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ON A CONSERVED PENROSE-FIFE TYPE SYSTEM*

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Abstract. We deal with a class of Penrose-Fife type phase field models for phase transitions, where the phase dynamics is ruled by a Cahn-Hilliard type equation. Suitable assumptions on the behaviour of the heat flux as the absolute temperature tends to zero and to $+\infty$ are considered. An existence result is obtained by a double approximation procedure and compactness methods. Moreover, uniqueness and regularity results are proved as well.

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1. Introduction

In this paper we address a well-posedness problem for a system of evolution equations in three space dimensions modelling a phase transition process. A material which occupies a bounded open region $\Omega \subset \mathbb{R}^3$ is supposed to be cooled or heated, and by means of this it changes its phase, e.g. from liquid to solid or vice versa. The evolution equations which describe the process involve the absolute temperature ϑ and an order parameter χ, which is of use for distinguishing one phase from another [31]. In the present paper, such a parameter is a conserved quantity, i.e., the integral of χ over Ω remains constant during the phase transition process. The system we study is

$$\partial_t(\vartheta + \lambda\chi) - \Delta\alpha(\vartheta) = g \quad \text{in } Q := \Omega \times (0, T), \tag{1.1}$$

$$\partial_t\chi - \Delta w = 0 \quad \text{in } Q, \tag{1.2}$$

$$w = -\Delta\chi + \beta(\chi) + \sigma'(\chi) - \frac{\lambda}{\vartheta_c} + \frac{\lambda}{\vartheta} \quad \text{in } Q. \tag{1.3}$$

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In the above equations

- $T > 0$ is a positive time,
- ϑ_c is a critical value of the absolute temperature around which the phase transition occurs; $w\colon Q \to \mathbb{R}$ is the so-called chemical potential,
- α and σ are constitutive smooth functions, with α monotone, while λ is a positive constant (latent heat),
- β is a maximal monotone graph in $\mathbb{R} \times \mathbb{R}$,
- g is a given source term.

The above system (where (1.3) has to be read as a differential inclusion if β is not a one-valued function) turns out to be of Penrose-Fife type [25], [26] and the subsystem (1.2)–(1.3) that rules the phase dynamics can be viewed as a generalization of the well-known Cahn-Hilliard equation or system (see, e.g., [1], [5]), namely

$$\partial_t \chi = -\operatorname{div} \mathbf{j}, \quad \mathbf{j} = -\nabla \frac{\delta \mathcal{F}}{\delta \chi}$$

where the generalized force $\delta\mathcal{F}/\delta\chi$ is the functional derivative of the Ginzburg-Landau functional ($\hat{\beta}$ being a primitive of β, see, e.g., [4], [5], [24])

$$\mathcal{F}(\vartheta, \chi) = \int_\Omega \left\{ \frac{1}{2} |\nabla \chi|^2 + \hat{\beta}(\chi) + \sigma(\chi) + \lambda \chi \Big(\frac{1}{\vartheta} - \frac{1}{\vartheta_c} \Big) \right\} \mathrm{d}x.$$

In equation (1.1) a key role is played by the choice of α. Indeed, (1.1) reflects the energy balance and α is related to the form of the heat flux $\mathbf{q}$, namely

$$\mathbf{q} = -\nabla \alpha(\vartheta). \tag{1.4}$$

In literature, several classes of heat flux laws are considered. Regarding non conserved models (i.e., second order phase dynamics), several papers [8], [12], [13], [16], [17], [19], [20], [30] deal with the case $\alpha(\vartheta) \approx -1/\vartheta$. On the other hand, less papers [10], [21] consider the Fourier law $\alpha(\vartheta) \approx \vartheta$, which is more satisfactory for high temperatures but leads to a more difficult problem. Such considerations suggest introducing a class of intermediate laws. In this direction we refer to [7], [8], [9], [11]. More precisely, in [8] it is assumed that $\alpha'(\vartheta) \approx \vartheta^{-2p}$ as $\vartheta \searrow 0$ and that $\alpha'(\vartheta) \approx \vartheta^{-2q}$ as $\vartheta \nearrow +\infty$, where $p \geqslant 1/2$ and $q \in [0, 1/2]$. The authors consider the third type boundary condition for temperature and prove an existence result. The particular case $p = 1$ and $q = 0$ is treated in [9], where a regularity and uniqueness result is proved, while [11] and [7] regard well-posedness results for different boundary conditions for $\alpha(\vartheta)$ (Dirichlet and Neumann, respectively) and use intermediate assumptions on α, namely $p = 1$ and $q \in [0, 1/2]$.

Regarding conserved models, many results are known in the case $\alpha(\vartheta) \approx -1/\vartheta$ (cf. [14], [15], [18], [29]), while just the paper [28] deals with the Fourier law, as far as we know. Finally, we mention [27], which assumes $p = 1$ and $q = 0$ as in [9] and accounts for memory effects as well (the no-memory case being a particular one).

In this paper we deal with the conserved case assuming essentially the same framework as in [11] and [7] as far as α is concerned (just the case $q = 1/2$ is excluded, indeed), and couple equations (1.1)–(1.3) with homogeneous Neumann conditions for χ and w (as usual) and third type boundary conditions for $u := \alpha(\vartheta)$, namely

$$\partial_n\chi = \partial_n w = 0 \qquad \text{and} \qquad \partial_n u + \gamma u = h \tag{1.5}$$

where ∂_n is the normal derivative and the boundary function h and the positive constant γ are given. However, the Dirichlet condition for u could be considered, similarly to [11]. On the other hand, a Neumann boundary condition for u (i.e., $\gamma = 0$ in (1.5)) looks much more delicate.

We prove an existence result using a double approximation procedure. The solution we obtain comes out to be unique whenever $\alpha(\vartheta)$ is at most a linear perturbation of $-1/\vartheta$ for small ϑ. With weaker assumptions on α, we prove also an existence and uniqueness result of smoother solutions.

The paper is organized as follows. Section 2 contains the statements of the problem and of the main results. Section 3 is devoted to the existence results for the approximating problems, obtained via a Faedo-Galerkin method. In Section 4 some auxiliary lemmas are proved. Section 5 contains the a priori estimates for the approximating problems and the proof of a first existence result. The last two sections address the problems of uniqueness of the solution obtained in Section 5, and of the existence and uniqueness of smoother solutions, respectively.

2. Statement of the problem

In this section, we take some care in describing the problem we are going to deal with. Moreover, we list our assumptions and state our results. We start with the assumption on the structure of the system.

As in Introduction, λ and γ are fixed positive constants. Moreover, we are given two C^1 functions $\alpha\colon (0,+\infty) \to \mathbb{R}$ and $\sigma\colon \mathbb{R} \to \mathbb{R}$ and a nonnegative function $\hat\beta\colon \mathbb{R} \to [0,+\infty]$ satisfying the conditions listed below, where the constants c_0 and c_∞ are finite and strictly positive.

$$\alpha \text{ is strictly increasing and concave and } \alpha(1) = 0, \tag{2.1}$$

$$\lim_{r\to 0^+} r^2\alpha'(r) = c_0 \text{ and } \lim_{r\to +\infty} r^{2q}\alpha'(r) = c_\infty \text{ with } 0 \leqslant q < 1/2, \tag{2.2}$$

$$\sigma' \text{ is Lipschitz continuous,} \tag{2.3}$$

$$\hat{\beta} \text{ is convex, proper, lower semicontinuous, with } \hat{\beta}(0) = 0 \tag{2.4}$$

Note that assumptions (2.2) ensure that the inequalities

$$\alpha'(r) \geqslant \frac{C_1}{r^2} \quad \text{and} \quad \alpha'(r) \geqslant \frac{C_1}{r^{2q}} \quad \forall\, r > 0 \tag{2.5}$$

hold true for some $C_1 > 0$. The same assumptions (2.2) imply that

$$\lim_{r \to 0^+} r\alpha(r) = -c_0 \quad \text{and} \quad \lim_{r \to +\infty} r^{2q-1}\alpha(r) = \frac{c_\infty}{1-2q}. \tag{2.6}$$

We define functions $\hat{\alpha}$ and ϱ and a graph β in $\mathbb{R} \times \mathbb{R}$ by

$$\hat{\alpha}(r) := \int_1^r \alpha(r')\,\mathrm{d}r' \quad (r > 0), \qquad \varrho := \alpha^{-1}, \quad \text{and} \quad \beta := \partial\hat{\beta} \tag{2.7}$$

and note that β is maximal monotone. The same symbol β will be used for the maximal monotone operators induced on L^2 spaces.

Next, we list our assumptions on the data. To this aim, we introduce a notation. In the sequel, Ω is a bounded connected open set in $\mathbb{R}^3$ with a C^2 boundary Γ and T is a given final time. For the sake of convenience we set also

$$H := L^2(\Omega), \quad V := H^1(\Omega), \quad \text{and} \quad W := \{v \in H^2(\Omega)\colon\ \partial_n v = 0\}. \tag{2.8}$$

Throughout the paper we think of the Hilbert triplet (V, H, V') obtained by identifying H with a subspace of V' in the usual way. Moreover, the symbol $\langle \cdot, \cdot \rangle$ stands for the duality product between V' and V. Note that $\langle u, v \rangle = (u, v)$ for any $u \in H$ and $v \in V$, where (u, v) is the scalar product in H. For the sake of simplicity, we denote with $\|\cdot\|_H$ both the norm in H and the norm in any power of H. Accordingly, we simply write, e.g., $L^2(0, T; H)$ meaning any power of it.

In order to present the problem we are going to deal with it in an abstract form including the boundary conditions, defining $A, B \in \mathcal{L}(V; V')$ by the formulas (where u and v vary in V)

$$\langle Au, v \rangle = \int_\Omega \nabla u \cdot \nabla v + \gamma \int_\Gamma uv \quad \text{and} \quad \langle Bu, v \rangle = \int_\Omega \nabla u \cdot \nabla v. \tag{2.9}$$

We note that the formula

$$\|v\|_V^2 = \langle Av, v \rangle \tag{2.10}$$

defines a norm in V which is equivalent to the usual one. Moreover, B is not one-to-one and its range is not the whole of V'. Precisely, if $f \in V'$ and f_Ω denotes its (generalized) mean value, i.e.,

$$f_\Omega := \frac{1}{|\Omega|} \langle f, 1 \rangle \tag{2.11}$$

then the range of B is the subspace V_0' of V' defined by

$$V_0' := \{ f \in V' : \; f_\Omega = 0 \} \tag{2.12}$$

and the restriction of B to the subspace

$$V_0 := V \cap V_0' = \{ v \in V : \; v_\Omega = 0 \} \tag{2.13}$$

maps V_0 onto V_0' isomorphically. Hence, we can introduce its inverse operator $\mathcal{N}$. Note that $\mathcal{N} f$ is the solution with zero mean value of a (generalized) Neumann problem and that $\mathcal{N}$ is determined by the conditions

$$\mathcal{N} \colon \; V_0' \to V_0 \quad \text{and} \quad B\mathcal{N} f = f \quad \forall\, f \in V_0'. \tag{2.14}$$

Moreover, the following relations hold:

$$\langle Bv, \mathcal{N} f \rangle = \langle f, v \rangle \quad \forall\, v \in V, \; \forall\, f \in V_0', \tag{2.15}$$

$$\langle f_1, \mathcal{N} f_2 \rangle = \langle f_2, \mathcal{N} f_1 \rangle = \int_\Omega (\nabla \mathcal{N} f_1) \cdot (\nabla \mathcal{N} f_2) \quad \forall\, f_1, f_2 \in V_0'. \tag{2.16}$$

In particular, the formula

$$\|f\|_*^2 := \langle f, \mathcal{N} f \rangle = \int_\Omega |\nabla \mathcal{N} f|^2$$

defines a norm in V_0' which is equivalent to the one induced by the usual norm of V'. Such a norm can be extended to the whole of V' by the formula

$$\|f\|_*^2 := \langle f, \mathcal{N}(f - f_\Omega) \rangle + f_\Omega^2. \tag{2.17}$$

We point out that the inequalities

$$|\,\langle f, v \rangle\,| \leqslant c \|f\|_* \|v\|_V \quad \text{and} \quad \|v\|_H^2 \leqslant c \|v\|_{L^4(\Omega)}^2 \leqslant \eta \|\nabla v\|_H^2 + c_\eta \|v\|_*^2 \tag{2.18}$$

hold for any $f \in V'$, $v \in V$ and $\eta > 0$. Here, c is a constant that depends only on Ω and γ, while c_η depends on η in addition.

