Sébastien Novo; Antonín Novotný

A remark on the smoothness of bounded regions filled with a steady compressible and isentropic fluid

Applications of Mathematics, Vol. 50 (2005), No. 4, 331--339

Persistent URL: http://dml.cz/dmlcz/134610

A REMARK ON THE SMOOTHNESS OF BOUNDED REGIONS FILLED WITH A STEADY COMPRESSIBLE AND ISENTROPIC FLUID

Sébastien Novo, Antonín Novotný, La Garde

(Received February 18, 2003)

Abstract. For convenient adiabatic constants, existence of weak solutions to the steady compressible Navier-Stokes equations in isentropic regime in smooth bounded domains is well known. Here we present a way how to prove the same result when the bounded domains considered are Lipschitz.

Keywords: Navier-Stokes equations, compressible fluid, weak solution

MSC 2000: 35Q30, 76N10

1. Introduction

In this note we investigate the existence of the so-called renormalized bounded energy weak solutions to the steady Navier-Stokes system of equations which describes the flow of a compressible and isentropic fluid in a bounded region $\Omega \subset \mathbb{R}^3$ with Lipschitz boundary. These equations read

$$\operatorname{div}(\varrho \boldsymbol{u}) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$\operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u} - \mu_1 \Delta \boldsymbol{u} - (\mu_1 + \mu_2)\nabla \operatorname{div} \boldsymbol{u} + \nabla \varrho^\gamma = \varrho \boldsymbol{f} + \boldsymbol{g} \quad \text{in } \Omega. \tag{1.2}$$

The unknown quantities are the scalar field $\varrho(x)$, $x \in \Omega$, which represents the density of the fluid and has to be non-negative, and the vector field $\boldsymbol{u}(x) = (u^1(x), u^2(x), u^3(x))$, $x \in \Omega$, which represents the velocity of the fluid. The quantities $\boldsymbol{f}(x) = (f^1(x), f^2(x), f^3(x))$ and $\boldsymbol{g}(x) = (g^1(x), g^2(x), g^3(x))$ at the right-hand side of equation (1.2) are two given vector fields defined on Ω. They correspond respectively to volumic and non volumic external forces acting on the fluid. The viscosity coefficients μ_1 and μ_2 are assumed to be constant and to satisfy the physically reasonable constraints

$$\mu_1 > 0, \quad \frac{2}{3}\mu_1 + \mu_2 \geqslant 0, \tag{1.3}$$

and the adiabatic constant γ is supposed to be such that

$$\gamma > \frac{3}{2} \quad \text{if} \;\; \operatorname{curl} \boldsymbol{f} = \boldsymbol{0}, \qquad \gamma > \frac{5}{3} \quad \text{otherwise.} \tag{1.4}$$

To complete equations (1.1)–(1.2) we require the so-called no-slip boundary conditions

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega \tag{1.5}$$

and prescribe the total mass of the fluid in the volume Ω

$$\int_\Omega \varrho \, \mathrm{d}x = M > 0. \tag{1.6}$$

Before we recall the meaning of a renormalized bounded energy weak solution to the problem (1.1), (1.2), (1.5) and (1.6), let us introduce some notation used throughout the text. By a domain $\mathcal{O} \subset \mathbb{R}^3$ we mean a connected open set. As usual, $\mathcal{D}(\mathcal{O})$ denotes the space of infinitely differentiable functions with compact support in $\mathcal{O}$ endowed with the usual topology inducing its dual $\mathcal{D}'(\mathcal{O})$, the space of distributions on $\mathcal{O}$; $W^{1,p}(\mathcal{O})$, $p \in [1, \infty]$, is the Sobolev space of functions whose generalized derivatives up to order 1 belong to the Lebesgue space of integrable functions $L^p(\mathcal{O})$. $W_0^{1,p}(\mathcal{O})$ is the completion of $\mathcal{D}(\mathcal{O})$ with respect to the norm $\|v\|_{1,p,\mathcal{O}} = \sum\limits_{|\alpha| \leqslant 1} \|D^\alpha v\|_{0,p,\mathcal{O}}$ where $\|\cdot\|_{0,p,\mathcal{O}}$ denotes the L^p-norm. The subspace of functions in $L^p(\mathcal{O})$ with zero mean value over $\mathcal{O}$ will be denoted by $\tilde{L}^p(\mathcal{O})$. The characteristic function of a set $A \subset \mathbb{R}^3$ will always be denoted by 1_A. Often, in the text, we will not make any distinction between a function defined on a domain $\mathcal{O}$ and its extension by zero outside $\mathcal{O}$.

