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ON FULLY DEVELOPED FLOWS OF FLUIDS WITH A PRESSURE DEPENDENT VISCOSITY IN A PIPE

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Abstract. Stokes recognized that the viscosity of a fluid can depend on the normal stress and that in certain flows such as flows in a pipe or in channels under normal conditions, this dependence can be neglected. However, there are many other flows, which have technological significance, where the dependence of the viscosity on the pressure cannot be neglected. Numerous experimental studies have unequivocally shown that the viscosity depends on the pressure, and that this dependence can be quite strong, depending on the flow conditions. However, there have been few analytical studies that address the flows of such fluids despite their relevance to technological applications such as elastohydrodynamics. Here, we study the flow of such fluids in a pipe under sufficiently high pressures wherein the viscosity depends on the pressure, and establish an explicit exact solution for the problem. Unlike the classical Navier-Stokes solution, we find the solutions can exhibit a structure that varies all the way from a plug-like flow to a sharp profile that is essentially two intersecting lines (like a rotated V). We also show that unlike in the case of a Navier-Stokes fluid, the pressure depends both on the radial and the axial coordinates of the pipe, logarithmically in the radial coordinate and exponentially in the axial coordinate. Exact solutions such as those established in this paper serve a dual purpose, not only do they offer solutions that are transparent and provide the solution to a specific but simple boundary value problems, but they can be used also to test complex numerical schemes used to study technologically significant problems.

Keywords: pressure dependent viscosity, implicit constitutive theory, Poiseuille flow

MSC 2000: 76A99

0. Introduction

A large class of fluids can be described for a very wide class of flows by the incompressible Navier-Stokes model, thereby ensuring a central place in the mechanics of fluids to it. Here, by the Navier-Stokes model, we mean the classical linearly viscous fluid model wherein the viscosity is assumed to be a constant. The compressible Navier-Stokes model is characterized by two material moduli that can both depend on the density. Of course, when thermal effects become significant, we allow for the material moduli to depend on the temperature. Stokes [1] recognized that the material moduli of the fluid could depend on the pressure; in a compressible fluid the pressure is given through an equation of state (a constitutive equation). In many fluids, while the density changes but slightly, the viscosity of the fluid can change by several orders of magnitude. We are then in a situation of an incompressible fluid whose viscosity can depend on the pressure which is now the mean normal stress. In such fluids the constitutive representation for the stress takes on a rather interesting twist. The constitutive relation does not provide an explicit relation for the stress and the symmetric part of the velocity gradient, but has the following form in isothermal processes:

$$\mathbf{T} = -p\mathbf{1} + \mu(p)\mathbf{D} = +\Big(\frac{1}{3}\operatorname{tr}\mathbf{T}\Big)\mathbf{1} + [\mu(\operatorname{tr}\mathbf{T})]\mathbf{D}, \tag{1}$$

where p denotes the Lagrange multiplier due to the constraint of incompressibility, μ denotes the viscosity, tr the trace operator, and $\mathbf{D}$ is given by

$$\mathbf{D} = \frac{1}{2}\big[(\operatorname{grad}\mathbf{v}) + (\operatorname{grad}\mathbf{v})^T\big], \tag{2}$$

where grad denotes the eulerian spatial gradient. Of course, (1) can be rewritten so that we can express $\mathbf{D}$ explicitly in terms of $\mathbf{T}$. In general, the viscosity μ can depend on the principal invariants of $\mathbf{D}$ and $\mathbf{T}$ (i.e., $\operatorname{tr}\mathbf{T}$), and in this general case, we cannot express $\mathbf{D}$ explicitly in terms of $\mathbf{T}$ either, and we have a truly implicit relationship between $\mathbf{T}$ and $\mathbf{D}$ of the form

$$\mathbf{f}(\mathbf{T}, \mathbf{D}) = \mathbf{0}. \tag{3}$$

The above model has built into it, the kinematical constraint

$$\operatorname{tr}\mathbf{D} = \operatorname{div}\mathbf{v} = 0. \tag{4}$$

While, in general, the viscosity can be a function of the pressure p, however, in many practical situations, the flow conditions are such that it is essentially a constant, leading to the classical incompressible Navier-Stokes model.