More generally, throughout the paper, we will use the symbol c for different constants which depend only on Ω, on the final time T, and on the constants and the norms of the functions involved in the assumptions of our statements. A notation like c_η allows the constant to depend on the positive parameter η, in addition. Hence, the meaning of c and c_η may change from line to line and even in the same chain of inequalities. On the contrary, symbols like $C_1, C_2, \ldots$ denote precise constants (e.g., defined in some statement).

Now, we come back to the problem we want to deal with. As far as the data of the problem are concerned, we are given four functions g, h, ϑ_0 and χ_0 satisfying

$$g \in L^2(Q) \ \text{ and } \ h \in L^\infty(0,T;L^{q_\bullet}(\Gamma)) \ \text{ where } \ q_\bullet := \frac{4}{3(1-2q)}, \tag{2.19}$$

$$\vartheta_0 \in L^\infty(\Omega), \ \ \vartheta_0 > 0 \ \text{ a.e. in } \Omega, \ \text{ and } \ 1/\vartheta_0 \in L^\infty(\Omega), \tag{2.20}$$

$$\chi_0 \in H^1(\Omega) \ \text{ and } \ \hat\beta(\chi_0) \in L^1(\Omega), \tag{2.21}$$

$$\text{the mean value of } \chi_0 \text{ belongs to the interior of } D(\beta) \tag{2.22}$$

where $D(\beta)$ is the domain of β (see, e.g., [3, p. 20]). Noting that $q_\bullet \geqslant 4/3$ and owing to [23, Thm. 4.2, p. 84], we have

$$\left| \int_\Gamma h(t) v \right| \leqslant \|h(t)\|_{L^{4/3}(\Gamma)} \|v\|_{L^4(\Gamma)} \leqslant c \|h(t)\|_{L^{q_\bullet}(\Gamma)} \|v\|_{H^1(\Omega)} \tag{2.23}$$

for a.a. $t \in (0,T)$ and for any $v \in H^1(\Omega)$. Hence, the formula

$$\langle f(t), v \rangle := \int_\Omega g(t) v + \gamma \int_\Gamma h(t) v \quad \text{for a.a. } t \in (0,T) \text{ and } v \in V \tag{2.24}$$

is meaningful and yields $f \in L^2(0,T;V')$. Then our problem consists in finding a quintuple $(\vartheta, \chi, u, w, \xi)$ satisfying the regularity conditions and the equations written below:

$$\vartheta \in L^\infty(0,T;H) \cap L^2(0,T;W^{1,q_*})(\Omega)) \cap H^1(0,T;V') \ \text{ where } \ q_* := \frac{2}{q+1}, \tag{2.25}$$

$$\vartheta > 0 \ \text{ a.e. in } Q \ \text{ and } \ 1/\vartheta \in L(0,T;V), \tag{2.26}$$

$$\chi \in L^2(0,T;W) \cap L^\infty(0,T;V) \cap H^1(0,T;V'), \tag{2.27}$$

$$u \in L^2(0,T;V), \ \ w \in L^2(0,T;V), \ \text{ and } \ \xi \in L^2(Q). \tag{2.28}$$

The equations to be satisfied are

$$\partial_t(\vartheta(t) + \lambda\chi(t)) + Au(t) = f(t) \ \text{ for a.a. } t \in (0,T), \tag{2.29}$$

$$u = \alpha(\vartheta) \ \text{ a.e. in } Q, \tag{2.30}$$

$$\partial_t \chi(t) + Bw(t) = 0 \ \text{ for a.a. } t \in (0,T), \tag{2.31}$$

$$w(t) = B\chi(t) + \xi(t) + \sigma'(\chi(t)) + \lambda/\vartheta(t) \text{ for a.a. } t \in (0,T), \tag{2.32}$$

$$\xi \in \beta(\chi) \text{ a.e. in } Q, \tag{2.33}$$

$$\vartheta(0) = \vartheta_0 \text{ and } \chi(0) = \chi_0 \tag{2.34}$$

where the new function σ' is obtained by adding $-\lambda/\vartheta_c$ to the one of (1.3). Now, we state our results.

Theorem 2.1. *Assume* (2.1)–(2.7), (2.8)–(2.9), (2.19)–(2.21) *and* (2.24). *Then there exists at least one solution* $(\vartheta, \chi, u, w, \xi)$ *to problem* (2.29)–(2.34) *satisfying the regularity requirements* (2.25)–(2.28).

In order to prove a uniqueness result for the solution to (2.25)–(2.34), we need to reinforce our assumptions on α. More precisely, note that the first of (2.2) can be rewritten as

$$r^2\alpha'(r) = c_0 + o(1) \quad \text{as } r \searrow 0. \tag{2.35}$$

Instead of (2.35) we require that

$$r^2\alpha'(r) = c_0 + O(r^2) \quad \text{as } r \searrow 0. \tag{2.36}$$

We will prove the following statement.

Theorem 2.2. *In the same setting as in Theorem* 2.1, *assume that* (2.36) *holds. Then the components* ϑ, χ *and* u *of the solution to* (2.29)–(2.34) *in Theorem* 2.1 *are unique.*

Remark 2.3. Uniqueness for the components χ and w is not guaranteed, in general, unless β is single-valued, as in [6, Remark 2.3]. Moreover, we note that assumptions (2.2), (2.36) imply that the function ℓ defined by

$$\ell(r) := \alpha(r) + \frac{c_0}{r} \tag{2.37}$$

is globally Lipschitz continuous on $(0, +\infty)$. This is what is actually used in Section 6 to prove Theorem 2.2.

In the last section, we prove further regularity and one more uniqueness result. To do that, we use the following assumption on α:

$$r^2\alpha'(r) = c_0 + O(r) \quad \text{as } r \searrow 0. \tag{2.38}$$

Such a condition is stronger than (2.35), but weaker than (2.36), obviously. Regarding the data, we assume that they are smoother and compatible as follows:

$$g \in W^{1,1}(0,T;H) \quad \text{and} \quad h \in W^{1,1}(0,T;L^{q_\bullet}(\Gamma)), \tag{2.39}$$

$$u_0 := \alpha(\vartheta_0) \in V, \tag{2.40}$$

$$\chi_0 \in W \quad \text{and} \quad (B\chi_0 + \beta(\chi_0)) \cap V \neq \emptyset. \tag{2.41}$$

The exponent $q_\bullet$ is defined in (2.19) and the last assumption means that there exists ξ_0 satisfying

$$\xi_0 \in H, \quad \xi_0 \in \beta(\chi_0) \text{ a.e. in } \Omega, \quad \text{and} \quad B\chi_0 + \xi_0 \in V. \tag{2.42}$$

Theorem 2.4. *In the same setting as in Theorem* 2.1, *assume* (2.38) *and* (2.39)–(2.41) *in addition. Then problem* (2.29)–(2.34) *has a solution satisfying*

$$\vartheta \in W^{1,\infty}(0,T;V'), \tag{2.43}$$

$$\chi \in H^1(0,T;V) \cap W^{1,\infty}(0,T;V') \cap L^\infty(0,T;W), \tag{2.44}$$

$$u \in L^\infty(0,T;V), \tag{2.45}$$

$$w \in L^\infty(0,T;V). \tag{2.46}$$

Moreover, the components ϑ, χ and u of such a solution are unique and ϑ satisfies

$$\vartheta^{1-2q} \in L^\infty(0,T;V) \quad \textit{and} \quad \vartheta \in L^\infty(0,T;W^{1,\bar{q}}(\Omega)) \tag{2.47}$$

where $\bar{q} := 2/(2q+1)$.

As said in Introduction, our results are proved in Sections 5–7, while the next sections are devoted to the approximating problems and to technical tools. We remind the reader that $Q = \Omega \times (0,T)$ and introduce the notation

$$Q_t := \Omega \times (0,t) \quad \text{and} \quad \Sigma_t := \Gamma \times (0,t)$$

for arbitrary $t \in (0,T)$. Moreover, we widely use the elementary inequality

$$ab \leqslant \eta a^2 + \frac{1}{4\eta} b^2 \quad \forall\, a,b \in \mathbb{R} \;\; \forall\, \eta > 0$$

without further notice.

3. Approximating problem

We follow some ideas of [8] in choosing the approximation, but we use different notation and a different method in showing an existence result. In the sequel, ε and δ are positive parameters that will tend to zero. However, the limit procedure will be performed later on, and ε and δ are fixed elements of $(0,1)$ throughout the whole section.

We replace the graphs β and ϱ by smooth functions, denoted by β_ε and ϱ_δ, respectively. The first, β_ε, is the Yosida regularization of β and is Lipschitz continuous with a Lipschitz constant $1/\varepsilon$ (see, e.g., [3, p. 28]), while ϱ_δ is constructed in the following way. We fix a C^1 cut-off function $\zeta\colon [0,+\infty) \to \mathbb{R}$ satisfying $\zeta(s) = 1$ for $s \leqslant 1$, $\zeta(s) = 0$ for $s \geqslant 2$, and $\zeta'(s) < 0$ for $s \in (1,2)$, and define

$$\varrho_\delta(s) := 1 + \int_0^s \varrho'(r)\zeta(\delta|r|)\,\mathrm{d}r, \quad s \in \mathbb{R}. \tag{3.1}$$

Note that ϱ_δ is nondecreasing and strictly positive. Moreover, in order to simplify the notation, we define

$$z_{\varepsilon\delta}(r,s) := \beta_\varepsilon(r) + \sigma'(r) + \frac{\lambda}{\varrho_\delta(s)}, \quad (r,s) \in \mathbb{R}^2 \tag{3.2}$$

and note that $z_{\varepsilon\delta}$ is Lipschitz continuous in $\mathbb{R}^2$ and satisfies

$$|z_{\varepsilon\delta}(r,s)| \leqslant c_{\varepsilon\delta}(|r|+1) \quad \forall\, r,s \in \mathbb{R}. \tag{3.3}$$

Finally, we approximate $\alpha(\vartheta_0)$, χ_0 and f by three functions

$$u_{0\delta} \in V, \quad \chi_{0\varepsilon} \in V \quad \text{and} \quad f_\varepsilon \in L^2(0,T;H). \tag{3.4}$$

More requirements on $u_{0\delta}$, $\chi_{0,\varepsilon}$ and f_ε will be introduced in the next section. Now, for fixed ε and δ, we look for a triple (u,χ,w) satisfying the conditions

$$u,\chi \in L^\infty(0,T;V) \cap H^1(0,T;H) \quad \text{and} \quad w \in L^2(0,T;V) \tag{3.5}$$

and fulfilling the equations

$$\partial_t(\varepsilon u + \varrho_\delta(u) + \lambda\chi) + Au = f_\varepsilon, \tag{3.6}$$

$$\partial_t\chi + Bw = 0, \tag{3.7}$$

$$w = \varepsilon\partial_t\chi + B\chi + z_{\varepsilon\delta}(\chi,u), \tag{3.8}$$

$$u(0) = u_{0\delta}, \quad \chi(0) = \chi_{0\varepsilon}. \tag{3.9}$$

Finite dimensional approximation. As it is not obvious that problem (3.6)–(3.9) has a solution, we derive an existence result for it using a Faedo-Galerkin scheme. As we are keeping ε and δ fixed, we do not point out the dependence of some functions on such parameters. Let $\{y_n\}$ be a complete orthonormal set of eigenfunctions for B, namely

$$By_n = \mu_n y_n \quad \text{and} \quad \|y_n\|_H = 1 \ \text{ for } \ n = 1, 2, \ldots,$$
$$(y_n, y_m) = 0 \ \text{ for } \ n \neq m,$$
$$\text{the set } \ \{y_n \colon \ n = 1, 2, \ldots\} \ \text{ spans a dense subspace of } \ V$$

where $\mu_1 < \mu_2 \leqslant \mu_3 \leqslant \ldots$ are the corresponding eigenvalues. We note at once that

$$\mu_1 = 0 \quad \text{and} \quad y_1 \ \text{ is a constant,} \tag{3.10}$$

$$\mathcal{N} v = \sum_{i=2}^{n} \frac{\eta_i}{\mu_i} y_i \quad \text{if} \ \ v = \sum_{i=2}^{n} \eta_i y_i \quad (n \geqslant 2). \tag{3.11}$$

For $n = 1, 2, \ldots$, we denote by V_n the subspace of V spanned by $y_1, \ldots, y_n$.