Consider functions $b : \mathbb{R}^+ \to \mathbb{R}$ satisfying

$$b \in C^0([0, \infty)) \cap C^1((0, \infty)), \; \exists\, c > 0, \; \exists\, \lambda_0 < 1, \; \forall\, t \in (0, 1], \quad |b'(t)| \leqslant c t^{-\lambda_0}, \tag{1.7}$$

and behaving at infinity as follows:

$$\exists\, c > 0, \; \exists\, \lambda_1, \lambda_2 \in \mathbb{R}, \; \forall\, t \geqslant 1, \quad |b'(t)| \leqslant c t^{\lambda_1}, \quad |t b'(t) - b(t)| \leqslant c t^{\lambda_2}. \tag{1.8}$$

Let $p \in [\frac{3}{2}, \infty)$. A couple of functions $(\varrho, \boldsymbol{u})$ will be called a renormalized bounded energy weak solution to the problem (1.1), (1.2), (1.5) and (1.6) if

(i) $\varrho \in L^p(\Omega)$, $\varrho \geqslant 0$ a.e. in Ω and satisfies (1.6), $\boldsymbol{u} \in W_0^{1,2}(\Omega)]^3$;

(ii) equation (1.1) holds in the sense of distributions on $\mathbb{R}^3$;

(iii) $(\varrho, \boldsymbol{u})$ is a renormalized solution of the continuity equation in the sense of distributions on $\mathbb{R}^3$. More precisely, for any function b satisfying (1.7) and (1.8) with

$$-1 < \lambda_1 \leqslant \frac{p}{2} - 1 \quad \text{and} \quad 0 < \lambda_2 \leqslant \frac{p}{2}, \tag{1.9}$$

we have

$$\operatorname{div}(b(\varrho)\boldsymbol{u}) + \{\varrho b'(\varrho) - b(\varrho)\} \operatorname{div} \boldsymbol{u} = 0 \quad \text{in } \mathcal{D}'(\mathbb{R}^3); \tag{1.10}$$

(iv) equation (1.2) holds in the sense of distributions on Ω;

(v) the following energy inequality holds:

$$\int_\Omega \{\mu_1 |\nabla \boldsymbol{u}|^2 + (\mu_1 + \mu_2)(\operatorname{div} \boldsymbol{u})^2\} \, \mathrm{d}x \leqslant \int_\Omega (\varrho \boldsymbol{f} + \boldsymbol{g}) \cdot \boldsymbol{u} \, \mathrm{d}x. \tag{1.11}$$

At this stage, we are ready to state a result similar to [5, Theorem 1.1] where the domain considered is a bounded Lipschitz one.

Theorem 1.1. *Assume that* $\Omega \subset \mathbb{R}^3$ *is a bounded domain of class* $C^{0,1}$, $\boldsymbol{f}, \boldsymbol{g} \in [L^\infty(\Omega)]^3$, *the viscosity coefficients* μ_1 *and* μ_2 *satisfy* (1.3), *the adiabatic constant* γ *satisfies* (1.4) *and* $M > 0$. *Then there exists a renormalized bounded energy weak solution* $(\varrho, \boldsymbol{u})$ *to the problem* (1.1), (1.2), (1.5) *and* (1.6) *such that* $\varrho \in L^{s(\gamma)}(\Omega)$ *where*

$$s(t) = \begin{cases} 3(t-1) & \text{if } t < 3, \\ 2t & \text{if } t \geqslant 3. \end{cases} \tag{1.12}$$

Theorem 1.1 is an improvement of [5, Theorem 1.1] which is needed as a technical tool in our foregoing paper [6] where we deal with the existence of weak solutions to the steady compressible and isentopic Navier-Stokes equations considered in domains with several outlets at infinity.

2. Outline of the proof

In order to prove [5, Theorem 1.1], our starting point were the results of P.-L. Lions [4, Theorem 6.7 and Section 6.10]. More precisely, we have used the following theorem:

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^3$ be a bounded domain of class $C^{2,\nu}$, $\nu \in (0,1]$, let $\boldsymbol{f}, \boldsymbol{g} \in [L^\infty(\Omega)]^3$, let the viscosity coefficients μ_1 and μ_2 satisfy* (1.3)*, let $\beta > \frac{5}{3}$, $\delta \in (0,1]$ and $M > 0$. Then there exists a couple $(\varrho, \boldsymbol{u})$ with the following properties: $\varrho \in L^{s(\beta)}(\Omega)$, $\varrho \geqslant 0$ a.e. in Ω, $\int_\Omega \varrho \,\mathrm{d}x = M$, $\boldsymbol{u} \in [W_0^{1,2}(\Omega)]^3$,*

$$\operatorname{div}(\varrho \boldsymbol{u}) = 0 \quad \textit{in } \mathcal{D}'(\mathbb{R}^3),$$
$$\operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu_1 \Delta \boldsymbol{u} - (\mu_1 + \mu_2) \nabla \operatorname{div} \boldsymbol{u} + \nabla \{\varrho^\gamma + \delta \varrho^\beta\} = \varrho \boldsymbol{f} + \boldsymbol{g} \quad \textit{in } [\mathcal{D}'(\Omega)]^3.$$

Moreover,

$$\int_\Omega \{\mu_1 |\nabla \boldsymbol{u}|^2 + (\mu_1 + \mu_2)(\operatorname{div} \boldsymbol{u})^2\} \,\mathrm{d}x \leqslant \int_\Omega (\varrho \boldsymbol{f} + \boldsymbol{g}) \cdot \boldsymbol{u} \,\mathrm{d}x.$$

We claim that this theorem holds as well when Ω is a bounded Lipschitz domain. Once this result is known, proof of Theorem 1.1 follows word by word by the argumentation of [5], letting $\delta \to 0^+$ in Theorem 2.1. In the sequel, we shall therefore explain how to prove Theorem 2.1 for domains with only Lipschitz boundary.

To prove Theorem 2.1, P.-L. Lions investigated the following approximation of the original problem:

$$\alpha \varrho + \operatorname{div}(\varrho \boldsymbol{u}) = \alpha h \quad \text{in } \Omega, \tag{2.1}$$

$$\frac{1}{2} \alpha h \boldsymbol{u} + \frac{3}{2} \alpha \varrho \boldsymbol{u} + \operatorname{div}(\varrho \boldsymbol{u} \otimes \boldsymbol{u}) - \mu_1 \Delta \boldsymbol{u} - (\mu_1 + \mu_2) \nabla \operatorname{div} \boldsymbol{u} + \nabla \{\varrho^\gamma + \delta \varrho^\beta\} = \varrho \boldsymbol{f} + \boldsymbol{g} \quad \text{in } \Omega, \tag{2.2}$$

$$\boldsymbol{u} = \boldsymbol{0} \quad \text{on } \partial\Omega, \tag{2.3}$$

$$\int_\Omega \varrho \,\mathrm{d}x = \int_\Omega h \,\mathrm{d}x \tag{2.4}$$

where $\alpha \in (0,1]$ and $h \in L^\infty(\Omega)$, $h \geqslant 0$ a.e. in Ω. He proved the following lemma:

Lemma 2.1. *Assume that the assumptions of Theorem 2.1 are satisfied. Let $\alpha \in (0,1]$ and let $h \in L^\infty(\Omega)$, $h \geqslant 0$ a.e. in Ω. Then there exists a pair of functions $(\varrho_\alpha, \boldsymbol{u}_\alpha)$ enjoying the following properties:*

(i) *$\varrho_\alpha \in L^{2\beta}(\Omega)$, $\varrho_\alpha \geqslant 0$ a.e. in Ω, $\int_\Omega \varrho_\alpha \,\mathrm{d}x = \int_\Omega h \,\mathrm{d}x$, $\boldsymbol{u}_\alpha \in [W_0^{1,2}(\Omega)]^3$;*

(ii) *there holds*

$$\alpha \varrho_\alpha + \operatorname{div}(\varrho_\alpha \boldsymbol{u}_\alpha) = \alpha h \quad \textit{in } \mathcal{D}'(\mathbb{R}^3); \tag{2.5}$$

(iii) *for any function $b\colon \mathbb{R}_+ \to \mathbb{R}$ belonging to the class of functions $C^1([0,\infty))$ which satisfy* (1.8) *and* (1.9) *with $p = 2\beta$,*

$$\operatorname{div}(b(\varrho_\alpha) \boldsymbol{u}_\alpha) + \{\varrho_\alpha b'(\varrho_\alpha) - b(\varrho_\alpha)\} \operatorname{div} \boldsymbol{u}_\alpha = \alpha (h - \varrho_\alpha) b'(\varrho_\alpha) \quad \textit{in } \mathcal{D}'(\mathbb{R}^3); \tag{2.6}$$