In the case of solids, the frictional resistance between two solids sliding relative to each other depends clearly on the force normal to the sliding planes. Thus, it should not come as a total surprise that such a situation could occur in a fluid as well. We would expect the frictional resistance between adjacent layers at a great depth below the surface of a fluid body such as an ocean to be far greater than that between adjacent layers near the upper surface. In other words, we would naturally expect the viscosity to depend on the normal stresses. In the flows that one usually encounters in pipes and channels, due to pressure gradients, the variation in pressures are not sufficiently great to cause significant changes in viscosity. However, if the fluid is forced through very narrow regions from a domain of much greater thickness, such as the geometry relevant to elastohydrodynamics, the increase in the pressures is so dramatic that it can influence the viscosity significantly. In flow geometries such as those that are encountered in elastohydrodynamic lubrication, this is indeed the case. In fact, the changes in the pressures are so great that they can cause glass transition. While in flows in pipes that one normally encounters, the variations in the pressure may not be large enough to warrant including the pressure dependence of viscosity, it is possible that in long pipes the pressure difference may be large enough to make a difference. However, at present we are not aware of any experiments of the laminar flow of fluids in which the pressure varies significantly which we could use to compare with our theoretical predictions. In fact, it would be interesting to carry out an experiment to assess the effect of the pressure dependence of viscosity on pipe flows, as we know from other experiments (see discussion that follows where references [4]–[10] as well as [2] are mentioned) that the viscosity does depend on the pressure. This is an interesting open problem from our perspective.

There has been a great amount of work on the response of both fluids and solids at high pressure and a detailed discussion of most of the relevant literature up until 1930 can be found in the authoritative book by Bridgman [2]. He systematically studied the changes in the response characteristics of fluids and solids when subjected to high pressure, especially with regard to their thermal, optical and rheological properties. Here, we shall be concerned purely with the fact that the viscosity of fluids can change significantly with pressure.

Equations of the type (4) do not define a constitutive relation, they define a class of constitutive relations as there could be more than one explicit relationship between $\mathbf{T}$ and $\mathbf{D}$ that satisfy (1) and (4). A more general implicit constitutive specification that allows for memory effects is

$$\mathbf{f}\left(\mathbf{T}, \dot{\mathbf{T}} \ldots, \overset{(n)}{\mathbf{T}}, \quad \mathbf{D}, \dot{\mathbf{D}} \ldots, \overset{(n)}{\mathbf{D}}\right) = \mathbf{0}, \tag{5}$$

where (n) denotes n material time derivatives of the quantity that occurs underneath.

Implicit constitutive relations can account for more than one constitutive relation between the stress and the kinematical quantities and there could be a bifurcation with respect to the material response (see Rajagopal and Wineman [3]).

There are important practical applications that one encounters, elastohydrodynamic lubrication for example, wherein the pressure dependence of the viscosity assumes ominous significance. There is a considerable body of experimental literature that indicates without a shadow of doubt the dependence of the viscosity on pressure (see Cutler et al. [4], Griest et al. [5], Johnson and Cameron [6], Johnson and Tevaarwerk [7], Greenwood [8], Bair and Winer [9]). It has been found that the variation of the viscosity with pressure could even be exponential, with the viscosity varying by several orders of magnitude (see also Szeri [10] for a discussion of the variation of viscosity with pressure in elastohydrodynamics).

It is also well known that in many fluids, the viscosity depends on the shear rate. Fluids in which the viscosity increases with shear rate are called shear thickening fluids and those in which the viscosity decreases with shear rate, shear thinning fluids. For instance, in narrow blood vessels, the viscosity of blood can shear thin by an order of magnitude (this type of response is typical of many biological fluids). Thus, we shall consider fluids in which the stress is given by the constitutive representation

$$\mathbf{T} = -p\mathbf{1} + 2[\hat{\mu}(p, \mathbf{D})]\mathbf{D}. \tag{6}$$

When $\hat{\mu}$ does not depend on p, but only on $\mathbf{D}$, the fluid is referred to as a generalized Newtonian fluid. Such fluids have been studied at great length by numerous authors and the recent book by Málek et al. [10] discusses many mathematical issues concerning them. The solutions to special problems depend on the form of the function $\hat{\mu}$. Here, for the sake of simplicity, we pick a special structure for $\hat{\mu}(p, \mathbf{D})$, a structure which reduces to power-law fluid models (such models are used extensively to describe many polymeric materials, food products, etc.) in the absence of pressure dependence. Thus, we choose a model of the form

$$\mathbf{T} = -p\mathbf{1} + 2\mu(p)|\mathbf{D}|^{\gamma-2}\mathbf{D}, \tag{7}$$

where $|\cdot|$ denotes the usual trace norm, and γ is the power-law exponent. When $\gamma = 2$ and $\mu(p)$ is constant, the model reduces to the classical Navier-Stokes model (1.4).