Now, for fixed n, we state the finite dimensional approximating problem. We look for $3n$ functions $a_i, b_i, c_i \colon \ [0, T] \to \mathbb{R}$ $(i = 1, \ldots, n)$ such that the functions defined by

$$u_n(t) = \sum_{j=1}^{n} a_j(t) y_j, \quad \chi_n(t) = \sum_{j=1}^{n} b_j(t) y_j \quad \text{and} \quad w_n(t) = \sum_{j=1}^{n} c_j(t) y_j$$

provide a solution on $[0, T]$ to the system $(i = 1, \ldots, n)$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \varepsilon u_n(t) + \varrho_\delta(u_n(t)) + \lambda \chi_n(t), y_i \rangle + \langle A u_n(t), y_i \rangle = \langle f_\varepsilon(t), y_i \rangle, \tag{3.12}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \langle \chi_n(t), y_i \rangle + \langle B w_n(t), y_i \rangle = 0, \tag{3.13}$$

$$\langle w_n(t), y_i \rangle = \varepsilon \frac{\mathrm{d}}{\mathrm{d}t} \langle \chi_n(t), y_i \rangle + \langle B \chi_n(t) + z_{\varepsilon\delta}(\chi_n(t), u_n(t)), y_i \rangle, \tag{3.14}$$

satisfying the Cauchy conditions $(i = 1, \ldots, n)$

$$\langle u_n(0), y_i \rangle = \langle u_{0\delta}, y_i \rangle \quad \text{and} \quad \langle \chi_n(0), y_i \rangle = \langle \chi_{0\varepsilon}, y_i \rangle. \tag{3.15}$$

Solution to the finite dimensional problem. The above system has the form

$$M_1(\mathbf{a}(t)) \mathbf{a}'(t) + \lambda \mathbf{b}'(t) + M_2 \mathbf{a}(t) = \mathbf{f}(t), \tag{3.16}$$

$$\mathbf{b}'(t) = -M_3 \mathbf{c}(t), \tag{3.17}$$

$$\mathbf{c}(t) = \varepsilon \mathbf{b}'(t) + M_3 \mathbf{b}(t) + \mathbf{z}(\mathbf{b}(t), \mathbf{a}(t)). \tag{3.18}$$

Here the matrices $M_\ell = (M_{\ell ij})$ and the functions $\mathbf{z} = (z_i)$ and $\mathbf{f} = (f_i)$ are defined by

$$M_{1ij}(\mathbf{a}) := (\varepsilon y_j + \varrho'_\delta(S(\mathbf{a}, y_1, \dots, y_n))y_j, y_i),$$
$$M_{2ij} := \langle A y_j, y_i \rangle, \quad M_3 := \mathrm{diag}(\mu_1, \dots, \mu_n),$$
$$z_i(\mathbf{b}, \mathbf{a}) := z_{\varepsilon\delta}(S(\mathbf{b}, y_1, \dots, y_n), S(\mathbf{a}, y_1, \dots, y_n)),$$
$$f_i(t) := \langle f_\varepsilon(t), y_i \rangle$$

with the notation $S(\mathbf{a}, y_1, \dots, y_n) = \sum_{k=1}^{n} a_k y_k$ and a similar one with $\mathbf{b}$ in place of $\mathbf{a}$. Clearly, M_1 and $\mathbf{z}$ are globally Lipschitz continuous functions of their arguments. Moreover, we can eliminate $\mathbf{c}$ in (3.17) with help of (3.18), then we can solve the obtained equation for $\mathbf{b}'$ since the constant matrix $I + \varepsilon M_3$ is positive definite, and replace $\mathbf{b}'$ by its expression in (3.16). Finally, a simple computation shows that $(M_1(\mathbf{a})\mathbf{p}) \cdot \mathbf{p} \geqslant \varepsilon |\mathbf{p}|^2$ for any $\mathbf{a} \in \mathbb{R}^n$ and $\mathbf{p} \in \mathbb{R}^n$, so that the first equation can be put into its normal form. Hence, we reduce our study to a Cauchy problem for a system like

$$(\mathbf{a}'(t), \mathbf{b}'(t)) = \mathbf{F}(t, \mathbf{a}(t), \mathbf{b}(t))$$

where $\mathbf{F}$ is a continuous function satisfying a global Lipschitz conditions with respect to $(\mathbf{a}, \mathbf{b})$ uniformly in t. Therefore, the approximating problem (3.12)–(3.15) has a unique global solution

$$u_n \in C^1([0,T]; V_n), \quad \chi_n \in C^1([0,T]; V_n) \quad \text{and} \quad w_n \in C^1([0,T]; V_n).$$

The remaining part of this section is devoted to passing to the limit in n in order to prove that the approximating problem (3.6)–(3.9) has a solution. To this aim, we derive a number of a priori estimates. Thus, it is convenient to introduce the function

$$R_\delta(s) := \int_0^s r \varrho'_\delta(r)\, \mathrm{d}r, \quad s \in \mathbb{R}. \tag{3.19}$$

We note that R_δ is nonnegative.

First a priori estimate. We write (3.12) for $t = s \in (0,T)$ and $i = 1, \dots, n$. Then, we multiply the obtained equations by $a_i(s)$, sum over i, and integrate over $(0,t)$ with respect to s. The resulting equation reads

$$\frac{\varepsilon}{2} \int_\Omega |u_n(t)|^2 + \int_{Q_t} \varrho'_\delta(u_n) u_n \partial_t u_n + \lambda \int_{Q_t} \partial_t \chi_n u_n + \int_0^t \langle A u_n(s), u_n(s) \rangle \,\mathrm{d}s$$
$$= \frac{\varepsilon}{2} \int_\Omega |u_n(0)|^2 + \int_0^t \langle f_\varepsilon(s), u_n(s) \rangle \,\mathrm{d}s.$$

We deal with (3.13) and (3.14) in a similar way, but we take just $i = 2, \dots, n$ and avoid the value $i = 1$. Moreover, we multiply these equations by $b_i'(s)/\mu_i$ and $-b_i'(s)$, respectively, and sum the resulting equalities. Then two terms cancel. After summation over i and integration over $(0,t)$, we obtain with help of (3.11)

$$\int_0^t \langle \partial_t \chi_n(s), \mathcal{N} \partial_t \chi_n(s) \rangle \, \mathrm{d}s + \varepsilon \int_{Q_t} |\partial_t \chi_n|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_n(t)|^2 + \int_{Q_t} z_{\varepsilon\delta}(\chi_n, u_n) \partial_t \chi_n = \frac{1}{2} \int_\Omega |\nabla \chi_n(0)|^2.$$

Now, we add this equation to the previous one multiplied by a parameter $\eta_1 \in (0,1)$, and start estimating. As $u_n(0)$ and $\nabla \chi_n(0)$ are projections, we have for any $\eta > 0$

$$\begin{aligned}
&\frac{\eta_1 \varepsilon}{2} \int_\Omega |u_n(t)|^2 + \eta_1 \int_\Omega R_\delta(u_n(t)) + \eta_1 \int_0^t \|u_n(s)\|_V^2 \, \mathrm{d}s \\
&\qquad + \int_0^t \|\partial_t \chi(s)\|_*^2 \, \mathrm{d}s + \varepsilon \int_{Q_t} |\partial_t \chi_n|^2 + \frac{1}{2} \int_\Omega |\nabla \chi_n(t)|^2 \\
&\quad \leqslant \frac{\eta_1 \varepsilon}{2} \int_\Omega |u_n(0)|^2 + \eta_1 \int_\Omega R_\delta(u_n(0)) + \frac{1}{2} \int_\Omega |\nabla \chi_n(0)|^2 \\
&\qquad + c\eta_1 \int_0^t (\|\partial_t \chi_n(s)\|_* + \|f_\varepsilon(s)\|_*) \|u_n(s)\|_V \, \mathrm{d}s \\
&\qquad + \frac{\varepsilon}{2} \int_{Q_t} |\partial_t \chi_n|^2 + c_{\varepsilon\delta} \int_{Q_t} |z_{\varepsilon\delta}(\chi_n(s), u_n(s))|^2 \\
&\quad \leqslant \|u_{0\delta}\|_H^2 + \eta_1 c_{\varepsilon\delta} + \|\chi_{0\varepsilon}\|_V^2 + \eta_1 \eta \int_0^t \|u_n(s)\|_V^2 \, \mathrm{d}s \\
&\qquad + \eta_1 c_\eta \int_0^t \|\partial_t \chi_n(s)\|_*^2 \, \mathrm{d}s + \eta_1 c_\eta \|f_\varepsilon\|_{L^2(0,T;V')}^2 \\
&\qquad + \frac{\varepsilon}{2} \int_{Q_t} |\partial_t \chi_n|^2 + c_{\varepsilon\delta} \int_{Q_t} |z_{\varepsilon\delta}(\chi_n(s), u_n(s))|^2.
\end{aligned}$$

By choosing first η and then η_1 small enough, we deduce

$$\begin{aligned}
&\int_\Omega |u_n(t)|^2 + \int_0^t \|u_n(s)\|_V^2 \, \mathrm{d}s \\
&\qquad + \int_0^t \|\partial_t \chi(s)\|_*^2 \, \mathrm{d}s + \int_{Q_t} |\partial_t \chi_n|^2 + \int_\Omega |\nabla \chi_n(t)|^2 \\
&\quad \leqslant c_{\varepsilon\delta} \|u_{0\delta}\|_H^2 + c_{\varepsilon\delta} \|\chi_{0\varepsilon}\|_V^2 + c_{\varepsilon\delta} \|f_\varepsilon\|_{L^2(0,T;V')}^2 \\
&\qquad + c_{\varepsilon\delta} \int_{Q_t} |z_{\varepsilon\delta}(\chi_n(s), u_n(s))|^2 + c_{\varepsilon\delta}
\end{aligned} \tag{3.20}$$

and we have to estimate just the last integral. Thanks to (3.3), we are led to estimating the integral of $|\chi_n|^2$, and this can be done as follows. We have

$$\|\chi_n(s)\|_H^2 \leqslant 2\|\chi_{0\varepsilon}\|_H^2 + 2T\int_0^s \|\partial_t\chi_n(\tau)\|_H^2\,\mathrm{d}\tau$$

whence

$$\int_{Q_t}|\chi_n|^2 \leqslant 2T\|\chi_{0\varepsilon}\|_H^2 + 2T\int_0^t\Big(\int_{Q_s}|\partial_t\chi_n|^2\Big)\,\mathrm{d}s. \tag{3.21}$$

Thus, combining (3.20) and (3.21), we obtain an inequality similar to (3.20) with the last integral of (3.21) in place of the $z_{\varepsilon\delta}$ term. So, we are in position to apply the Gronwall lemma and obtain a number of a priori estimates. Then, a further use of (3.21) yields a bound for the norm of χ_n in $L^2(Q)$ as well. Hence, we have

$$\|u_n\|_{L^\infty(0,T;H)} + \|u_n\|_{L^2(0,T;V)} + \|\partial_t\chi_n\|_{L^2(0,T;H)} + \|\chi_n\|_{L^\infty(0,T;V)} \leqslant c_{\varepsilon\delta}. \tag{3.22}$$

Second a priori estimate. For $i = 1,\dots,n$, we write (3.12) for $t = s$ and multiply by $a_i'(s)$. After summation and integration as before, we obtain

$$\begin{aligned}\varepsilon\int_{Q_t}|\partial_t u_n|^2 + \int_{Q_t}\varrho_\delta'(u_n)|\partial_t u_n|^2 + \frac{1}{2}\|u_n(t)\|_V^2 \\ = \frac{1}{2}\|u_{0\delta}\|_V^2 + \int_{Q_t}(f_\varepsilon - \lambda\partial_t\chi_n)\partial_t u_n.\end{aligned}$$

Owing to (3.4) and (3.22), we deduce

$$\|\partial_t u_n\|_{L^2(0,T;H)} + \|u_n\|_{L^\infty(0,T;V)} \leqslant c_{\varepsilon\delta}. \tag{3.23}$$

Third a priori estimate. For $i = 1,\dots,n$, we write (3.13) and (3.14) for $t = s$, multiply them by $\varepsilon c_i(s)$ and by $c_i(s)$, respectively, and sum the resulting equalities. Note that two terms cancel out. After summation over i and integration in time, we obtain

$$\varepsilon\int_{Q_t}|\nabla w_n|^2 + \int_{Q_t}|w_n|^2 = \int_{Q_t}\nabla\chi_n\cdot\nabla w_n + \int_{Q_t}z_{\varepsilon\delta}(\chi_n,u_n)w_n,$$

whence easily

$$\|w_n\|_{L^2(0,T;V)} \leqslant c_{\varepsilon\delta}. \tag{3.24}$$

Existence for the approximating problem. At this point, it is straightforward to pass to the limit as $n \nearrow \infty$, at least for a subsequence, using the standard weak and strong compactness arguments. Indeed, the nonlinearities involve Lipschitz continuous functions and the strong convergence in $L^2(Q)$ is sufficient to identify the limits of the nonlinear terms. Thus, we have proved that the approximating problem (3.6)–(3.9) has at least one solution satisfying the regularity requirements (3.5).