(iv) *there holds*

$$\frac{1}{2}\alpha h\boldsymbol{u}_\alpha + \frac{3}{2}\alpha\varrho_\alpha\boldsymbol{u}_\alpha + \operatorname{div}(\varrho_\alpha\boldsymbol{u}_\alpha\otimes\boldsymbol{u}_\alpha) - \mu_1\Delta\boldsymbol{u}_\alpha - (\mu_1+\mu_2)\nabla\operatorname{div}\boldsymbol{u}_\alpha + \nabla\{\varrho_\alpha^\gamma + \delta\varrho_\alpha^\beta\} = \varrho_\alpha\boldsymbol{f} + \boldsymbol{g} \quad \text{in } [\mathcal{D}'(\Omega)]^3; \tag{2.7}$$

(v) $(\varrho_\alpha, \boldsymbol{u}_\alpha)$ *fulfils the energy inequality*

$$\begin{aligned}\alpha\int_\Omega (h+\varrho_\alpha)|\boldsymbol{u}_\alpha|^2\,\mathrm{d}x + \int_\Omega \{\mu_1|\nabla\boldsymbol{u}_\alpha|^2 + (\mu_1+\mu_2)(\operatorname{div}\boldsymbol{u}_\alpha)^2\}\,\mathrm{d}x \\ + \frac{\gamma\alpha}{\gamma-1}\int_\Omega(\varrho_\alpha-h)(\varrho_\alpha^{\gamma-1}-h^{\gamma-1})\,\mathrm{d}x + \frac{\delta\beta\alpha}{\beta-1}\int_\Omega(\varrho_\alpha-h)(\varrho_\alpha^{\beta-1}-h^{\beta-1})\,\mathrm{d}x \\ \leqslant \int_\Omega(\varrho_\alpha\boldsymbol{f}+\boldsymbol{g})\cdot\boldsymbol{u}_\alpha\,\mathrm{d}x + \frac{\gamma\alpha}{\gamma-1}\int_\Omega(h-\varrho_\alpha)h^{\gamma-1}\,\mathrm{d}x + \frac{\delta\beta\alpha}{\beta-1}\int_\Omega(h-\varrho_\alpha)h^{\beta-1}\,\mathrm{d}x.\end{aligned} \tag{2.8}$$

In the sequel, we are going to explain how to prove the same result when Ω is only a bounded Lipschitz domain. To this end, we shall need the following lemma concerning the approximation of a bounded domain by a decreasing sequence of smooth bounded domains.

Lemma 2.2. *Let $N \geqslant 2$ and let $\Omega\subset\mathbb{R}^N$ be a bounded Lipschitz domain. Then there exists a sequence of bounded domains $\{\Omega_n\}_{n\in\mathbb{N}^*}$ satisfying*

(i) $\Omega_n \in C^\infty$;

(ii) $\overline{\Omega}\subset\Omega_{n+1}\subset\overline{\Omega_{n+1}}\subset\Omega_n$ *and* $\lim\limits_{n\to\infty}|\Omega_n\setminus\Omega| = 0$.

Proof. Let $\omega_n = \{x;\, \operatorname{dist}(x,\Omega) < \frac{1}{n}\}$. Clearly $\omega_{n+1}\subset\subset\omega_n$ and hence there exists a function $\varphi_n\in\mathcal{D}(\omega_n,[0,1])$ such that $\varphi_n\equiv 1$ on $\overline{\omega_{n+1}}$. Thus, according to the Morse-Sard Lemma (see [3]), for almost all $t\in(0,1)$,

$$\{\varphi_n = t\}\cap\{J\varphi_n = 0\} = \emptyset \tag{2.9}$$

where $J\varphi_n$ denotes the Jacobian of φ_n. We choose $t_n\in(0,1)$ such that (2.9) is satisfied and put $\Omega_n = \{\varphi_n > t_n\}$. Then it is easy to check that Ω_n possesses the properties (ii). The property (i) is a consequence of the Implicit Functions Theorem. □

Now, let $\Omega\subset\mathbb{R}^3$ be a bounded Lipschitz domain, let $\boldsymbol{f},\boldsymbol{g}\in[L^\infty(\Omega)]^3$ and let $h\in L^\infty(\Omega)$, $h\geqslant 0$ a.e. in Ω. Then, according to Lemma 2.1, for any $n\in\mathbb{N}^*$, there exists a pair of functions $(\varrho_n,\boldsymbol{u}_n)$ enjoying the following properties: $\varrho_n\in L^{2\beta}(\Omega_n)$, $\varrho_n\geqslant 0$ a.e. in Ω_n, $\int_{\Omega_n}\varrho_n\,\mathrm{d}x = \int_\Omega h\,\mathrm{d}x$, $\boldsymbol{u}_n\in[W_0^{1,2}(\Omega_n)]^3$; equations (2.5)–(2.7) and energy inequality (2.8) hold with ϱ_n, $\boldsymbol{u}_n$ and Ω_n instead of ϱ_α, $\boldsymbol{u}_\alpha$ and Ω respectively.

