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A NOTE ON A GENERALIZATION OF DILIBERTO'S THEOREM FOR CERTAIN DIFFERENTIAL EQUATIONS OF HIGHER DIMENSION*

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Abstract. In the theory of autonomous perturbations of periodic solutions of ordinary differential equations the method of the Poincaré mapping has been widely used. For the analysis of properties of this mapping in the case of two-dimensional systems, a result first obtained probably by Diliberto in 1950 is sometimes used. In the paper, this result is (partially) extended to a certain class of autonomous ordinary differential equations of higher dimension.

Keywords: Poincaré mapping, variational equation, moving orthogonal system

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1. Introduction

The theory of dynamical systems and an important part of the qualitative theory of differential equations is based on the concept of the *phase space* S and the concept of the *flow*. By a flow we understand any mapping $\varphi\colon A\times S\to S$, where A is $\mathbb{R}$ or $\mathbb{Z}$ and S is a metric or topological space such that, for any $t_1,t_2\in A$ and $x\in S$

1. φ is continuous,
2. $\varphi(t_1,\varphi(t_2,x))=\varphi(t_1+t_2,x)$, (the group property)
3. $\varphi(0,x)=x$.

For example, if $f\colon\mathbb{R}^n\to\mathbb{R}^n$ is a continuous function such that an autonomous system $\dot{\mathfrak{x}}=f(\mathfrak{x})$ (where the dot denotes the derivative with respect to t) is complete

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and its solutions $t \mapsto \varphi(t, x)$ are uniquely determined by the initial condition x at $t = 0$, then the function $\varphi\colon (t, x) \mapsto \varphi(t, x)$ is a flow. Here $A = \mathbb{R}$ and $S = \mathbb{R}^n$.

A very powerful tool, depending on the concept of a flow, is the "method of sections" invented by Poincaré [12], nowadays known as the method of the *Poincaré mapping* or the *transition mapping*, which makes it possible to replace some differential equations by a suitable mapping and, at the same time, to reduce the dimension of the phase space by one by cutting the flow transversaly by one or more hyperplanes.

This idea is really far reaching—from the realms of global analysis [11] to those of the generalized Poincaré operators of differential inclusions in the sense of Carathéodory, when the flow is a set-valued map [4], [5].

While in some trivial cases the Poincaré mapping can be explicitly calculated, in most cases it is not so and the same is true for its differential. Up to now, for an n-dimensional case, we have had only indirect information about the characteristic eigenvalues of the Poincaré mapping attached to a p-periodic solution. More precisely, we know that the eigenvalues are equal to the characteristic multipliers (with one exception) of the corresponding *variational equation*. And there is no way how to compute the values of those characteristic multipliers...

Because there is such an evident connection between the Poincaré mapping and the variational equation, it is clear that a naive hypothesis for the differential equation considered

$$\dot{\mathfrak{x}} = f(\mathfrak{x}), \tag{1}$$

is

H1 $f\colon \mathrm{Dom}(f) \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is a C^1 function with an open domain.

Then the Poincaré mapping is continuously differentiable, which immediately leads to the question about the differential of such a mapping. This question was, in the case of $n = 2$, successfully solved by Andronov [6] and, in a sense, by Diliberto [8].

Instead of the original Diliberto's formulation we give here a new transcription taken from Chicone [7], where some minor numerical errors were corrected (J is the special symplectic matrix $J := \begin{bmatrix} 0, & 1 \\ -1, & 0 \end{bmatrix}$).