R e m a r k 3.1. Let (u,χ,w) be a solution to the approximating problem (3.6)–(3.9). Then $\partial_t\chi(t)$ belongs to the range of B for a.a. $t\in(0,T)$. Hence, it has zero mean value. Therefore, $\mathcal{N}\chi(t)$ is well defined for a.a. $t\in(0,T)$ and

$$\langle\chi(t),1\rangle=\langle\chi_{0\varepsilon},1\rangle\quad\forall\,t\in[0,T]. \tag{3.25}$$

Moreover, we note that all the terms of equation (3.8) but $B\chi$ belong to $L^2(0,T;H)$. We deduce that $B\chi\in L^2(0,T;H)$ as well, whence $\chi\in L^2(0,T;W)$ (see (2.8)). Finally, $\varrho_\delta(u)$ and $1/\varrho_\delta(u)$ belong at least to $L^2(0,T;V)$ since ϱ_δ and $1/\varrho_\delta$ are Lipschitz continuous.

4. Auxiliary results

In this section, we make some details of the approximation more precise and prove some auxiliary results. In order not to be too boring, we simplify some statements (and the corresponding proofs) and just say "for δ small enough". This means that the statements hold provided that $\delta\leqslant\delta_0$, where δ_0 is some positive number depending only on Ω, on the final time T, and on the constants and the norms of the functions involved in our general assumptions.

First of all, we recall the definition of ϱ_δ and R_δ given in (3.1) and in (3.19), respectively, and define

$$m_\delta:=\varrho_\delta(-2/\delta)\quad\text{and}\quad M_\delta:=\varrho_\delta(2/\delta). \tag{4.1}$$

Note that these values coincide with $\min\varrho_\delta$ and $\max\varrho_\delta$, respectively, and that the restriction of ϱ_δ to the interval $(-2/\delta,2/\delta)$ is invertible. So, if we set

$$\alpha_\delta:=\text{inverse of the restriction of }\varrho_\delta\text{ to }(-2/\delta,2/\delta) \tag{4.2}$$

the domain of α_δ is the open interval (m_δ,M_δ). Moreover, for $s\in\mathbb{R}$ we set

$$\hat\beta_\varepsilon(s):=\int_0^s\beta_\varepsilon(r)\,\mathrm{d}r\quad\text{and}\quad\overline{R}_\delta(s):=\int_0^s\Big(1-\frac{1}{\varrho_\delta(r)}\Big)\,\mathrm{d}r, \tag{4.3}$$

$$\varrho_{\varepsilon\delta}(s):=\varepsilon s+\varrho_\delta(s)\quad\text{and}\quad I_{\varepsilon\delta}(s):=\varrho_\delta(\varrho_{\varepsilon\delta}^{-1}(s)) \tag{4.4}$$

and observe that the function $I_{\varepsilon\delta}$ is well defined since $\varrho_{\varepsilon\delta}$ is invertible and its range is the whole of $\mathbb{R}$, thanks to the inequalities $\varrho_\delta'\geqslant0$ and $m_\delta\leqslant\varrho_\delta\leqslant M_\delta$.

For the reader's convenience, we list some properties that easily follow from our definitions (for the last inequality it suffices to compute the derivative of $I_{\varepsilon\delta}$; as far as $\hat\beta_\varepsilon$ is concerned, we refer, e.g., to [3, Prop. 2.11, p. 39]). We have

(4.5) ϱ' is nondecreasing in $(-\infty,+\infty)$ and ϱ'_δ is nondecreasing in $(-\infty,1/\delta)$,

(4.6) $$0 \leqslant \varrho'_\delta(s) \leqslant \varrho'(s) \ \ \forall\, s\in\mathbb{R} \ \text{ and } \ \varrho_\delta(s)=\varrho(s) \ \ \forall\, s\in[-1/\delta,1/\delta],$$

(4.7) $$0<\varrho_\delta(s)\leqslant \max\{1,\varrho(s)\} \ \ \forall\, s\in\mathbb{R},$$

(4.8) $$\alpha_\delta(r)\leqslant\alpha(r) \ \ \forall\, r\in(m_\delta,1) \ \text{ and } \ \alpha_\delta(r)\geqslant\alpha(r) \ \ \forall\, r\in(1,M_\delta),$$

(4.9) $$\hat\beta_\varepsilon \ \text{ is convex and } \ 0\leqslant\hat\beta_\varepsilon(s)\leqslant\hat\beta(s) \ \ \forall\, s\in D(\hat\beta),$$

(4.10) $$R_\delta \ \text{ and } \ \overline{R}_\delta \ \text{ are nonnegative,}$$

(4.11) $$I_{\varepsilon\delta} \ \text{ is nondecreasing in } \ (-\infty,+\infty),$$

(4.12) $$I_{\varepsilon\delta}(1)=1 \ \text{ and } \ |I_{\varepsilon\delta}(r)-I_{\varepsilon\delta}(r')|\leqslant|r-r'| \ \ \forall\, r,r'\in\mathbb{R}.$$

Now, we state the properties we need the approximating data to possess. As far as the source term is concerned, we construct f_ε by setting

(4.13) $$f_\varepsilon := g+h_\varepsilon$$

where h_ε is defined in the following way. First note that (2.23) ensures that the formula

$$\langle h_*(t),v\rangle=\int_\Gamma h(t)v \quad \forall\, v\in V$$

defines $h_*(t)\in V'$ for a.a. $t\in(0,T)$. Hence, the variational equation

(4.14) $$\varepsilon\int_\Omega \nabla h_\varepsilon(t)\cdot\nabla v+(1+\varepsilon)\int_\Omega h_\varepsilon(t)v=\int_\Gamma h(t)v \quad \forall\, v\in V$$

defines $h_\varepsilon(t)\in V$ and $\|h_\varepsilon(t)\|_V\leqslant\varepsilon^{-1}\|h_*(t)\|_*$ holds. Then the second assumption of (2.19) implies that $h_\varepsilon\in L^\infty(0,T;V)$, whence $f_\varepsilon\in L^2(0,T;H)$. Moreover,

(4.15) $$f_\varepsilon\to f \ \text{ strongly in } \ L^2(0,T;V') \quad \text{and} \quad \|f_\varepsilon\|_{L^2(0,T;V')}\leqslant c$$

for some c independent of ε, as we see in a moment.

Let us come to the Cauchy data for (3.9). When dealing with an existence result, we can simply choose

(4.16) $$\chi_{0\varepsilon}=\chi_0$$

but we have to relate the choice of $u_{0\delta}$ to the one of ε. Precisely, we assume that

(4.17) $$u_{0\delta}\in V, \quad \varrho_\delta(u_{0\delta})\to\vartheta_0 \ \text{ a.e. in } \Omega, \quad \text{and} \quad |u_{0\delta}|\leqslant C_0 \ \text{ a.e. in } \Omega,$$

(4.18) $$\varepsilon\int_\Omega \overline{R}_\delta(u_{0\delta})+\varepsilon\|u_{0\delta}\|_V^2\leqslant C_0$$

where C_0 does not depend on the pair (ε,δ). Clearly, (2.20) ensures that it is possible to choose $u_{0\delta}$ satisfying (4.17) just by regularizing $\alpha(\vartheta_0)$. Once $\{u_{0\delta}\}$ is constructed, one can choose ε in order that (4.18) hold as well.

R e m a r k 4.1. Clearly, ε has to become smaller and smaller, depending on δ, if ϑ_0 satisfies just assumption (2.20). However, if ϑ_0 is smoother, ε can be treated as an independent parameter. Indeed, if we assume in addition that $\vartheta_0 \in V$, we can simply take $u_{0\delta} = \alpha(\vartheta_0)$ and (4.17)–(4.18) hold for some constant C_0 and for any $\varepsilon \in (0,1)$. In this case, one can let δ tend to zero keeping ε fixed and the limit procedure we have used in order to prove Theorem 2.1 works with minor changes showing the existence of a quintuple $(\vartheta, \chi, u, w, \xi)$ satisfying conditions (2.25)–(2.28) (at least) and solving the system

$$\partial_t(\varepsilon u(t) + \vartheta(t) + \lambda\chi(t)) + Au(t) = f_\varepsilon(t) \quad \text{for a.a. } t \in (0,T), \tag{4.19}$$

$$u = \alpha(\vartheta) \quad \text{a.e. in } Q, \tag{4.20}$$

$$\partial_t \chi(t) + Bw(t) = 0 \quad \text{for a.a. } t \in (0,T), \tag{4.21}$$

$$w(t) = \varepsilon\partial_t\chi(t) + B\chi(t) + \xi(t) + \sigma'(\chi(t)) + \lambda/\vartheta(t) \quad \text{for a.a. } t \in (0,T), \tag{4.22}$$

$$\xi = \beta_\varepsilon(\chi) \quad \text{a.e. in } Q, \tag{4.23}$$

$$\vartheta(0) = \vartheta_0 \quad \text{and} \quad \chi(0) = \chi_{0\varepsilon}. \tag{4.24}$$

Then, it will be clear that the same procedure can be used to prove that such a solution tends to a solution to (2.29)–(2.34) as $\varepsilon \searrow 0$, at least for a subsequence.

Lemma 4.2. *Conditions* (4.15) *hold.*

P r o o f. Clearly, it suffices to prove that

$$h_\varepsilon \to h_* \quad \text{strongly in } L^2(0,T;V') \tag{4.25}$$

and the argument that proves such a convergence is quite standard. Let $(\cdot,\cdot)_1$ and $\|\cdot\|_1$ be the usual scalar product and the norm in V and consider the associated Riesz isomorphism $J\colon V \to V'$, the dual scalar product $(\cdot,\cdot)_{1*}$ in V', and the dual norm $\|\cdot\|_{1*}$ in V'. Then

$$\langle Ju, v\rangle = (u,v)_1 = (Ju, Jv)_{1*} \quad \text{and} \quad \big\langle Ju, J^{-1}v\big\rangle = \langle u, v\rangle \quad \forall\, u, v \in V$$

and (4.14) reads (for a.a. $t \in (0,T)$)

$$\varepsilon(h_\varepsilon(t), v)_1 + \langle h_\varepsilon(t), v\rangle = \int_\Gamma h(t)v := \langle h_*(t), v\rangle \quad \forall\, v \in V.$$

Taking $v = J^{-1}h_\varepsilon(t)$, we obtain

$$\varepsilon\|h_\varepsilon(t)\|_H^2 + \|h_\varepsilon(t)\|_{1*}^2 = \big\langle h_*(t), J^{-1}h_\varepsilon(t)\big\rangle \leqslant \|h_*(t)\|_{1*}\|h_\varepsilon(t)\|_{1*}.$$

We deduce

$$\|h_\varepsilon(t)\|_{1*} \leqslant \|h_*(t)\|_{1*} \quad \text{and} \quad \varepsilon^{1/2}\|h_\varepsilon(t)\|_H \leqslant \|h_*(t)\|_{1*}$$

whence immediately

$$h_\varepsilon(t) \to h_*(t) \text{ weakly in } V' \text{ as } \varepsilon \searrow 0 \quad \text{and} \quad \limsup_{\varepsilon\searrow 0} \|h_\varepsilon(t)\|_{1*} \leqslant \|h_*(t)\|_{1*}$$

for a.a. $t \in (0,T)$. Therefore, the convergence is strong and (4.25) follows by the dominated convergence theorem (see (2.23) and (2.19)). □

Lemma 4.3. *There exists $C_2 > 0$ such that*

$$\alpha_\delta'(r) \geqslant \frac{C_2}{r^2} \quad \textit{and} \quad \alpha_\delta'(r) \geqslant \frac{C_2}{r^{2q}} \;\; \forall\, r \in (m_\delta, M_\delta), \tag{4.26}$$

for δ small enough.