Our ultimate goal in this note is to pass to the limit $n \to \infty$. To this end, we first need some estimates. In order to prove these estimates, we will use the following result due to Bogovskiĭ [1].

Lemma 2.3. *Let $G \subset \mathbb{R}^3$ be a bounded Lipschitz domain. Then there exists a linear operator $\boldsymbol{\mathcal{B}}_G = (\mathcal{B}_G^1, \mathcal{B}_G^2, \mathcal{B}_G^3)$ such that*

$$\forall\, p \in (1,\infty),\ \boldsymbol{\mathcal{B}}_G\colon\ \tilde{L}^p(G) \to [W_0^{1,p}(G)]^3, \quad \forall\, \mathcal{F} \in \tilde{L}^p(G),\ \operatorname{div} \boldsymbol{\mathcal{B}}_G(\mathcal{F}) = \mathcal{F} \ \text{ a.e. in } G,$$
$$\forall\, \mathcal{F} \in \tilde{L}^p(G),\ \forall\, p \in (1,\infty),\ \|\nabla \boldsymbol{\mathcal{B}}_G(\mathcal{F})\|_{0,p,G} \leqslant c(G,p)\|\mathcal{F}\|_{0,p,G}.$$

From the energy inequality (2.8) satisfied by $(\varrho_n, \boldsymbol{u}_n)$, it is not difficult to convince oneself that Hölder's, Sobolev's and Young's inequalities lead to

$$\|\nabla \boldsymbol{u}_n\|_{0,2,\Omega_n} \leqslant c(\Omega, \boldsymbol{f}, \boldsymbol{g}, h)(1 + \|\varrho_n\|_{0,\frac{6}{5},\Omega}). \tag{2.10}$$

Notice that the $L^{\frac{6}{5}}$-norm of the density ϱ_n occurring on the right-hand side of (2.10) is taken over Ω. This fact will play an essential role in the sequel. Next, according to the properties of $(\varrho_n, \boldsymbol{u}_n)$ and Lemma 2.3, it is not difficult to check that the extension by zero outside Ω of the function $\boldsymbol{\varphi} = \boldsymbol{\mathcal{B}}_\Omega\big(\varrho_n^\beta - 1/|\Omega| \int_\Omega \varrho_n^\beta \,\mathrm{d}y\big)$ is an admissible test function of the momentum equation (2.2) satisfied by $(\varrho_n, \boldsymbol{u}_n)$. By standard computations which essentially consist in several integrations by parts, Hölder's inequality, some interpolations, the Poincaré inequality, Sobolev's inequality and Lemma 2.3 (see [5, Lemma 4.2] for similar computations), we finally conclude that

$$\|\varrho_n\|_{0,2\beta,\Omega} \leqslant c(\Omega, \boldsymbol{f}, \boldsymbol{g}, h). \tag{2.11}$$

Since $2\beta > \frac{6}{5}$, this new information inserted in (2.10) implies that

$$\|\nabla \boldsymbol{u}_n\|_{0,2,\Omega_n} \leqslant c(\Omega, \boldsymbol{f}, \boldsymbol{g}, h). \tag{2.12}$$

Consequences of estimates (2.11) and (2.12) are summarized in the following statement.

Lemma 2.4. *There exist functions ϱ_α, $\overline{\varrho_\alpha^\gamma}$, $\overline{\varrho_\alpha^\beta}$, $\boldsymbol{u}_\alpha$ and a subsequence of $\{(\varrho_n, \boldsymbol{u}_n)\}_{n\in\mathbb{N}^*}$ such that*