Recently Hron, Málek and Rajagopal [11] have studied the flow of a fluid modeled by (7) between two parallel plates as well as between two rotating cylinders. They have shown that the solution of the governing equation for the problem considered for such a fluid is significantly different from the solution for the Navier-Stokes equation. For instance, it is possible to find solutions for the counterpart of Poiseuille for such fluids with a velocity profile that varies from that which resembles a plug flow to one that is essentially triangular in shape, profiles that are in marked departure from the usual parabolic profile for the classical Navier-Stokes model. More importantly, they have shown that for the case of unidirectional flow between two parallel plates that are moving, *multiple solutions are possible* depending on the value of the power-law exponent. *Such multiplicity of solutions is not possible in the classical Navier-Stokes model.* The analysis of Hron, Málek and Rajagopal [11] suggests that similar departure from the Navier-Stokes solution is possible in other flow situations, and we investigate such a possibility in this paper.

Hron, Málek and Rajagopal [11] assume two specific forms for $\hat{\mu}(p)$:

$$\mu(p) = \mathrm{e}^{\alpha p} \tag{8}$$

and

$$\mu(p) = \alpha p. \tag{9}$$

They show that unidirectional flows are not possible between parallel plates in the case of a model of the type (8), a secondary flow being necessary. On the other hand, they show that unidirectional flows are possible in the case of (9). In this paper, we will restrict our investigation to viscosities of the form (9).

Rigorous existence theorems for flows have been established for the models considered in this work (see Málek, Nečas and Rajagopal [12]) and we refer the interested reader to the same.

In this paper, we consider the fully developed flow of a fluid modeled by (7), in a pipe. As in the study by Hron, Málek and Rajagopal [11], we find that the solutions can depart markedly from the Navier-Stokes solution, depending on the values for the power-law parameter and the value of α. Though the governing equations are nonlinear, we find that by introducing appropriate transformations, the equations can be solved exactly, and an exact explicit solution can be established. Profiles with pronounced plug-like structure are possible. Also, the maximum velocity that is attained can be considerably different than that for the Navier-Stokes fluid.

1. Governing equations

We shall now proceed to develop the governing equations for the flow of a fluid modeled by (7) in a cylindrical pipe. We first observe that dimensional considerations require that

$$\dim \alpha = T^{\gamma - 1}, \tag{10}$$

where T is the dimension for time. The ab-initio point of interest in the present investigation is to examine whether the high-pressure field to which the fluid is subjected can adjust itself to support unidirectional flows in cylindrical pipes of circular cross-sections. Accordingly we assume that the flow is steady, laminar, and adopting a cylindrical coordinate system (r, ϑ, z) to describe the flow, we seek a velocity field of the form

$$\mathbf{v} = w(r)\mathbf{e}_z. \tag{11}$$

In virtue of the constraint of incompressibility, we have

$$\operatorname{div} \mathbf{v} = 0, \tag{12}$$

and the above equation is automatically satisfied by the form (7) for the velocity field. Substituting (7) into the balance of linear momentum

$$\varrho \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}t} = \varrho \mathbf{b} + \operatorname{div} \mathbf{T}, \tag{13}$$

observing that $\mathrm{d}\mathbf{v}/\mathrm{d}t \equiv \mathbf{0}$ in view of (11) and neglecting body forces, we obtain

$$-\operatorname{grad} p + \mu |\mathbf{D}|^{\gamma-2} \Delta \mathbf{v} + 2|\mathbf{D}|^{\gamma-2}[\mathbf{D} \operatorname{grad} \mu(p)] + 2\mu(p)[\mathbf{D} \operatorname{grad} |\mathbf{D}|^{\gamma-2}] = \mathbf{0}. \tag{14}$$