Theorem 1.1. *Let $\varphi(t, x)$ denote the flow of the differential equation* (1), $n = 2$. *Let $f(x) \neq 0$, and let $Y(t)$ be the fundamental matrix of the homogeneous variational equation $\dot{y} = Df(\varphi(t, x))y$ such that $Y(0)$ is the identity matrix. Ten for all $t \in \mathbb{R}$ we have $Y(t)f(x) = f(\varphi(t, x))$, $Y(t)f^\top(x) = a(t, x)f(\varphi(t, x)) + b(t, x)f^\top(\varphi(t, x))$,* *where* $f^\top(x) := Jf(x)$ *and*

$$b(t,x) = \frac{\|f(x)\|^2}{\|f(\varphi(t,x))\|^2} \exp \int_0^t \operatorname{div} f(\varphi(s,x))\,\mathrm{d}s,$$

$$a(t,x) = \int_0^t \{2\kappa(s,x)\|f(\varphi(s,x))\| - \operatorname{rot} f(\varphi(s,x))\}\,\mathrm{d}s.$$

Here $\kappa(s,x)$ *denotes the scalar curvature along the plane curve given by the flow* $\varphi(\cdot,x)$.

Chicone [7] was able to obtain a nice geometric identification for the quantities $a(t,x)$ (in terms of the local transition time) and $b(t,x)$ (in terms of the local transition mapping) and he posed the problem (Problem 5.13) about possible generalization of Diliberto's theorem to the case of $n > 2$.

2. Diliberto's theorem

Clearly there is no way how to obtain the full generalization of Diliberto's result, because his success is based on an a priori knowledge of one nontrivial solution of the variational equation (e.g. [10], p. 276).

In this paper we extend the theorem to three dimensional systems equipped with some kind of two-dimensional invariant manifold. We would like to note that we have been able to extend our considerations to n-dimensional systems [2] and that the proof will appear elsewhere.

To keep our presentation as simple as possible, we will suppose that

H2 all solutions of (1) are defined on $[0,\infty)$,

and that (1) has a very special kind of an invariant manifold, namely a level set of a first integral. Therefore our third hypothesis is

H3 $g\colon \mathrm{Dom}(f) \subseteq \mathbb{R}^3 \to \mathbb{R}$ is a nondegenerate first integral [9, p. 114] of (1), $g \in C^2$.

If $\varphi(t,x)$ is the solution of (1) such that $\varphi(0,x) = x$ then $g(\varphi(t,x)) = g(x)$. Hence the equation $g(x) = c$, $c \in \mathbb{R}$ yields a *foliation* of $\mathrm{Dom}(f)$ with two-dimensional submanifolds $M^2(c) := \{x \in \mathrm{Dom}(f)\colon\ g(x) = c\}$ embedded in $\mathbb{R}^3$ as *leaves*. In the sequel we will work with exactly one leaf of $M^2(c)$, so we will write simply M instead of $M^2(c)$.

As g is C^2 and nondegenerate at any $x \in M$, the one-dimensional normal space N_xM and the two-dimensional tangent space T_xM of M at $x \in M$ are defined. Clearly $N_xM := \{\alpha n(x)\colon\ \alpha \in \mathbb{R}\}$, where $n(x) := \operatorname{grad} g(x)/\|\operatorname{grad} g(x)\|$ is the unit normal vector (the surface normal) and $n(x)$ is C^1 on M. Because $\dim T_xM = 2$, we may suppose ($[\cdot|\cdot]$ denotes cross-product)

H4 there are two C^1 functions $a_1(x)$, $a_2(x)$ on M such that $\|a_1(x)\| = \|a_2(x)\| = 1$ and $a_2(x) = [n(x)|a_1(x)]$.

The choice of $a_1(x)$, $a_2(x)$ is not unique, we will give some examples after Theorem 2.2.

In our proof of Diliberto's theorem we repeatedly use the fact that the dimension of the phase space is $n = 3$. This is the only dimension, where a beautiful relation between vectors and skew-symmetric matrices exists. This relation allows us to replace the cross-product by matrix multiplication and vice versa. We will recall this relation as a lemma.

Lemma 2.1. *Let* $a, b \in \mathbb{R}^3$, $a = (a^1, a^2, a^3)$. *Then* $[a|b] = R(a)b$, *where*

$$R(a) = \begin{bmatrix} 0\,, & -a^3, & a^2 \\ a^3, & 0\,, & -a^1 \\ -a^2, & a^1, & 0 \end{bmatrix}.$$

Finally, putting $f_\perp(x) := [n(x)|f(x)]$ for any $x \in M$ and denoting the usual inner product by $\langle\cdot|\cdot\rangle$ we see that $f(x) = \langle a_1|f\rangle a_1 + \langle a_2|f\rangle a_2$, $f_\perp(x) = -\langle a_2|f\rangle a_1 + \langle a_1|f\rangle a_2$. Now we may state Diliberto's theorem for three-dimensional systems with a first integral.