$$\varrho_n \rightharpoonup \varrho_\alpha \ \text{ in } L^{2\beta}(\mathbb{R}^3), \quad \varrho_\alpha \geqslant 0 \ \text{ a.e. in } \Omega, \quad \varrho_\alpha = 0 \ \text{ a.e. in } \mathbb{R}^3 \setminus \overline{\Omega},$$
$$\varrho_n^\gamma \rightharpoonup \overline{\varrho_\alpha^\gamma} \ \text{ in } L^{2\beta/\gamma}(\mathbb{R}^3), \quad \varrho_n^\beta \rightharpoonup \overline{\varrho_\alpha^\beta} \ \text{ in } L^2(\mathbb{R}^3),$$
$$\boldsymbol{u}_n \rightharpoonup \boldsymbol{u}_\alpha \ \text{ in } [W^{1,2}(\mathbb{R}^3)]^3, \quad \boldsymbol{u}_\alpha = \boldsymbol{0} \ \text{ a.e. in } \mathbb{R}^3 \setminus \overline{\Omega},$$
$$\forall\, p \in [1,6),\ \boldsymbol{u}_n \to \boldsymbol{u} \ \text{ in } [L^p(\Omega)]^3,$$
$$\varrho_n \boldsymbol{u}_n \rightharpoonup \varrho \boldsymbol{u} \ \text{ in } [L^{6\beta/(\beta+3)}(\mathbb{R}^3)]^3, \quad \varrho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n \rightharpoonup \varrho \boldsymbol{u} \otimes \boldsymbol{u} \ \text{ in } [L^{6\beta/(2\beta+3)}(\mathbb{R}^3)]^{3\times 3}.$$

Moreover, we have

$$\alpha\varrho_\alpha + \operatorname{div}(\varrho_\alpha \boldsymbol{u}_\alpha) = \alpha h \quad \textit{in } \mathcal{D}'(\mathbb{R}^3), \tag{2.13}$$

$$\frac{1}{2}\alpha h \boldsymbol{u}_\alpha + \frac{3}{2}\alpha\varrho_\alpha \boldsymbol{u}_\alpha + \operatorname{div}(\varrho_\alpha \boldsymbol{u}_\alpha \otimes \boldsymbol{u}_\alpha) - \mu_1 \Delta \boldsymbol{u}_\alpha - (\mu_1+\mu_2)\nabla \operatorname{div} \boldsymbol{u}_\alpha + \nabla\{\overline{\varrho_\alpha^\gamma} + \delta\overline{\varrho_\alpha^\beta}\} = \varrho_\alpha \boldsymbol{f} + \boldsymbol{g} \quad \textit{in } [\mathcal{D}'(\Omega)]^3. \tag{2.14}$$

Since Ω is a bounded Lipschitz domain in $\mathbb{R}^3$, it is clear that $\boldsymbol{u}_\alpha \in [W_0^{1,2}(\Omega)]^3$. Then, in order to check that ϱ_α satisfies (2.4), consider the sequence of functions $\{\Phi_n\}_{n\in\mathbb{N}^*} \subset \mathcal{D}(\Omega)$ defined by

$$0 \leqslant \Phi_n \leqslant 1, \quad \Phi_n(x) = \begin{cases} 1 & \text{if } x \in \{y \in \Omega,\ \operatorname{dist}(y, \partial\Omega) \geqslant \frac{2}{n}\}, \\ 0 & \text{if } x \in \{y \in \Omega,\ \operatorname{dist}(y, \partial\Omega) \leqslant \frac{1}{n}\}, \end{cases} \quad |\nabla\Phi_n| \leqslant 2n \text{ in } \Omega.$$

Equation (2.1) with a test function Φ_n yields

$$\int_\Omega (\varrho_\alpha - h)\Phi_n \, \mathrm{d}x = 1/\alpha \int_\Omega \varrho_\alpha \boldsymbol{u}_\alpha \cdot \nabla\Phi_n \, \mathrm{d}x.$$

On the one hand, as n tends to infinity, it is obvious that the left-hand side of this equality tends to $\int_\Omega (\varrho_\alpha - h)\,\mathrm{d}x$. On the other hand, the right-hand side is bounded by

$$c\|\varrho_\alpha\|_{0,2,\operatorname{supp}\nabla\Phi_n} \|\boldsymbol{u}_\alpha(\operatorname{dist}(x,\partial\Omega))^{-1}\|_{0,2,\Omega}. \tag{2.15}$$

In accordance with the definition of Φ_n, one has $|\operatorname{supp}\nabla\Phi_n| \to 0$ as $n \to \infty$. Consequently, using Hardy's inequality

$$\|\boldsymbol{u}_\alpha(\operatorname{dist}(x,\partial\Omega))^{-1}\|_{0,2,\Omega} \leqslant c\|\nabla\boldsymbol{u}_\alpha\|_{0,2,\Omega}, \quad \boldsymbol{u}_\alpha \in [W_0^{1,2}(\Omega)]^3$$

and the summability of ϱ_α, we get the convergence to zero of (2.15).