Using $\mu = \alpha p$ in (14) and introducing non-dimensional variables

$$r^* = \frac{r}{a}, \quad z^* = \frac{z}{a}, \quad w^* = \frac{w}{a\alpha^{-1/(\gamma-1)}}, \quad p^* = \frac{p}{p_0}, \tag{15}$$

where p_0 is the pressure at a typical section (say $z = 0$), the corresponding component equations of (14), after dropping the asterisks for the sake of clarity, are

$$-\frac{\partial p}{\partial r} + 2^{1-\gamma/2} \frac{\partial p}{\partial z} \left| \frac{\mathrm{d}w}{\mathrm{d}r} \right|^{\gamma-2} \frac{\mathrm{d}w}{\mathrm{d}r} = 0, \tag{16}$$

$$\frac{1}{r} \frac{\partial p}{\partial \vartheta} = 0, \tag{17}$$

and

$$\begin{aligned} -\frac{\partial p}{\partial z} + 2^{1-\gamma/2} p \left| \frac{\mathrm{d}w}{\mathrm{d}r} \right|^{\gamma-2} \left(\frac{\mathrm{d}^2 w}{\mathrm{d}r^2} + \frac{1}{r} \frac{\mathrm{d}w}{\mathrm{d}r} \right) + 2^{1-\gamma/2} \frac{\partial p}{\partial r} \left| \frac{\mathrm{d}w}{\mathrm{d}r} \right|^{\gamma-2} \frac{\mathrm{d}w}{\mathrm{d}r} \\ + 2^{1-\gamma/2} p \left(\frac{\partial}{\partial r} \left| \frac{\mathrm{d}w}{\mathrm{d}r} \right|^{\gamma-2} \right) \frac{\mathrm{d}w}{\mathrm{d}r} = 0. \end{aligned} \tag{18}$$

Equation (17) suggests that $p = p(r, z)$. The appropriate boundary conditions are

$$w = 0 \quad \text{at} \quad r = 1, \tag{19}$$

$$\frac{\mathrm{d}w}{\mathrm{d}r} = 0 \quad \text{at} \quad r = 0. \tag{20}$$

As w decreases with increasing r, we have

$$\left|\frac{\mathrm{d}w}{\mathrm{d}r}\right| = -\frac{\mathrm{d}w}{\mathrm{d}r} \quad \text{in } 0 \leqslant r \leqslant 1. \tag{21}$$

Using (21) in the equations (16) and (17) and eliminating $\partial p/\partial r$, we arrive at the equation

$$\begin{aligned} -\frac{\partial p}{\partial z}\left\{1-(-1)^{2\gamma-4}\,2^{2-\gamma}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{2\gamma-2}\right\} \\ +(-1)^{\gamma-2}\,2^{1-\gamma/2}p\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{\gamma-2}\left\{(\gamma-1)\frac{\mathrm{d}^2w}{\mathrm{d}r^2}+\frac{1}{r}\frac{\mathrm{d}w}{\mathrm{d}r}\right\}=0.\end{aligned} \tag{22}$$

The above equation can be solved to yield

$$\frac{\dfrac{\partial p}{\partial z}}{p} = \frac{(-1)^{\gamma-2}\,2^{1-\gamma/2}\left(\dfrac{\mathrm{d}w}{\mathrm{d}r}\right)^{\gamma-2}\left\{(\gamma-1)\dfrac{\mathrm{d}^2w}{\mathrm{d}r^2}+\dfrac{1}{r}\dfrac{\mathrm{d}w}{\mathrm{d}r}\right\}^{2\gamma-2}}{\left\{1-(-1)^{2\gamma-4}\,2^{2-\gamma}\dfrac{\mathrm{d}w}{\mathrm{d}r}\right\}}. \tag{23}$$

Note that the left-hand side of (23) is a function of r, z, while the right-hand side is purely a function of r. We shall seek a solution in which both sides of (23) equal to a constant k. In general, we could have both of them equal to an arbitrary function $h(r)$. The constant k may be positive, negative or zero. Choosing

$$k=-\lambda, \quad \lambda>0, \tag{24}$$

to derive physically significant solutions for this flow problem, we obtain

$$\frac{\partial p}{\partial z} = -\lambda p, \tag{25}$$

and

$$\begin{aligned}(\gamma-1)\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{\gamma-2}\frac{\mathrm{d}^2w}{\mathrm{d}r^2}+\frac{1}{r}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{\gamma-1}-\lambda(-1)^{\gamma-2}\,2^{1-\gamma/2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{2\gamma-2} \\ =-(-1)^{2-\gamma}2^{\gamma/2-1}\lambda.\end{aligned} \tag{26}$$

The equation (25) can be solved to yield

$$p(r,z)=L(r)\mathrm{e}^{-\lambda z}. \tag{27}$$

The arbitrary function $L(r)$ is determined by knowing the pressure distribution at a given cross-section. Rewriting the equation (26) in the form