Proof. We distinguish three cases. Assume $r < 1$. Then the first inequality of (4.8) holds. On the other hand, both $\alpha_\delta(r)$ and $\alpha(r)$ are negative. Hence,

$$\varrho_\delta'(\alpha_\delta(r)) \leqslant \varrho'(\alpha_\delta(r)) \leqslant \varrho'(\alpha(r))$$

by (4.5)–(4.6). We deduce $\alpha_\delta'(r) \geqslant \alpha'(r)$, and (4.26) follows from (2.5). Assume now $r \geqslant 1$ and $\alpha(r) < 1/\delta$. Then $\alpha_\delta'(r) = \alpha'(r)$ and (4.26) follows by (2.5) as well. Finally, assume $\alpha(r) \geqslant 1/\delta$. Then we have also $s_\delta := \alpha_\delta(r) \in [1/\delta, 2/\delta)$. On the other hand, we have

$$\varrho_\delta'(s) \leqslant \varrho'(s) \leqslant \varrho'(2/\delta) \quad \text{and} \quad \varrho_\delta(s) \geqslant \varrho_\delta(1/\delta) = \varrho(1/\delta)$$

for any $s \in [1/\delta, 2/\delta)$. We deduce

$$r^2\alpha_\delta'(r) = \frac{\varrho_\delta^2(s_\delta)}{\varrho_\delta'(s_\delta)} \geqslant \frac{\varrho^2(1/\delta)}{\varrho'(2/\delta)} \quad \text{and} \quad r^{2q}\alpha_\delta'(r) \geqslant \frac{\varrho^{2q}(1/\delta)}{\varrho'(2/\delta)}$$

and the result follows once we prove that

$$\liminf_{s\to+\infty} \frac{\varrho^2(s)}{\varrho'(2s)} > 0 \quad \text{and} \quad \liminf_{s\to+\infty} \frac{\varrho^{2q}(s)}{\varrho'(2s)} > 0.$$

Using (2.2) and (2.6), we easily see that the limit

$$\lim_{s\to+\infty} \frac{\varrho'(s)}{s^\mu} = \lim_{r\to+\infty} \frac{1}{(\alpha(r))^\mu\alpha'(r)}$$

is finite and strictly positive provided that $\mu = 2q/(1-2q)$. With such a choice of μ, it follows that the limit

$$\lim_{s\to+\infty} \frac{\varrho(s)}{s^{\mu+1}} = \lim_{s\to+\infty} \frac{\varrho'(s)}{(\mu+1)s^{\mu}}$$

is finite and strictly positive, too. On the other hand, we have

$$\frac{\varrho^2(s)}{\varrho'(2s)} = \frac{\varrho^2(s)}{s^{2\mu+2}} \frac{(2s)^{\mu}}{\varrho'(2s)} 2^{-\mu} s^{\mu+2} \quad \text{and} \quad \frac{\varrho^{2q}(s)}{\varrho'(2s)} = \frac{\varrho^{2q}(s)}{s^{2q\mu+2q}} \frac{(2s)^{\mu}}{\varrho'(2s)} 2^{-\mu}.$$

Combining these equalities and noting that $\mu + 2 > 0$, we complete the proof. □

Lemma 4.4. *The limit*

$$C_3 := \lim_{\delta\searrow 0} \frac{m_\delta}{\delta} \tag{4.27}$$

exists and is finite and positive.

Proof. Writing $1 = \varrho(0)$ as the integral of ϱ', we easily obtain

$$\frac{m_\delta}{\delta} = \frac{\varrho(-2/\delta)}{\delta} + \frac{1}{\delta}\int_{-2/\delta}^{0} \varrho'(s)(1-\zeta(\delta|s|))\,\mathrm{d}s$$

and we are led to compute two limits. Thanks to (2.6), the first limit is given by

$$\lim_{\delta\searrow 0} \frac{\varrho(-2/\delta)}{\delta} = -\frac{1}{2}\lim_{r\searrow 0} r\alpha(r) = \frac{c_0}{2}.$$

To deal with the second limit, we change the integration variable and conclude that

$$\begin{aligned}\lim_{\delta\searrow 0} \frac{1}{\delta}\int_{-2/\delta}^{0} \varrho'(s)(1-\zeta(\delta|s|))\,\mathrm{d}s &= \lim_{\delta\searrow 0}\int_0^2 \frac{r^2}{\delta^2}\varrho'(-r/\delta)\frac{1-\zeta(r)}{r^2}\,\mathrm{d}r \\ &= c_0\int_0^2 \frac{1-\zeta(r)}{r^2}\,\mathrm{d}r\end{aligned}$$

thanks to the dominated convergence theorem. □

Lemma 4.5. *There exist $C_4, C_5 > 0$ such that*

$$s\Big(1 - \frac{1}{\varrho_\delta(s)}\Big) \geqslant \frac{C_4}{\varrho_\delta^2(s)} - C_5 \quad \forall\, s \in \mathbb{R} \tag{4.28}$$

for δ small enough.

Proof. First, we prepare some inequalities. Setting

$$C := \inf_{0<r\leqslant 1/2} r\alpha(r)(r-1)$$

and noting that C is positive thanks to (2.1)–(2.2), we have

$$\alpha(r)\Big(1 - \frac{1}{r}\Big) \geqslant \frac{C}{r^2} \quad \text{for } 0 < r \leqslant 1/2.$$

On the other hand, a simple application of Lemma 4.4 yields

$$\lim_{\delta \searrow 0} \frac{2}{\delta}\Big(1 - \frac{1}{m_\delta}\Big) m_\delta^2 = -2C_3.$$

Therefore, for δ small enough, we have

$$-\frac{2}{\delta}\Big(1 - \frac{1}{m_\delta}\Big) m_\delta^2 \geqslant C_3$$

as well as $m_\delta < 1$. At this point, we consider the following three cases: $\varrho_\delta(s) < 1/2$, $1/2 \leqslant \varrho_\delta(s) \leqslant 1$, $\varrho_\delta(s) > 1$. In the first case, we have two possibilities: $s \leqslant -2/\delta$ and $s > -2/\delta$. If $s \leqslant -2/\delta$, then $\varrho_\delta(s) = m_\delta$, whence

$$s\Big(1 - \frac{1}{\varrho_\delta(s)}\Big) \geqslant -\frac{2}{\delta}\Big(1 - \frac{1}{m_\delta}\Big).$$

We deduce that

$$s\Big(1 - \frac{1}{\varrho_\delta(s)}\Big) \geqslant \frac{C_3}{m_\delta^2} = \frac{C_3}{\varrho_\delta^2(s)}.$$

Consider now the second possibility $s > -2/\delta$ and set $r := \varrho_\delta(s)$. Then $s = \alpha_\delta(r)$ and $m_\delta < r \leqslant 1/2$, whence $\alpha_\delta(r) \leqslant \alpha(r)$ by (4.8). Thus, we have

$$s\Big(1 - \frac{1}{\varrho_\delta(s)}\Big) = \alpha_\delta(r)\Big(1 - \frac{1}{r}\Big) \geqslant \alpha(r)\Big(1 - \frac{1}{r}\Big) \geqslant \frac{C}{r^2} = \frac{C}{\varrho_\delta^2(s)}.$$

Next, suppose $1/2 \leqslant \varrho_\delta(s) \leqslant 1$. We can assume that $\delta < -1/\alpha(1/2)$. This means $-1/\delta < \alpha(1/2)$. It follows that $\alpha_\delta(1/2) = \alpha(1/2)$ and $\alpha(1/2) \leqslant s \leqslant 0$. Hence

$$s\Big(1 - \frac{1}{\varrho_\delta(s)}\Big) \geqslant s \geqslant \alpha(1/2) \geqslant \frac{1}{\varrho^2(s)} + \alpha(1/2) - \frac{1}{4}.$$

In the last case we have $\varrho_\delta(s) > 1$, whence $s > 0$. Then

$$s\Big(1 - \frac{1}{\varrho_\delta(s)}\Big) \geqslant 0 \geqslant \frac{1}{\varrho^2(s)} - 1.$$

Therefore, inequality (4.28) holds in any case provided that C_4 is small enough and C_5 is large enough. $\square$

5. Existence

In this section we prove Theorem 2.1. Our proof starts from the approximating problem. If $(u_{\varepsilon\delta}, \chi_{\varepsilon\delta}, w_{\varepsilon\delta})$ is any solution to (3.6)–(3.9) satisfying the regularity conditions (3.5), we set

$$\vartheta_{\varepsilon\delta} := \varrho_\delta(u_{\varepsilon\delta}) \quad \text{and} \quad \xi_{\varepsilon\delta} := \beta_\varepsilon(\chi_{\varepsilon\delta}). \tag{5.1}$$

Our aim is to derive a number of a priori estimates and show that the quintuple $(\vartheta_{\varepsilon\delta}, \chi_{\varepsilon\delta}, u_{\varepsilon\delta}, w_{\varepsilon\delta}, \xi_{\varepsilon\delta})$ converges to a solution $(\vartheta, \chi, u, w, \xi)$ to problem (2.29)–(2.34) in a suitable topology as ε and δ tend to zero, at least for a subsequence. It is useful to introduce the auxiliary function

$$\Theta_{\varepsilon\delta} := \varepsilon u_{\varepsilon\delta} + \vartheta_{\varepsilon\delta}. \tag{5.2}$$

However, in performing the calculation, we avoid the subscripts ε and δ as far as the approximate solution and the functions (5.1)–(5.2) are concerned, and we restore the full notation just once the a priori estimates are obtained. In the following, we use (4.16).