Next, we have to prove that $\varrho_\alpha^s = \overline{\varrho_\alpha^s}$ a.e. in Ω, $s = \gamma, \beta$. In other words, we have to prove e.g. at least the strong convergence of the sequence of densities $\{\varrho_n\}_n$ in $L^1(\Omega)$ which, in accordance with the bound (2.11), the weak lower semicontinuity of norms and interpolation, will imply that $\varrho_n \to \varrho_\alpha$ in $L^p(\Omega)$, $p \in [1, 2\beta)$. Let us briefly describe the main lines how to get this proof. First, following the ideas of P.-L. Lions [4, Chapter 6], the following weak compactness result for the effective pressure $p(\varrho_\alpha) - (2\mu_1 + \mu_2)\operatorname{div}\boldsymbol{u}_\alpha$ can be proved: for any function $b \in C^1([0,\infty))$ satisfying (1.8) and (1.9) with $p = 2\beta$ and $\lambda_1 = 0$, one has

$$\overline{p(\varrho_\alpha)b(\varrho_\alpha)} - (2\mu_1+\mu_2)\overline{b(\varrho_\alpha)\operatorname{div}\boldsymbol{u}_\alpha} = \overline{p(\varrho_\alpha)}\,\overline{b(\varrho_\alpha)} - (2\mu_1+\mu_2)\overline{b(\varrho_\alpha)}\operatorname{div}\boldsymbol{u}_\alpha \text{ a.e. in } \Omega$$

where $p(\varrho) = \varrho^\gamma + \delta\varrho^\beta$ and overlined quantities stand for weak limits of the corresponding sequences. Next, using the transport theory of DiPerna and P.-L. Lions [2] applied to the continuity equation (2.5), one can prove the following lemma.

Lemma 2.5. *Let $p \geqslant 2$, let λ_1, λ_2 satisfy* (1.9)*. Assume that $\varrho \in L^p_{\mathrm{loc}}(\mathbb{R}^3)$, $\varrho \geqslant 0$ a.e. in $\mathbb{R}^3$, $\boldsymbol{u} \in [W^{1,2}_{\mathrm{loc}}(\mathbb{R}^3)]^3$, and $f \in L^{q'}_{\mathrm{loc}}(\mathbb{R}^3)$, $1 \leqslant q \leqslant p/\lambda_1$ if $\lambda_1 > 0$, $1 < q < +\infty$ if $\lambda_1 \leqslant 0$, satisfy*

$$\operatorname{div}(\varrho\boldsymbol{u}) \geqslant f \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{2.16}$$

Then for any non decreasing function $b \in C^1([0,+\infty))$ with growth conditions (1.8) *at infinity we have*

$$\operatorname{div}(b(\varrho)\boldsymbol{u}) + \{\varrho b'(\varrho) - b(\varrho)\}\operatorname{div}\boldsymbol{u} = f b'(\varrho) \quad \text{in } \mathcal{D}'(\mathbb{R}^3). \tag{2.17}$$

If $f \equiv 0$, the assumptions on b can be relaxed to (1.7)–(1.9)*.*

Applying Lemma 2.5 with $b(t) = (t+l)^\theta$, $l > 0$, $0 < \theta < 1$, to the continuity equation (2.5), one obtains

$$\begin{aligned}\alpha\theta(\varrho_n + l)^\theta + \operatorname{div}((\varrho_n + l)^\theta\boldsymbol{u}_n) + (\theta - 1)(\varrho_n + l)^\theta \operatorname{div}\boldsymbol{u}_n \\ \geqslant \alpha\theta h(\varrho_n + l)^{\theta-1} + \theta l(\varrho_n + l)^{\theta-1}\operatorname{div}\boldsymbol{u}_n + \alpha\theta l(\varrho_n + l)^{\theta-1} \\ \geqslant \alpha\theta h(\varrho_n + l)^{\theta-1} + \theta l(\varrho_n + l)^{\theta-1}\operatorname{div}\boldsymbol{u}_n \quad \text{in } \mathcal{D}'(\mathbb{R}^3).\end{aligned}$$