Theorem 2.2. *Let hypotheses* H1, H2, H3, H4 *be fulfilled and* $\varphi(t,x)$ *denote the solution of the differential equation* (1), $\varphi(0,x) = x \in M$, $n = 3$. *Let* $f(x) \neq 0$, *and let* $Y(t)$ *be the fundamental matrix of the variational equation* $\dot{y} = Df(\varphi(t,x))y$ *such that* $Y(0)$ *is the identity matrix. Then for all* $t \in \mathbb{R}$ *we have* $Y(t)f(x) = f(\varphi(t,x))$, $Y(t)f_\perp(x) = a(t,x)f(\varphi(t,x)) + b(t,x)f_\perp(\varphi(t,x))$, *and*

$$(2) \qquad b(t,x) = \frac{\|f(x)\|^2}{\|f(\varphi(t,x))\|^2} \exp \int_0^t \{\langle a_1|(Df)a_1\rangle + \langle a_2|(Df)a_2\rangle\}(\varphi(s,x))\,\mathrm{d}s,$$

$$a(t,x) = \int_0^t \{(2\kappa_g\|f\| - \langle (Df)a_1|a_2\rangle + \langle (Df)a_2|a_1\rangle)b\}(\varphi(s,x))\,\mathrm{d}s,$$

where κ_g *is the geodesic curvature of* $\varphi(\cdot,x)$ *equipped with natural parametrization and the other functions are evaluated at* $\varphi(s,x)$.

P r o o f. Let $x \in M$. Then, due to the invariance of M, $\varphi(t,x) \in M$ for $t \geqslant 0$. With a small abuse of notation $f(x), f_\perp(x) \in T_xM$ (more correctly we should write here $(x, f(x))$, $(x, f_\perp(x))$), hence $Y(t)f(x), Y(t)f_\perp(x) \in T_{\varphi(t,x)}M$ for $t \geqslant 0$. But for any $\tilde{x} \in M$, $T_{\tilde{x}}M$ is a two-dimensional vector space and $\langle f(x)|f_\perp(x)\rangle = 0$, hence for any $t \geqslant 0$

$$Y(t)f_\perp(x) = a(t,x)f(\varphi(t,x)) + b(t,x)f_\perp(\varphi(t,x)).$$

Since x is fixed, due to uniqueness we may put $\mathcal{A}(t) := a(t,x)$, $\mathcal{B}(t) := b(t,x)$, so $\mathcal{A}\colon \mathbb{R} \to \mathbb{R}$ and $\mathcal{B}\colon \mathbb{R} \to \mathbb{R}$. As $Y(t)f_\perp(x)$ is the solution of $\dot{y} = Df(\varphi(t,x))y$, $y(0) = f_\perp(x)$, we have

$$\begin{aligned} &Df(\varphi(t,x))[\mathcal{A}(t)f(\varphi(t,x)) + \mathcal{B}(t)f_\perp(\varphi(t,x))] \\ &\qquad = \mathcal{A}'(t)f(\varphi(t,x)) + \mathcal{A}(t)Df(\varphi(t,x))f(\varphi(t,x)) \\ &\qquad\quad + \mathcal{B}'(t)f_\perp(\varphi(t,x)) + \mathcal{B}(t)Df_\perp(\varphi(t,x))f(\varphi(t,x)) \end{aligned}$$

or

$$\begin{aligned} \mathcal{B}(t)Df(\varphi(t,x))f_\perp(\varphi(t,x)) &= \mathcal{A}'(t)f(\varphi(t,x)) + \mathcal{B}'(t)f_\perp(\varphi(t,x)) \\ &\quad + \mathcal{B}(t)Df_\perp(\varphi(t,x))f(\varphi(t,x)). \end{aligned} \tag{3}$$