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}\left\{r\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{\gamma-1}\right\}-\lambda(-1)^{\gamma-2}\,2^{1-\gamma/2}\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{2\gamma-2}+(-1)^{2-\gamma}\,2^{\gamma/2-1}\lambda=0, \tag{28}$$

we recognize that it can be expressed as a generalized Riccati equation (see Ince [13]) by employing the substitution

$$\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{\gamma-1} = -\frac{1}{\lambda}(-1)^{2-\gamma}\,2^{-1+\gamma/2}\left(\frac{1}{u}\frac{\mathrm{d}u}{\mathrm{d}r}\right), \tag{29}$$

which transforms it into the following linear equation of the second order:

$$\frac{\mathrm{d}^2u}{\mathrm{d}r^2} + \frac{1}{r}\frac{\mathrm{d}u}{\mathrm{d}r} - \lambda^2 u = 0. \tag{30}$$

The general solution of the above Bessel equation is

$$u(r) = AI_0(\lambda r) + BK_0(\lambda r), \tag{31}$$

where $I_0(\lambda r)$ and $K_0(\lambda r)$ are modified Bessel functions of the first and second kind respectively, A and B being arbitrary constants. Accordingly, $(\mathrm{d}w/\mathrm{d}r)^{\gamma-1}$ becomes

$$\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{\gamma-1} = -(-1)^{2-\gamma}\,2^{-1+\gamma/2}\left\{\frac{AI_1(\lambda r) - BK_1(\lambda r)}{AI_0(\lambda r) + BK_0(\lambda r)}\right\}, \tag{32}$$

where $I_1(\lambda r)$ and $K_1(\lambda r)$ are modified Bessel functions of the first and second kind, respectively. Imposing the boundary condition (20) $\mathrm{d}w/\mathrm{d}r = 0$ at $r = 0$ in the above expression and noting that $I_0(0) = 1$, $I_1(0) = 0$ and that $K_v(z)$ has a logarithmic singularity at $z = 0$ (see McLachlan [14]), we get $B = 0$, and thus

$$\left(\frac{\mathrm{d}w}{\mathrm{d}r}\right)^{\gamma-1} = (-1)^{1-\gamma}\,2^{-1+\gamma/2}\frac{I_1(\lambda r)}{I_0(\lambda r)}. \tag{33}$$

Extracting $\mathrm{d}w/\mathrm{d}r$ from the above relation, integrating it with respect to r and using the no-slip boundary condition (19), we get the final solution for the velocity field as

$$w(r) = 2^{(\gamma-2)/(2(\gamma-1))}\int_r^1 \left\{\frac{I_1(\lambda r)}{I_0(\lambda r)}\right\}^{1/(\gamma-1)}\mathrm{d}r. \tag{34}$$

From equation (28) we can find explicit solutions for particular choices of γ. If $\gamma = 2$, i.e., if $\mathbf{T}$ is linear in $\mathbf{D}$, then

$$w(r) = \int_r^1 \frac{I_1(\lambda r)}{I_0(\lambda r)}\,\mathrm{d}r. \tag{35}$$

The integral on the right-hand side of equation (35) is exactly integrable. The corresponding exact solution for w is

$$w(r) = -\frac{1}{\lambda}\ln\left\{\frac{I_0(\lambda r)}{I_0(\lambda)}\right\}. \tag{36}$$

Considering further choices of γ, $\gamma = \frac{3}{2}, \frac{4}{3}$, which are used in certain models (see Malevsky and Yen [15]), we obtain

$$w(r) = \frac{1}{2^{1/2}} \int_r^1 \left\{ \frac{I_1(\lambda r)}{I_0(\lambda r)} \right\}^2 \mathrm{d}r \quad \text{for } \gamma = \frac{3}{2}, \tag{37}$$

$$w(r) = \frac{1}{2} \int_r^1 \left\{ \frac{I_1(\lambda r)}{I_0(\lambda r)} \right\}^3 \mathrm{d}r \qquad \text{for } \gamma = \frac{4}{3}. \tag{38}$$

For the more general case $\gamma = (n+1)/n$, $n \in \mathbb{N}$, we get

$$w(r) = \frac{1}{2^{(n-1)/2}} \int_r^1 \left\{ \frac{I_1(\lambda r)}{I_0(\lambda r)} \right\}^n \mathrm{d}r. \tag{39}$$