Letting $n \to \infty$, one gets

$$\begin{aligned}\alpha\theta\overline{(\varrho_\alpha + l)^\theta} + \operatorname{div}(\overline{(\varrho_\alpha + l)^\theta}\boldsymbol{u}_\alpha) \geqslant (1-\theta)\overline{(\varrho_\alpha + l)^\theta \operatorname{div}\boldsymbol{u}_\alpha} + \alpha\theta h\overline{(\varrho_\alpha + l)^{\theta-1}} \\ + \theta l\overline{(\varrho_\alpha + l)^{\theta-1}\operatorname{div}\boldsymbol{u}_\alpha} \quad \text{in } \mathcal{D}'(\mathbb{R}^3).\end{aligned}$$

Applying Lemma 2.5 with $b(t) = t^{1/\theta}$ to the last equation, then using the weak compactness result for the effective pressure with $b(t) = (t+l)^\theta$, and finally letting $l \to 0^+$, one concludes that

$$\begin{aligned}\alpha\big(\overline{\varrho_\alpha^\theta}\big)^{1/\theta} + \operatorname{div}\Big\{\big(\overline{\varrho_\alpha^\theta}\big)^{1/\theta}\boldsymbol{u}_\alpha\Big\} \\ \geqslant \alpha h + \frac{(1-\theta)}{\theta(2\mu_1 + \mu_2)}\big\{\overline{p(\varrho_\alpha)\varrho_\alpha^\theta} - \overline{p(\varrho_\alpha)}\,\overline{\varrho_\alpha^\theta}\big\}\big(\overline{\varrho_\alpha^\theta}\big)^{1/\theta-1} \quad \text{in } \mathcal{D}'(\mathbb{R}^3).\end{aligned}$$

This fact combined with the continuity equation (2.5) implies

$$\alpha r_\alpha + \operatorname{div}(r_\alpha\boldsymbol{u}_\alpha) \geqslant \frac{(1-\theta)}{\theta(2\mu_1 + \mu_2)}\big\{\overline{p(\varrho_\alpha)\varrho_\alpha^\theta} - \overline{p(\varrho_\alpha)}\,\overline{\varrho_\alpha^\theta}\big\}\big(\overline{\varrho_\alpha^\theta}\big)^{1/\theta-1} \quad \text{in } \mathcal{D}'(\mathbb{R}^3)$$

where $r_\alpha = \left(\overline{\varrho_\alpha^\theta}\right)^{1/\theta} - \varrho_\alpha \leqslant 0$ a.e. in $\mathbb{R}^3$. Then, by standard arguments of convex analysis, one obtains $\varrho_\alpha^s = \overline{\varrho_\alpha^s}$ a.e. in Ω, $s = \gamma, \beta$. This yields the strong convergence $\varrho_n \to \varrho_\alpha$ in $L^1(\Omega)$.

Finally, it remains to show inequality (2.8). It comes from the similar energy inequality (2.8) satisfied by $(\varrho_n, \boldsymbol{u}_n)$ supplemented by Lemma 2.4, the strong convergence of densities and the weak semicontinuity of the convex positive quadratic form

$$\boldsymbol{v} \in [W^{1,2}(\Omega)]^3 \mapsto \int_\Omega \{\mu_1 |\nabla \boldsymbol{v}|^2 + (\mu_1 + \mu_2)(\operatorname{div} \boldsymbol{v})^2\} \, \mathrm{d}x.$$

References

[1] *M. E. Bogovskiĭ*: The solution of some problems of vector analysis, associated with the operators div and grad. Trudy Semin. S. L. Soboleva *1* (1980), 5–40. (In Russian.)

[2] *R. J. DiPerna, P.-L. Lions*: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. *98* (1989), 511–547.

[3] *L. C. Evans, R. F. Gariepy*: Measure Theory and Fine Properties of Functions. Studies in Advanced Mathematics. CRC Press, Boca Raton, 1992.

[4] *P.-L. Lions*: Mathematical Topics in Fluid Mechanics, Vol. 2. Compressible Models. Lecture Series in Mathematics and its Applications. Clarendon Press, Oxford, 1998.

[5] *S. Novo, A. Novotný*: On the existence of weak solutions to the steady compressible Navier-Stokes equations when the density is not square integrable. J. Math. Kyoto Univ. *42* (2002), 531–550.

[6] *S. Novo, A. Novotný*: On the existence of weak solutions to the steady compressible Navier-Stokes equations in domains with conical outlets. J. Math. Fluid Mech. *7* (2005), 1–24.

Authors' address: *S. Novo, A. Novotný*, La Garde Laboratoire d'Analyse Non linéaire Appliquée et Modélisation, Université de Toulon et du Var, B.P. 20132, 83957 La Garde, France, e-mails: `novotny@univ-tln.fr`, `seb.novo@cegetel.net`.