Multiplying this equation by $f_\perp(\varphi(t,x))$ we obtain (henceforth omitting arguments to keep formulae in a reasonable range)

$$\mathcal{B}'\langle f_\perp|f_\perp\rangle = \mathcal{B}\{\langle (Df)f_\perp|f_\perp\rangle - \langle (Df_\perp)f|f_\perp\rangle\}. \tag{4}$$

Clearly $\langle f_\perp|f_\perp\rangle = \|f\|^2$. For the second inner product on the right-hand side of (4) we obtain

$$\langle (Df_\perp)f|f_\perp\rangle = \Big\langle \frac{\mathrm{d}}{\mathrm{d}t}f_\perp \,\Big|\, f_\perp\Big\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\langle f_\perp|f_\perp\rangle = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|f\|^2,$$

so $\mathcal{B}(t)$ fulfils the differential equation

$$\mathcal{B}' = \mathcal{B}\Big\{\langle a_1|(Df)a_1\rangle + \langle a_2|(Df)a_2\rangle - \frac{\mathrm{d}}{\mathrm{d}t}\log\|f\|^2\Big\}, \tag{5}$$

and after integration from 0 to t we obtain the first equation from (2).

Multiplying (3) by f we obtain

$$\mathcal{A}'\|f\|^2 = \mathcal{B}\{\langle (Df)f_\perp|f\rangle - \langle (Df_\perp)f|f\rangle\}. \tag{6}$$

For the second inner product in (6) we have $\langle (Df_\perp)f|f\rangle = -\langle (Df)f|f_\perp\rangle$, hence

$$\begin{aligned} \mathcal{A}'\|f\|^2 &= \mathcal{B}\{\langle (Df)f_\perp|f\rangle + \langle (Df)f|f_\perp\rangle\} \\ &= \mathcal{B}\{2\langle f_\perp|(Df)f\rangle + \langle f_\perp|((Df)^* - (Df))f\rangle\}. \end{aligned} \tag{7}$$

In order to manage the term $\langle f_\perp|(Df)f\rangle$, let us denote by $\psi\colon\ s \mapsto \psi(s,x)$ the curve $\varphi\colon\ t \mapsto \varphi(t,x)$ in its natural parametrization,

$$s = \int_0^t \|f(\varphi(\xi,x))\|\,\mathrm{d}\xi.$$

Then $\varphi(t,x) = \psi(s(t),x)$, $\varphi_t(t,x) = \psi_s(s(t),x)\dot{s}(t)$, $\|\psi_s(s,x)\| \equiv 1$ and $\dot{s}(t) > 0$. Therefore

$$\langle f_\perp|(Df)f\rangle = \langle [n|\varphi_t]|\varphi_{tt}\rangle = (n|\varphi_t|\varphi_{tt}) = (n|\psi_s|\psi_{ss})(\dot{s})^3 = \kappa_g\|\varphi_t\|^3,$$

where $\kappa_g(t,x) = \kappa_g(\xi(s(t),x))$ is the geodesic curvature [3, p. 173] of the curve $\varphi\colon t \mapsto \varphi(t,x)$ ($(\cdot|\cdot|\cdot)$ is a mixed or triple product of vectors). This together with (7) gives

$$\mathcal{A}' = \mathcal{B}\{2\kappa_g\|f\| - \langle (Df)a_2|a_1\rangle + \langle (Df)a_1|a_2\rangle\},$$

and after integration from 0 to t we obtain the second equation from (2). $\square$

Here and now Diliberto's theorem could be seen as a triviality, but the truth is (quite) opposite. In the next section we will demonstrate one of possible applications—computation of the derivative of the Poincaré mapping in a direction v for $v \in T_xM$, $v \notin \operatorname{span}\{f(x)\}$.