First a priori estimate. We recall (3.2) for the definition of $z_{\varepsilon\delta}$. After writing equations (3.6)–(3.8) at $t = s$, we test them by $1 - 1/\vartheta(s) + \eta_1 u(s)$, $\mathcal{N}\partial_t\chi(s)$ and $-\partial_t\chi(s)$, respectively, where η_1 is a positive parameter (see Remark 3.1). Then, we sum the resulting equalities and integrate over $(0, t)$. Noting that some terms cancel (see also (2.14)–(2.15)), we obtain

$$\begin{aligned} (5.3)\qquad & \varepsilon \int_\Omega \overline{R}_\delta(u(t)) + \frac{\varepsilon\eta_1}{2} \int_\Omega |u(t)|^2 + \int_\Omega (\vartheta(t) - 1 - \ln\vartheta(t)) + \eta_1 \int_\Omega R_\delta(u(t)) \\ & + \int_{Q_t} \nabla u \cdot \nabla(-1/\vartheta) + \int_{\Sigma_t} u_{\varepsilon\delta}(1 - 1/\vartheta) + \eta_1 \int_0^t \|u(s)\|_V^2 \,\mathrm{d}s \\ & + \int_0^t \|\partial_t\chi(s)\|_*^2 \,\mathrm{d}s + \varepsilon \int_{Q_t} |\partial_t\chi|^2 + \frac{1}{2} \int_\Omega |\nabla\chi(t)|^2 + \int_\Omega \hat{\beta}_\varepsilon(\chi(t)) \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon \int_\Omega \overline{R}_\delta(u) + \frac{\varepsilon\eta_1}{2}\int_\Omega |u_{0\delta}|^2 + \int_\Omega (\vartheta(0) - 1 - \ln\vartheta(0)) + \eta_1 \int_\Omega R_\delta(u_{0\delta}) \\
&\quad + \frac{1}{2}\int_\Omega |\nabla\chi_0|^2 + \int_\Omega \hat{\beta}_\varepsilon(\chi_0) + \int_{Q_t} \partial_t\chi\sigma'(\chi) \\
&\quad + \int_0^t \langle f_\varepsilon(s), 1 - 1/\vartheta(s) + \eta_1 u(s)\rangle \,\mathrm{d}s - \lambda \int_{Q_t} \partial_t\chi(1+\eta_1 u).
\end{aligned}
$$

Now, we note that the first two integrals on the left-hand side are nonnegative, and treat the terms that need some manipulation separately. In the sequel, η is a positive parameter. An elementary inequality yields

$$
\int_\Omega (\vartheta(t) - 1 - \ln\vartheta(t)) \geqslant \frac{1}{2}\|\vartheta(t)\|_{L^1(\Omega)} + \frac{1}{2}\|\ln\vartheta(t)\|_{L^1(\Omega)} - c.
$$

We just observe that the term involving R_δ is nonnegative and deal with the next integral. For the sake of convenience, we denote by Q_t^* the subset of Q_t where $|u| < 2/\delta$ and note that $u = \alpha_\delta(\vartheta)$ in Q_t^* and $\nabla\vartheta = 0$ a.e. in its complement. Accounting for the first inequality of (4.26), we have

$$
\begin{aligned}
\int_{Q_t} \nabla u \nabla(-1/\vartheta) &= \int_{Q_t^*} \nabla\alpha_\delta(\vartheta)\cdot\nabla(-1/\vartheta) = \int_{Q_t^*} \frac{\alpha_\delta'(\vartheta)}{\vartheta^2}\,|\nabla\vartheta|^2 \\
&\geqslant C_2 \int_{Q_t^*} \frac{|\nabla\vartheta|^2}{\vartheta^4} = C_2 \int_{Q_t} |\nabla(1/\vartheta)|^2.
\end{aligned}
$$

As far as the next term is concerned, Lemma 4.5 immediately yields

$$
\int_{\Sigma_t} u(1 - 1/\vartheta) \geqslant C_4 \int_{\Sigma_t} 1/\varrho_\delta^2(s) - c.
$$

As the remaining terms on the left-hand side are nonnegative, we deal with the right-hand side. The first two terms are bounded by a constant independent of δ thanks to (4.18), and the next one is estimated using (4.17) in an obvious way. The next term is easily controlled with help of (4.17), and the integral involving R_δ is treated as follows. Observing that

$$
\hat{\alpha}(\varrho(s)) = \int_1^{\varrho(s)} \alpha(r')\,\mathrm{d}r' = \int_0^s \alpha(\varrho(r))\varrho'(r)\,\mathrm{d}r = \int_0^s r\varrho'(r)\,\mathrm{d}r \quad \forall\, s\in\mathbb{R}
$$

and recalling (4.6) we deduce that $R_\delta(s) \leqslant \hat{\alpha}(\varrho(s))$ for any $s\in\mathbb{R}$. Hence

$$
\int_\Omega R_\delta(u) \leqslant \int_\Omega \hat{\alpha}(\varrho(u)) \leqslant c
$$

by (4.17). The next integral does not depend on δ and the one involving $\hat\beta$ is bounded thanks to (2.21), since

$$\int_\Omega \hat\beta_\varepsilon(\chi_0) \leqslant \int_\Omega \hat\beta(\chi_0).$$

We go on. Using (2.3), we have

$$\int_{Q_t} \partial_t\chi\sigma'(\chi) \leqslant \eta \int_0^t \|\partial\chi(s)\|_*^2 \,\mathrm{d}s + c_\eta \int_{Q_t} (1+|\chi|^2+|\nabla\chi|^2)\,\mathrm{d}s.$$

On the other hand, we can account for the second inequality of (2.18) and argue as we did to get (3.21), now using the V'-norm instead of the H-norm. We obtain

$$\begin{aligned}\int_{Q_t} |\chi|^2 &\leqslant \int_{Q_t} |\nabla\chi|^2 + c\int_0^t \|\chi(s)\|_*^2\,\mathrm{d}s \\ &\leqslant \int_{Q_t} |\nabla\chi|^2 + c\|\chi_0\|_*^2 + c\int_0^t \left(\int_0^s \|\partial_t\chi(\tau)\|_*^2\,\mathrm{d}\tau\right)\mathrm{d}s.\end{aligned}$$

The last but one integral in (5.3) is estimated by

$$\begin{aligned}&\int_0^t \langle f_\varepsilon(s), 1-1/\vartheta(s)+\eta_1 u(s)\rangle\,\mathrm{d}s \\ &\qquad\leqslant c\int_0^t \|f_\varepsilon(s)\|_*\,\mathrm{d}s + \eta\int_0^t \|1/\vartheta(s)\|_V^2\,\mathrm{d}s \\ &\qquad\quad + \frac{\eta_1}{4}\int_0^t \|u(s)\|_V^2\,\mathrm{d}s + c_{\eta,\eta_1}\int_0^t \|f_\varepsilon(s)\|_*^2\,\mathrm{d}s\end{aligned}$$

where we have used the first inequality of (2.18). Moreover, we observe that the above integrals containing f_ε are bounded by (4.15). Finally, we deal with the last term to be estimated using (2.18) again. We have

$$\begin{aligned}-\lambda\int_{Q_t} \partial_t\chi(1+\eta_1 u) &\leqslant \eta\int_0^t \|\partial_t\chi(s)\|_*^2\,\mathrm{d}s + \frac{\eta_1}{4}\int_0^t \|u(s)\|_V^2\,\mathrm{d}s \\ &\quad + c\eta_1\int_0^t \|\partial_t\chi(s)\|_*^2\,\mathrm{d}s + c_\eta.\end{aligned}$$

At this point, we use all the inequalities we have obtained in (5.3) and choose first η and then η_1 small enough. Hence we can apply the Gronwall lemma and deduce the basic estimate

$$\begin{aligned}(5.4)\qquad \|\vartheta_{\varepsilon\delta}\|_{L^\infty(0,T;L^1(\Omega))} &+ \|\ln\vartheta_{\varepsilon\delta}\|_{L^\infty(0,T;L^1(\Omega))} \\ &+ \|R_\delta(u_{\varepsilon\delta})\|_{L^\infty(0,T;L^1(\Omega))} + \|\hat\beta_\varepsilon(\chi_{\varepsilon\delta})\|_{L^\infty(0,T;L^1(\Omega))} \\ &+ \|1/\vartheta_{\varepsilon\delta}\|_{L^2(0,T;V)} + \|u_{\varepsilon\delta}\|_{L^2(0,T;V)} \\ &+ \|\partial_t\chi_{\varepsilon\delta}\|_{L^2(0,T;V')} + \varepsilon^{1/2}\|\partial_t\chi_{\varepsilon\delta}\|_{L^2(0,T;H)} \\ &+ \|\chi_{\varepsilon\delta}\|_{L^\infty(0,T;V)} \leqslant c.\end{aligned}$$

Recovering $\partial_t\Theta_{\varepsilon\delta} = \partial_t(\varepsilon u_{\varepsilon\delta} + \vartheta_{\varepsilon\delta})$ from (3.6), we conclude immediately that

$$\|\partial_t\Theta_{\varepsilon\delta}\|_{L^2(0,T;V')} \leqslant c. \tag{5.5}$$

Second a priori estimate. We write (3.6) at the time $t = s$. Then we test it by $\Theta(s)$ and integrate over $(0,t)$. We obtain

$$\begin{aligned}
&\frac{1}{2}\int_\Omega |\Theta(t)|^2 + \varepsilon\int_{Q_t}|\nabla u|^2 + \int_{Q_t}\nabla u\cdot\nabla\vartheta + \varepsilon\gamma\int_{\Sigma_t}u^2 + \varepsilon\gamma\int_{\Sigma_t}u(\vartheta-1)\\
&= \frac{1}{2}\int_\Omega|\Theta(0)|^2 - \int_{Q_t}\partial_t\chi\Theta - \varepsilon\gamma\int_{\Sigma_t}u + \varepsilon\int_{Q_t}f_\varepsilon u + \int_{Q_t}g\vartheta + \int_{Q_t}h_\varepsilon\vartheta.
\end{aligned}$$

Only a few terms need some manipulation. Denoting by Q_t^* the subset of Q_t where $|u| < 2/\delta$ as before and taking into account the second inequality of (4.26), we have

$$\begin{aligned}
\int_{Q_t}\nabla u\cdot\nabla\vartheta &= \int_{Q_t^*}\nabla\alpha_\delta(\vartheta)\cdot\nabla\vartheta = \int_{Q_t^*}\alpha_\delta'(\vartheta)|\nabla\vartheta|^2\\
&\geqslant C_2\int_{Q_t^*}\vartheta^{-2q}|\nabla\vartheta|^2 = \frac{C_2}{(1-q)^2}\int_{Q_t}|\nabla\vartheta^{1-q}|^2.
\end{aligned}$$

As the last term is nonnegative, we deal with the right-hand side. The first term is bounded by (4.17) and (4.18). To treat the next integral, we integrate by parts and estimate it as

$$\begin{aligned}
-\int_{Q_t}\partial_t\chi\Theta &= -\int_\Omega\chi(t)\Theta(t) + \int_\Omega\chi_0\Theta(0) + \int_{Q_t}\chi\partial_t\Theta\\
&\leqslant \|\chi(t)\|_H\,\|\Theta(t)\|_H + \|\chi_0\|_H\,\|\varepsilon u_{0\delta} + \varrho_\delta(u_{0\delta})\|_H\\
&\quad + \|\chi\|_{L^2(0,T;V)}\|\partial_t\Theta\|_{L^2(0,T;V')}\\
&\leqslant \eta\int_\Omega|\Theta(t)|^2 + c_\eta
\end{aligned}$$

due to (4.17), (4.18), (5.4) and (5.5). The next two integrals are controlled thanks to (5.4) and (4.15). The last but one integral is estimated as

$$\int_{Q_t}g\vartheta \leqslant \int_{Q_t}|\vartheta|^2 + c \leqslant \int_{Q_t}|\Theta|^2 + c$$

since $\vartheta = I_{\varepsilon\delta}(\Theta)$ (see (4.4)) and (4.12) holds. Finally, we deal with the last term. As we would like to speak of $\alpha_\delta(\vartheta)$, we introduce

$$\vartheta_n = \max\{m_\delta + 1/n, \min\{M_\delta - 1/n, \vartheta\}\} \quad \text{a.e. in } Q$$

for n large enough. Owing to (2.19), we have

$$\int_\Gamma h_\varepsilon(t)\vartheta_n(t) \leqslant c\|h_\varepsilon(t)\|_{L^{q_\bullet}(\Gamma)}\|\vartheta_n(t)\|_{L^{q'_\bullet}(\Gamma)} \leqslant c\|\vartheta_n(t)\|_{L^{q'_\bullet}(\Gamma)} \quad \text{for a.a. } t \in (0,T).$$

On the other hand,

$$\|v\|_{L^{q'_\bullet}(\Gamma)} \leqslant \|\nabla\alpha_\delta(v)\|_H^2 + c(1+\|v\|_H^2)$$

holds for any $v \in H$ satisfying $m_\delta < v < M_\delta$ a.e. in Ω and $\nabla\alpha_\delta(v) \in H$. Indeed, a similar result is proved in [7, Lemma 3.1] with α in place of α_δ and just under the natural condition $v > 0$ on the range of v. However, the only property of α that enters the proof is the second inequality of (2.5). So, as we have proved (4.26), the same proof holds for α_δ. Therefore, we can combine the last two inequalities and integrate with respect to time. We deduce that

$$\int_{\Sigma_t} h_\varepsilon\vartheta_n \leqslant \int_{Q_t} |\nabla\alpha_\delta(\vartheta_n)|^2 + c\int_{Q_t}(1+\vartheta_n^2).$$