The expression corresponding to the volumetric rate of flow discharge across any cross-reaction for any particular velocity field is computed using the formula

$$q = \int_0^1 w(r)\, \mathrm{d}(\pi r^2). \tag{40}$$

Using the expression (35) for $w(r)$ and integrating by parts, we get

$$q = \frac{\pi}{2^{(n-1)/2}} \int_0^1 r^2 \left\{ \frac{I_1(\lambda r)}{I_0(\lambda r)} \right\}^n \mathrm{d}r. \tag{41}$$

We shall now solve for the pressure field. Appealing to equation (16) and using the expression $p(r,z) = L(r)\mathrm{e}^{-\lambda z}$ given by (27) and $(\mathrm{d}w/\mathrm{d}r)^{\gamma-1}$ from the equation (33), we get

$$\frac{\mathrm{d}L}{\mathrm{d}r} + \lambda L \frac{I_1(\lambda r)}{I_0(\lambda r)} = 0. \tag{42}$$

Integrating the equation (42) we get

$$L(r) = M I_0(\lambda r), \tag{43}$$

where M is the arbitrary constant of integration. Hence

$$p(r,z) = M I_0(\lambda r) \mathrm{e}^{-\lambda z}. \tag{44}$$

2. Discussion

If the pressures at two typical cross-sections are known, i.e. if

$$p = p_0 \quad \text{at } r = 1,\ z = 0, \tag{45}$$

$$p = p_1 \quad \text{at } r = 1,\ z = \beta, \tag{46}$$

then we can determine the M and λ appearing in the above equation (44). Thus,

$$p(r, z) = p_0 I_0(\lambda r)\mathrm{e}^{-\lambda z}, \tag{47}$$

where

$$\lambda = \frac{1}{\beta}\ln(p_0/p_1). \tag{48}$$

It is interesting to note that the above expression for pressure as given by equation (47) is independent of the power-law index γ.

The pressure field with the kind of structure given by equations (47) ensures unidirectional flows in pipes of constant cross-section for a class of fluids given by the constitutive model $\mathbf{T} = -p\mathbf{1} + 2\alpha p|\mathbf{D}|^{\gamma-2}\mathbf{D}$, α being constant. From equation (48) we note that λ depends on the pressure ratio p_0/p_1 and the distance β between the corresponding sections. A change in the values of either of the two can cause λ to vary. Suppose the ratio p_0/p_1 has the value 3, and β varies from 0 to 5. Also, suppose λ can vary from 0.1 to 1.5. Based on such a variation of λ, we have sketched the velocity profiles for $\gamma = \frac{4}{3}$, using the general expression given in equation (39), in Fig. 1. For comparison, the parabolic profile representing the classical Pouiseuille solution is also plotted and is denoted as NS. It can be observed that the profiles tend to approach "V" shaped curves that were observed by Hron et al. (J. Hron, J. Málek and K. R. Rajagopal [11]) in the case of Poiseuille flow between parallel plates. However, this trend is reversed in the case of shear thinning fluids, wherein the profiles tend to get flatter in the vicinity of the axis, thereby suggesting the possibility of a plug flow near the inner core of the pipe. This feature is distinctly different from the flow behavior in parallel plates or channels as observed by Hron et al. [11], in their figures 1 (b) and 1 (c). The distribution of axial pressure, i.e., $p(0, z)$ is plotted in Fig. 2 for various cross-sections between $z = -1$ and $z = 1$. From this figure we infer that for large values of λ we need a very large pressure difference to maintain such a flow. The corresponding radial variation of the pressure at the section $z = 0$ are plotted in Fig. 3. We notice that the pressures are quite different from those for the Navier-Stokes fluid.