R e m a r k 1. The choice of $a_1(x)$, $a_2(x)$ is not unique. The most obvious, but often not the best, is the choice $a_1(x) := \|f(x)\|^{-1}f(x)$, $a_2(x) := \|f_\perp(x)\|^{-1}f_\perp(x)$.

There are many important systems with a linear first integral—one of them are kinetical systems

$$\dot{\mathfrak{x}} = AG(\mathfrak{x}), \tag{8}$$

an interesting class of nonlinear differential equations used in physical chemistry and biology (for general definition and basic asymptotic properties see e.g. [1]). In such systems linear first integrals appear as a result of conservation laws (for mass or for population).

On the assumptions of this paper the kinetical system (8) has one linear first integral $g(x) = \langle u|x\rangle = U\operatorname{col}(x)$, A is a $3\times m$ matrix, $\operatorname{rank}(A) = 2$ and $UA = 0$. If we suppose that the first two columns a_1 and a_2 of A are orthogonal, we can choose functions $a_i(x)$ as

$$a_i(x) = \frac{a_i}{\|a_i\|}, \quad i = 1,2.$$

3. The transition mapping

In this section we will suppose that the hypotheses H1, H2, H3 and H4 are fulfilled. If $x_1 \in M$ is a point such that $f(x_1) \neq 0$, then the solution $\varphi(t, x_1)$ of (1) is defined and belongs to M for $t \geqslant 0$. Let $x_2 = \varphi(p, x_1)$ for some $0 < p < \infty$. Let Σ_1 and Σ_2 be two planes such that $x_1 \in \Sigma_1$ and $x_2 \in \Sigma_2$, respectively, which are transverse to the vector field $f(x)$ at x_1 and x_2 (that is $\langle f(x_i) | n_i(x_i)\rangle \neq 0$, where $n_i(x)$ is a normal to Σ_i at x_i, $i = 1, 2$).

In particular, for $i = 1, 2$ we have $\Sigma_i \neq T_{x_i}M$, therefore the intersection $\Sigma_{x_i} \cap M$ is locally a C^1 curve with a parametrization $\sigma_i\colon (-\varepsilon_i, \varepsilon_i) \to \mathbb{R}^3$ such that $\sigma_i(0) = x_i$, $0 \neq \sigma_i'(0) =: v_i \in T_{x_i}M$. We will henceforth use the notation $\{\sigma_i\} := \{\sigma_i(t) \in \mathbb{R}^3\colon -\varepsilon_i < t < \varepsilon_i\}$ for $i = 1, 2$.

Due to H1 there is a real C^1 function τ defined on a neighborhood of x_1 such that $\tau(x_1) = p$, $\varphi(\tau(x_1), x_1) = \varphi(p, x_1) = x_2$ and $\varphi(\tau(x), x) \in \Sigma_2$. The function τ is called the *transition time* (or the *return time* if $\Sigma_1 = \Sigma_2$).

Let V_1 be a neighborhood of x_1 and $U_1 = V_1 \cap \Sigma_1$, then the mapping

$$\begin{aligned} &\Psi\colon U_1 \subseteq \Sigma_1 \to \Sigma_2, \\ &\Psi\colon x \mapsto \varphi(\tau(x), x), \end{aligned} \tag{9}$$

is called the *transition map* (or the *Poincaré map* or the *first return map* if $\Sigma_1 = \Sigma_2$). Obviously $\Psi(x_1) = x_2$. Our aim is to compute $D\Psi(x_1)v_1$.

To this end we will use the fact that $D\Psi(x_1)v_1 = \frac{\mathrm{d}}{\mathrm{d}s}\Psi(\sigma_1(s))|_{s=0}$. From (9) we obtain

$$\Psi(\sigma_1(s)) = \varphi(\tau(\sigma_1(s)), \sigma_1(s)).$$

Differentiating both sides with respect to s we obtain

$$\begin{aligned} D\Psi(\sigma_1(s))\sigma_1'(s) &= D_1\varphi(\tau(\sigma_1(s)), \sigma_1(s))\frac{\mathrm{d}}{\mathrm{d}s}\tau(\sigma_1(s)) + D_2\varphi(\tau(\sigma_1(s)), \sigma_1(s))\sigma_1'(s) \\ &= f(\varphi(\tau(\sigma_1(s)), \sigma_1(s)))(D\tau(\sigma_1(s))\sigma_1'(s)) \\ &\quad + D_2\varphi(\tau(\sigma_1(s)), \sigma_1(s))\sigma_1'(s). \end{aligned}$$