Letting n tend to ∞ and using (2.19) and (5.4) again, we obtain

$$\int_{\Sigma_t} h_\varepsilon\vartheta \leqslant \int_{Q_t} |\nabla u|^2 + c\int_{Q_t}(1+\vartheta^2) \leqslant \int_{Q_t}\vartheta^2 + c.$$

At this point, we combine all the inequalities we have established. After choosing η small enough, we can first apply the Gronwall lemma and then use (4.12) again. We conclude that

$$\|\Theta_{\varepsilon\delta}\|_{L^\infty(0,T;H)} + \|\vartheta_{\varepsilon\delta}\|_{L^\infty(0,T;H)} + \|\nabla\vartheta_{\varepsilon\delta}^{1-q}\| \leqslant c. \tag{5.6}$$

The above estimate and the argument used in [8, Lemma 4.8] imply that

$$\|\vartheta_{\varepsilon\delta}\|_{L^2(0,T;W^{1,q_*}(\Omega))} \leqslant c \tag{5.7}$$

where q_* is defined in (2.25). Hence, noting that $q_* \leqslant 2$, and owing to (5.2) and to (5.4), we deduce also

$$\|\Theta_{\varepsilon\delta}\|_{L^2(0,T;W^{1,q_*}(\Omega))} \leqslant c. \tag{5.8}$$

Third a priori estimate. We want to estimate the norm of ξ (see (5.1)) in $L^2(Q)$, and this can be done as in [6], with minor changes. For the reader's convenience, we perform at least the first calculation. We introduce the mean value

$$\bar{\xi}_{\varepsilon\delta}(t) := \frac{1}{|\Omega|}\langle\xi_{\varepsilon\delta}(t),1\rangle \quad \text{for a.a. } t \in (0,T) \tag{5.9}$$

and avoid the subscript for a while. Arguing for a.a. $t \in (0,T)$, we test (3.7) and (3.8) by $\mathcal{N}(\xi(t) - \bar{\xi}(t))$ and by $\bar{\xi}(t) - \xi(t)$, respectively. Now we sum the resulting equalities and note that two terms cancel out by (2.15). Recalling (3.2), we obtain

$$\begin{aligned}
\int_\Omega \partial_t \chi(t) \mathcal{N}(\xi(t) - \bar{\xi}(t)) &+ \varepsilon \int_\Omega \partial_t \chi(t)(\xi(t) - \bar{\xi}(t)) \\
&+ \int_\Omega \nabla \chi(t) \cdot \nabla(\xi(t) - \bar{\xi}(t)) + \int_\Omega \xi(t)(\xi(t) - \bar{\xi}(t)) \\
&+ \int_\Omega \sigma'(\chi(t))(\xi(t) - \bar{\xi}(t)) + \lambda \int_\Omega \frac{1}{\vartheta(t)} (\xi(t) - \bar{\xi}(t)) \\
&= 0 = \int_\Omega \bar{\xi}(t)(\xi(t) - \bar{\xi}(t)).
\end{aligned}$$

Using (2.16), noting that $\nabla \bar{\xi}(t) = 0$ and rearranging, we have

$$\int_\Omega \nabla \chi(t) \cdot \nabla \xi(t) + \int_\Omega |\xi(t) - \bar{\xi}(t)|^2 = \int_\Omega F(t)(\xi(t) - \bar{\xi}(t))$$

where we have set for the sake of convenience

$$F(t) := -\mathcal{N} \partial_t \chi(t) - \varepsilon \partial_t \chi(t) - \sigma'(\chi(t)) - \frac{\lambda}{\vartheta(t)}.$$

As the first integral is nonnegative, we deduce that

$$\int_\Omega |\xi(t) - \bar{\xi}(t)|^2 \leqslant \int_\Omega |F(t)|^2.$$

As (5.4) implies that F is bounded in $L^2(0,T;H)$, we conclude that

$$\int_Q |\xi_{\varepsilon\delta} - \bar{\xi}_{\varepsilon\delta}|^2 \leqslant c.$$

Hence, we are led to estimate the norm of the mean value in $L^2(0,T)$. This can be done using the ideas of [17]. The reader could refer to [6, p. 283] for a detailed application which holds in the present case without any change. Exactly in this step we need assumption (2.22). The conclusion is the estimate

$$\|\xi_{\varepsilon\delta}\|_{L^2(Q)} \leqslant c. \tag{5.10}$$

Fourth a priori estimate. We test both (3.7) and (3.8) with $w(t)$ and sum the resulting equalities. Then we integrate over $(0,T)$. We obtain

$$\int_Q (|\nabla w|^2 + w^2) = (\varepsilon - 1) \int_Q \partial_t \chi w + \int_Q \nabla \chi \cdot \nabla w + \int_Q z_{\varepsilon\delta}(\chi, u) w.$$

Using (5.4) and (5.10), and taking into account the definition of $z_{\varepsilon\delta}$ (see (3.2)) and (2.3), we get immediately

$$\|w_{\varepsilon\delta}\|_{L^2(0,T;V)} \leqslant c. \tag{5.11}$$

Conclusion. Recalling (5.4)–(5.5), (5.7)–(5.8) and (5.10)–(5.11) and using the well-known weak and weak star compactness results, we find a quintuple $(\vartheta, \chi, u, w, \xi)$ and a function ψ such that the following convergences hold:

$$\vartheta_{\varepsilon\delta} \to \vartheta \quad \text{weakly star in } L(0,T;H) \text{ and weakly in } L^2(0,T;W^{1,q_*}(\Omega)), \tag{5.12}$$

$$u_{\varepsilon\delta} \to u \quad \text{weakly in } L^2(0,T;V), \tag{5.13}$$

$$\chi_{\varepsilon\delta} \to \chi \quad \text{weakly in } L^2(0,T;W) \text{ and in } H^1(0,T;V') \text{ and weakly star in } L^\infty(0,T;V), \tag{5.14}$$

$$\Theta_{\varepsilon\delta} \to \vartheta \quad \text{weakly star in } L^\infty(0,T;H) \text{ and weakly in } L^2(0,T;W^{1,q_*}(\Omega)), \tag{5.15}$$

$$\partial_t\Theta_{\varepsilon\delta} \to \partial_t\vartheta \quad \text{weakly in } L^2(0,T;V'), \tag{5.16}$$

$$\xi_{\varepsilon\delta} \to \xi \quad \text{weakly in } L^2(Q), \tag{5.17}$$

$$w_{\varepsilon\delta} \to w \quad \text{weakly in } L^2(0,T;V), \tag{5.18}$$

$$1/\vartheta_{\varepsilon\delta} \to \psi \quad \text{weakly in } L^2(0,T;V) \tag{5.19}$$

at least for a subsequence, where we have already noticed that the limits in (5.12) and in (5.15) have to coincide due to (5.13) and the definition of $\Theta_{\varepsilon\delta}$ (see (5.2)). Our aim is to show that $(\vartheta, \chi, u, w, \xi)$ solves problem (2.29)–(2.34) and we start by identifying the limits of the nonlinear terms.

We recall that $q_* = 2/(q+1)$ and observe that $W^{1,q_*}(\Omega)$ is compactly embedded in $L^2(\Omega)$ since Ω is a 3D domain. Hence, owing to (5.15)–(5.16), (5.14) and to the Aubin lemma (see, e.g., [22, p. 58]), we deduce

$$\Theta_{\varepsilon\delta} \to \vartheta \text{ and } \chi_{\varepsilon\delta} \to \chi \quad \text{strongly in } L^2(0,T;H) \text{ and a.e. in } Q \tag{5.20}$$

at least after a further selection of a subsequence. Hence, (5.2), (5.13) and (5.20) yield

$$\vartheta_{\varepsilon\delta} \to \vartheta \quad \text{strongly in } L^2(0,T;H) \text{ and a.e. in } Q. \tag{5.21}$$

Therefore, the limits of the nonlinear terms can be identified by standard arguments. First, we note that the second convergence of (5.20) and (2.3) imply that $\sigma'(\chi_{\varepsilon\delta})$

converges to $\sigma'(\chi)$ strongly in $L^2(Q)$. Next, we see that (5.21) yields

$$\int_Q (\vartheta - \varrho(v))(u - v) = \lim_{\varepsilon,\delta\searrow 0} \int_Q (\vartheta_{\varepsilon\delta} - \varrho_\delta(v))(u_{\varepsilon\delta} - v) \geqslant 0$$

for any $v \in L^2(Q)$ satisfying $\varrho(v) \in L^2(Q)$, since $\varrho_\delta(v) \to \varrho(v)$ strongly in $L^2(Q)$ by (4.7) and the dominated convergence theorem. As ϱ induces a maximal monotone operator in $L^2(Q)$, we conclude that $\vartheta = \varrho(u)$ a.e. in Q. This implies that $\vartheta > 0$ a.e. in Q and that (2.30) holds. In particular, it follows that $1/\vartheta_{\varepsilon\delta} \to 1/\vartheta$ a.e. in Q. On the other hand, we have proved (5.19). Thus $\psi = 1/\vartheta$ a.e. in Q. Finally, starting from the second convergence of (5.20) and from (5.17) and arguing similarly, we prove (2.33).

At this point, it is straightforward to check that (2.29) and (2.31)–(2.32) hold. On the other hand, the initial conditions (2.34) hold as well, since (5.15)–(5.16) and (5.14) imply

$$\Theta_{\varepsilon\delta}(0) \to \vartheta(0) \quad \text{and} \quad \chi_{\varepsilon\delta}(0) \to \chi(0) \quad \text{weakly in } V'$$

and (4.16) and (4.17) imply that $\chi_{\varepsilon\delta}(0) = \chi_0$ and that $\Theta_{\varepsilon\delta}(0)$ converges to ϑ_0 in H. To complete the proof, we observe that the last condition (2.25) follows by comparison with (2.29) from the regularity already established.

6. First uniqueness result

In this section, we prove Theorem 2.2. We follow the argument of [7, Theorem 2.4] and assume that the constants c_0 and c_∞ in (2.2) are both equal to 1, without any loss of generality. Let $(\vartheta_i, \chi_i, u_i, w_i, \xi_i)$ be two solutions to (2.29)–(2.34), $i = 1, 2$. We subtract the two equalities (2.29), integrate from 0 to $s \in (0, T)$ and test the equation with $u(s) := u_1(s) - u_2(s) \in V$. Then, we integrate again, now from 0 to $t \in (0, T)$, and obtain

$$\int_{Q_t} \vartheta u + \lambda \int_{Q_t} \chi u + \frac{1}{2} \int_\Omega \left| \nabla \int_0^t u \right|^2 = 0, \tag{6.1}$$

where $\vartheta := \vartheta_1 - \vartheta_2$ and $\chi := \chi_1 - \chi_2$. Now, observe that $\chi(s) \in V_0'$ (cf. (2.12)) since χ_1 and χ_2 have the same initial datum χ_0 and their mean value is preserved. Hence, we take the difference of the two equations (2.31) at time s and use $\mathcal{N}\chi(s)$ as a test function. Owing to (2.15), we get

$$\langle \partial_t \chi(s), \mathcal{N}\chi(s) \rangle + \langle w(s), \chi(s) \rangle = 0, \tag{6.2}$$

where $w := w_1 - w_2$. Now we take the difference of the two equations (2.32) at time s and test it with $\chi(s)$. Using (2.9), we obtain

$$\langle w(s), \chi(s)\rangle = \int_\Omega |\nabla\chi|^2 + \langle \xi(s), \chi(s)\rangle + \langle \sigma'(s), \chi(s)\rangle + \lambda\Big\langle \frac{1}{\vartheta_1(s)} - \frac{1}{\vartheta_2(s)}, \chi(s)\Big\rangle,$$

where $\xi := \xi_1 - \xi_2$. Next we use this in (6.2), integrate from 0 to t, and add it to (6.1). Since $\langle \xi, \chi\rangle \geqslant 0$ a.e. in Q due to (2.4), using (2.17) and the Lipschitz continuity of σ' we obtain

$$\begin{aligned}(6.3)\qquad \int_{Q_t} \vartheta u + \lambda \int_{Q_t} \chi u + \frac{1}{2}\int_\Omega \Big|\nabla \int_0^t u\Big|^2 + \frac{1}{2}\|\chi(t)\|_*^2 + \int_{Q_t} |\nabla\chi|^2 \\ \leqslant c\int_{Q_t} |\chi|^2 - \lambda\int_{Q_t}\Big(\frac{1}{\vartheta_1(s)} - \frac{1}{\vartheta_2(s)}\Big)\chi(s).\end{aligned}$$