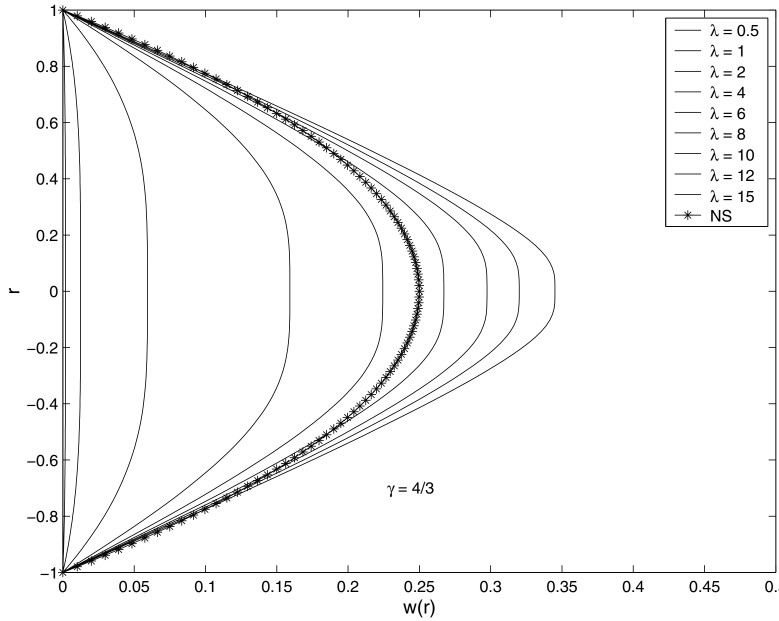

Figure 1. Velocity profiles for different pressure gradients for $\gamma = \frac{4}{3}$.

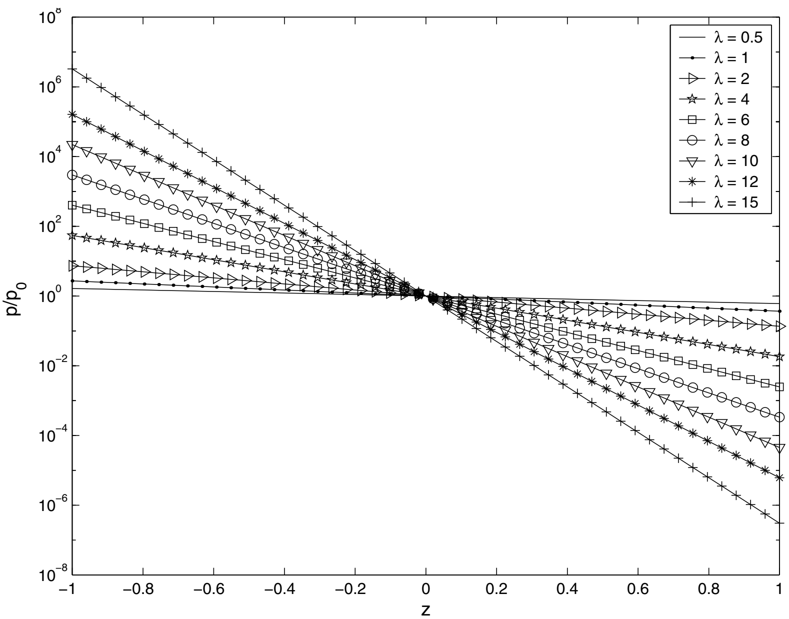

Figure 2. (p/p_0) versus z for various λ at $r = 0$.

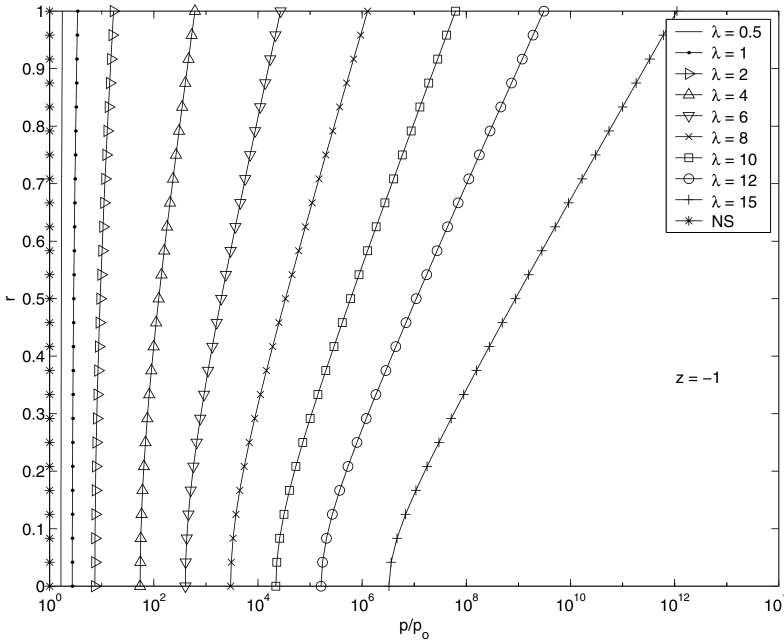

Figure 3. (p/p_0) versus γ for various λ^S at $z = -1$.

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