Putting $s = 0$ we obtain

$$D\Psi(x_1)v_1 = f(x_2)(D\tau(x_1)v_1) + D_2\varphi(p, x_1)v_1.$$

Because $\sigma_1(0) = x_1$ and $\{\sigma_1\} \in M$ we have $v_1 \in T_{x_1}M$. Therefore there are $\alpha, \beta \in \mathbb{R}$ such that

$$\begin{aligned} D\Psi(x_1)v_1 &= f(x_2)(D\tau(x_1)v_1) + D_2\varphi(p, x_1)[\alpha f(x_1) + \beta f_\perp(x_1)] \\ &= f(x_2)(D\tau(x_1)v_1) + \alpha D_2\varphi(p, x_1)f(x_1) + \beta D_2\varphi(p, x_1)f_\perp(x_1) \\ &= f(x_2)(D\tau(x_1)v_1) + \alpha D_2\varphi(p, x_1)f(x_1) + \beta[\mathcal{A}(p)f(x_2) + \mathcal{B}(p)f_\perp(x_2)], \end{aligned}$$

where $\mathcal{A}(t) := a(t, \varphi(t, x_1))$, $\mathcal{B}(t) := b(t, \varphi(t, x_1))$ are the functions used in Diliberto's theorem.

Finally, using $D_2\varphi(p, x_1)f(x_1) = f(\varphi(p, x_1)) = f(x_2)$ we obtain

$$D\Psi(x_1)v_1 = [D\tau(x_1)v_1 + \alpha + \beta\mathcal{A}(p)]f(x_2) + \beta\mathcal{B}(p)f_\perp(x_2).$$

Suppose that the second curve $\{\sigma_2\}$ is orthogonal to the curve $\varphi(t, x_1)$ at x_2, i.e. $\langle f(x_2)|v_2\rangle = 0$. Since $\{\sigma_i\} \subset \Sigma_i$, $\Psi\colon \{\sigma_1\} \to \{\sigma_2\}$ and $x_i \in \{\sigma_i\}$ for $i = 1, 2$, we have $D\Psi\colon T_{x_1}\{\sigma_1\} \to T_{x_2}\{\sigma_2\}$, but $T_{x_2}\{\sigma_2\} = \{\gamma v_2 : \gamma \in \mathbb{R}\}$ and $f(x_2) \notin T_{x_2}\{\sigma_2\}$.

Hence $D\tau(x_1)v_1 + \alpha + \beta\mathcal{A}(p) = 0$ and $D\Psi(x_1)v_1 = \beta\mathcal{B}(p)f_\perp(x_2)$ and we have proved the following theorem.

Theorem 3.1. *Let the hypotheses* H1, H2, H3 *and* H4 *be fulfilled. Let* $x_1 \in M$, $f(x_1) \neq 0$ *and* $x_2 = \varphi(p, x_1)$, *where* $0 < p < \infty$ *is the first time with this property. Let* Σ_1, Σ_2 *be planes such that* Σ_1 *is transverse to* $\varphi(t, x)$ *at* $t = 0$ *and* Σ_2 *is orthogonal to* $\varphi(t, x)$ *at* $t = p$. *Let* $\Psi\colon U \subset \Sigma_1 \to \Sigma_2$ *be the Poincaré mapping. If* $v_1 \in T_{x_1}\Sigma_1 \cap T_{x_1}M$, *then*

$$D\Psi(x_1)v_1 = \frac{\langle v_1|f_\perp(x_1)\rangle}{\|f(x_1)\|^2}\, b(p, x_2)f_\perp(x_2),$$

where $b(t, x)$ *is the function defined in* (2).

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