j(t), \quad j = 1, 2, \\ \varphi(t+T) &= \varphi(t) + 2\pi\nu, \quad \forall t \in [0, T] \end{aligned}$$

then the motion E/E' is called a *one-parameter closed planar homothetic motion* with the period $T > 0$ and the rotation number $\nu \in \mathbb{Z}$. During a closed motion, each of the points can pass through several times on its orbit in period $[0, T]$. We call the oriented surface area of the orbit curve multiplied by the number of passing through (Durchlaufzahl) the *orbit surface area*. To avoid the cases of pure translation and pure rotation we assume that

$$\dot{\varphi}(t) = d\varphi/dt \neq 0.$$

Under the one-parameter closed planar homothetic motions, if $P = (p_1, p_2)$ is the pole point of the motion at a time t , then the sliding velocity of a fixed point $X = (x_1, x_2) \in E$ with respect to E' is

$$(4) \quad d\mathbf{x}' = \{(x_1 - p_1) dh - (x_2 - p_2) h d\varphi\} \mathbf{e}_1 + \{(x_1 - p_1) h d\varphi + (x_2 - p_2) dh\} \mathbf{e}_2.$$

Furthermore, the area F_X described by the fixed point X , given by the Gauss area formula [10], is

$$(5) \quad F_X = \frac{1}{2} \oint (x'_1 dx'_2 - x'_2 dx'_1),$$

where the integration is taken along the closed orbit curve of X . Then we obtain

$$(6) \quad \begin{aligned} 2F_X &= (x_1^2 + x_2^2) \int_0^T h^2(t) d\varphi(t) - 2x_1 \int_0^T p_1(t) h^2(t) d\varphi(t) \\ &\quad - 2x_2 \int_0^T p_2(t) h^2(t) d\varphi(t) \end{aligned}$$

$$\begin{aligned}
& + \int_0^T \{u_1(t)p_1(t)h(t) \, d\varphi(t) + u_2(t)p_2(t)h(t) \, d\varphi(t) + u_1(t)p_2(t) \, dh(t) \\
& \qquad \qquad \qquad - u_2(t)p_1(t) \, dh(t)\} \\
& + x_1 \int_0^T \{u_2(t) \, dh(t) - 2p_2(t)h(t) \, dh(t) + h(t) \, du_2(t)\} \\
& + x_2 \int_0^T \{-u_1(t) \, dh(t) + 2p_1(t)h(t) \, dh(t) - h(t) \, du_1(t)\}.
\end{aligned}$$

Moreover, using the mean value theorem of integral calculus for the closed interval $0 \leq t \leq T$, there exists at least one point $t_0 \in [0, T]$ such that

$$(7) \quad \int_0^T h^2(t) \, d\varphi(t) = \int_0^T h^2(t) \dot{\varphi}(t) \, dt = 2h^2(t_0)\pi\nu.$$

By taking $\nu \neq 0$, the Steiner point $S = (s_1, s_2)$ for the closed planar homothetic motion can be written as

$$(8) \quad s_j = \frac{\int_0^T h^2(t)p_j(t) \, d\varphi(t)}{\int_0^T h^2(t) \, d\varphi(t)}, \quad j = 1, 2.$$

Thus, from Eqs. (6), (7) and (8) we get [3]

$$(9) \quad F_X = F_O + h^2(t_0)\pi\nu(x_1^2 + x_2^2 - 2x_1s_1 - 2x_2s_2) + \mu_1x_1 + \mu_2x_2,$$

where F_O is the area of the origin point of the moving coordinate system and

$$(10) \quad \begin{aligned} \mu_1 &= \frac{1}{2} \int_0^T \{-2h(t)p_2(t) \, dh(t) + h(t) \, du_2(t) + u_2(t) \, dh(t)\}, \\ \mu_2 &= \frac{1}{2} \int_0^T \{2h(t)p_1(t) \, dh(t) - h(t) \, du_1(t) - u_1(t) \, dh(t)\}. \end{aligned}$$

Eq. (9) is called the *Steiner area formula* for the one-parameter closed planar homothetic motion E/E' .

2. A GENERALIZATION OF HOLDITCH THEOREM TO
TWO DIFFERENT CLOSED CURVES

Theorem 1. *Let E_i/E' be one-parameter closed planar homothetic motions with the same orientation, the same number of passing through (Durchlaufzahl) and the rotation numbers ν_i ($i = 1, 2$). Under closed homothetic motions E_i/E' , if the endpoints A_i and B_i of lines $g_i = A_iB_i$ draw orbit curves $k_A, k_B \in E'$, respectively, then the points $X_i \in g_i$ ($\overline{A_iX_i} = \lambda_i a$, $\overline{X_iB_i} = \lambda_i b$) describe orbit curves k_i with orbit surface areas F_i . The difference $F = F_1 - F_2$ of the orbit surface areas F_i depends on the parameters $a, b, \lambda_i, \nu_i, h$, while it is independent of k_A and k_B .*

Proof. Let A_iB_i ($i = 1, 2$) have the directions of the real axes of the moving planes E_i . In this case we have $A_i = (0, 0)$, $B_i = (\lambda_i(a + b), 0)$, $X_i = (\lambda_i a, 0)$. Using the generalization of the Holditch Theorem given by Kuruoğlu and Yüce [11], for the orbit surface areas F_{A_i}, F_{B_i}, F_i of points A_i, B_i, X_i we get

$$(11) \quad F_1 = \frac{aF_{B_1} + bF_{A_1}}{a + b} - h^2(t_0)\pi\nu_1\lambda_1^2 ab$$

and

$$(12) \quad F_2 = \frac{aF_{B_2} + bF_{A_2}}{a + b} - h^2(t_0)\pi\nu_2\lambda_2^2 ab.$$

Hence, we obtain

$$(13) \quad F = F_1 - F_2 = \frac{a(F_{B_1} - F_{B_2}) + b(F_{A_1} - F_{A_2})}{a + b} + (\nu_2\lambda_2^2 - \nu_1\lambda_1^2)h^2(t_0)\pi ab.$$

Since the points A_i (B_i) draw the closed curves k_A (k_B) with the same orientation under the homothetic motions E_i/E' , we can write $F_{A_1} = F_{A_2}$ ($F_{B_1} = F_{B_2}$). Then we get

$$(14) \quad F = F_1 - F_2 = (\nu_2\lambda_2^2 - \nu_1\lambda_1^2)h^2(t_0)\pi\nu ab.$$

Special case 1: In the special case of the homothetic scale $h \equiv 1$, Eq. (14) yields

$$(15) \quad F = F_1 - F_2 = (\nu_2\lambda_2^2 - \nu_1\lambda_1^2)\pi\nu ab$$

which was given by Pottman [7].

Special case 2: If $k_A = k_B$, $\nu_2 = 1$ and $\lambda_1 = 0$ then Eq. (14) yields the Holditch theorem for the closed planar homothetic motion which was given by Tutar and Kuruoğlu [3].

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