Due to the definition (2.37) of the function ℓ, we have

$$\frac{1}{\vartheta_1(s)} - \frac{1}{\vartheta_2(s)} = \ell(\vartheta_1(s)) - \ell(\vartheta_2(s)) - [u_1(s) - u_2(s)].$$

Using this in (6.3), noting that two terms cancel out, forgetting a positive integral on the left-hand side, and taking into account Remark 2.3, we get

$$(6.4)\qquad \int_{Q_t} \vartheta u + \frac{1}{2}\|\chi(t)\|_*^2 + \int_{Q_t} |\nabla\chi|^2 \leqslant c\int_{Q_t} |\chi|^2 + \|\ell'\|_\infty \int_{Q_t} |\vartheta|\,|\chi|.$$

Thanks to (2.5), we can estimate the first term of the left-hand side of (6.4) from below in the following way:

$$(6.5)\qquad \int_{Q_t} \vartheta u = \int_{Q_t} (\vartheta_1 - \vartheta_2)(\alpha(\vartheta_1) - \alpha(\vartheta_2)) \geqslant C_1 \int_{Q_t} \frac{\vartheta^2}{1 + \vartheta_1^{2q} + \vartheta_2^{2q}}.$$

Moreover, regarding the last term in (6.4), we get

$$\begin{aligned}(6.6)\qquad \int_{Q_t} |\vartheta|\,|\chi| &\leqslant \frac{C_1}{2}\int_{Q_t} \frac{\vartheta^2}{1 + \vartheta_1^{2q} + \vartheta_2^{2q}} + c\int_{Q_t} (1 + \vartheta_1^{2q} + \vartheta_2^{2q})\chi^2 \\ &\leqslant \frac{C_1}{2}\int_{Q_t} \frac{\vartheta^2}{1 + \vartheta_1^{2q} + \vartheta_2^{2q}} + c\int_{Q_t} (1 + \vartheta_1 + \vartheta_2)\chi^2\end{aligned}$$

since $0 \leqslant q < 1/2$. Now, we estimate the last integral using the Hölder inequality and the regularity $\vartheta_i \in L^\infty(0,T;H)$ for $i = 1, 2$. Moreover, we recall the second inequality (2.18) and obtain for any $\eta > 0$

$$\begin{aligned}(6.7)\qquad \int_{Q_t} (1 + \vartheta_1 + \vartheta_2)\chi^2 &\leqslant \int_0^t (\|1\|_H + \|\vartheta_1\|_H + \|\vartheta_2\|_H)\|\chi\|_{L^4(\Omega)}^2 \\ &\leqslant c\int_0^t \|\chi\|_{L^4(\Omega)}^2 \leqslant \eta\int_{Q_t} |\nabla\chi|^2 + c_\eta \int_0^t \|\chi\|_*^2.\end{aligned}$$

Inserting (6.5)–(6.7) into (6.4) and choosing η small enough, we are now in position to apply the Gronwall lemma and get $\chi(t)=0$ for a.a. $t\in(0,T)$ and $\vartheta=0$ a.e. in Q, whence also $u=0$, obviously.

7. Further regularity and uniqueness

In this section we prove Theorem 2.4. As in the previous section, we assume that the constants c_0 and c_∞ are both equal to 1. We first address the proof of uniqueness of the regular solution and follow the procedure used in [9] and in [11]. We observe that (2.2) implies the existence of a positive constant c_1 such that

$$\varrho'(r)\geqslant\frac{c_1}{1+r^2}\quad\forall\, r\in\mathbb{R}. \tag{7.1}$$

Let $(\vartheta_i,\chi_i,u_i,w_i,\xi_i)$ be two solutions to (2.29)–(2.34), $i=1,2$. Arguing as in the previous section, we arrive at (6.4). Now, we get rid of ϑ_i using $\vartheta_i=\varrho(u_i)$ and $-1/\vartheta_i=u_i-\ell(\varrho(u_i))$ (see (2.37) with $c_0=1$). Moreover, we use (7.1) and the Lipschitz continuity of $\ell\circ\varrho$ stated in [11, Lemma 3.3], and forget the positive integral on the left-hand side. Hence, we have

$$c_1\int_{Q_t}\frac{|u|^2}{1+|u_1|^2+|u_2|^2}+\frac{1}{2}\|\chi(t)\|_*^2+\int_{Q_t}|\nabla\chi|^2\leqslant c\int_{Q_t}|\chi|^2+c\int_{Q_t}|u|\,|\chi|. \tag{7.2}$$

In order to estimate the last term in (7.2), we proceed exactly as in [9, (30)], using the fact that $u_i\in L^\infty(0,T;V)$. We obtain

$$\int_{Q_t}|u|\,|\chi|\leqslant\frac{1}{2}\int_{Q_t}\frac{c_1|u|^2}{1+|u_1|^2+|u_2|^2}+\frac{1}{4}\int_{Q_t}|\nabla\chi|^2+c\int_{Q_t}|\chi|^2. \tag{7.3}$$

Finally, we estimate the integral of $|\chi|^2$ both in (7.2) and in (7.3) using the second relation in (2.18). Hence, choosing η small enough and applying the Gronwall lemma, we obtain $\chi(t)=0$ for a.a. t and $u=0$ a.e. in Q, whence also $\vartheta=0$, obviously.

Now we prove the additional regularity stated in (2.43)–(2.45) by deriving further estimates for the solution $(\vartheta_\varepsilon,\chi_\varepsilon,u_\varepsilon,w_\varepsilon,\xi_\varepsilon)$ to (4.19)–(4.24). To do that, we have to make a proper choice of $\chi_{0\varepsilon}$. We need

$$\chi_{0\varepsilon}\in W,\quad \chi_{0\varepsilon}\to\chi_0\ \text{ in } V,\quad\text{and}\quad \{B\chi_{0\varepsilon}+\beta_\varepsilon(\chi_{0\varepsilon})\}\ \text{ bounded in } V \tag{7.4}$$

and one can use (2.41) and easily see that such conditions are fulfilled by choosing $\chi_{0\varepsilon}$ to be the unique solution $\overline{\chi}\in W$ to the nonlinear elliptic problem

$$\overline{\chi}+B\overline{\chi}+\beta_\varepsilon(\overline{\chi})=\chi_0+B\chi_0+\xi_0$$

where ξ_0 satisfies (2.42). Moreover, we observe that, under the stronger assumptions (2.39) on g and h, if f_ε is the approximation of f introduced in (4.13)–(4.14), then $f_\varepsilon \in W^{1,1}(0,T;H)$ and we have

$$f_\varepsilon \to f \quad \text{strongly in } W^{1,1}(0,T;V') \tag{7.5}$$

as one can easily see arguing as in Lemma 4.2, now on the time derivatives. Finally, we need a technical lemma regarding the regularity of $(\vartheta_\varepsilon, \chi_\varepsilon, u_\varepsilon, w_\varepsilon, \xi_\varepsilon)$.

Lemma 7.1. *Assume $\chi_{0\varepsilon} \in W$ and let $(\vartheta_\varepsilon, \chi_\varepsilon, u_\varepsilon, w_\varepsilon, \xi_\varepsilon)$ be a solution to* (4.19)–(4.24). *Then*

$$\partial_t \chi_\varepsilon, B\chi_\varepsilon, w_\varepsilon \in L^2(0,T;V) \cap H^1(0,T;H) \quad \text{and} \quad Au \in L^2(0,T;H). \tag{7.6}$$

Proof. In the following, $(\vartheta, \chi, u, w, \xi)$ stands for $(\vartheta_\varepsilon, \chi_\varepsilon, u_\varepsilon, w_\varepsilon, \xi_\varepsilon)$. We rewrite (4.22) as

$$\varepsilon\partial_t\chi + B\chi = z^\varepsilon - w$$

where

$$z^\varepsilon := \beta_\varepsilon(\chi) + \sigma'(\chi) + \frac{\lambda}{\vartheta} = \beta_\varepsilon(\chi) + \sigma'(\chi) + \lambda((\ell \circ \varrho)(u) - u) \tag{7.7}$$

and note that $z^\varepsilon \in L^2(0,T;V) \cap H^1(0,T;H)$ since $\chi, u \in L^2(0,T;V) \cap H^1(0,T;H)$ and β_ε, σ' and $\ell \circ \varrho$ are Lipschitz continuous. As $w \in L^2(0,T;V)$ by (3.5) and $\chi_0 \in W$, we derive

$$\partial_t\chi, B\chi \in L^2(0,T;V)$$

from the well-known regularity results. In particular, we are allowed to apply the operator B to such functions and deduce from (4.21)–(4.22)

$$(\mathrm{Id} + \varepsilon B)\partial_t\chi + \varepsilon^{-1}B(\mathrm{Id} + \varepsilon B)\chi = B(\varepsilon^{-1}\chi - z^\varepsilon)$$

where Id is the identity operator. Using also the fact that the operator

$$\mathrm{Id} + \varepsilon B\colon\ L^2(0,T;V) \to L^2(0,T;V')$$

is an isomorphism and that $(\mathrm{Id} + \varepsilon B)^{-1}$ and B commute, we can rewrite the above equation as

$$\partial_t\chi + \varepsilon^{-1}B\chi = B(\mathrm{Id} + \varepsilon B)^{-1}(\varepsilon^{-1}\chi - z^\varepsilon).$$

Noting that the right-hand side belongs to $H^1(0,T;H)$, we deduce the expected regularity for $\partial_t\chi$ and $B\chi$. Finally, the regularity of w and Au stated in (7.6) follows by comparison with (4.22) and (4.19), respectively. $\square$

Now we are ready to derive estimates we need in order to prove (2.43)–(2.45). First of all, we multiply (4.19) by $\partial_t u_\varepsilon$ and integrate over Q_t. We obtain

$$
\begin{aligned}
(7.8)\qquad \varepsilon\int_{Q_t}|\partial_t u_\varepsilon|^2+\int_{Q_t}\varrho_\varepsilon'(u_\varepsilon)|\partial_t u_\varepsilon|^2+\frac{1}{2}\|u_\varepsilon(t)\|_V^2 \\
=\frac{1}{2}\|u_0\|_V^2-\lambda\int_{Q_t}\partial_t\chi_\varepsilon\partial_t u_\varepsilon+\int_{Q_t}f_\varepsilon\partial_t u_\varepsilon.
\end{aligned}
$$

Next, due to Lemma 7.1, we can differentiate equation (4.21) with respect to t, test it with $\mathcal{N}(\partial_t\chi_\varepsilon)$ and integrate on $[0,t]$. It follows that

$$
(7.9)\qquad \frac{1}{2}\|\partial_t\chi_\varepsilon(t)\|_*^2+\int_{Q_t}\partial_t w_\varepsilon\partial_t\chi_\varepsilon=\frac{1}{2}\|\partial_t\chi_\varepsilon(0)\|_*^2.
$$

Again due to Lemma 7.1, we can differentiate equation (4.22) and test it with $\partial_t\chi_\varepsilon$. We get

$$
\begin{aligned}
(7.10)\qquad \int_{Q_t}\partial_t w_\varepsilon\partial_t\chi_\varepsilon=\frac{\varepsilon}{2}\int_\Omega|\partial_t\chi_\varepsilon(t)|^2-\frac{\varepsilon}{2}\int_\Omega|\partial_t\chi_\varepsilon(0)|^2 \\
+\int_{Q_t}|\nabla(\partial_t\chi_\varepsilon)|^2+\int_{Q_t}\partial_t z^\varepsilon\partial_t\chi_\varepsilon,
\end{aligned}
$$

where z^ε is defined in (7.7). Hence, the last integral is given by

$$
\begin{aligned}
\int_{Q_t}\partial_t z^\varepsilon\partial_t\chi_\varepsilon=\int_{Q_t}\beta_\varepsilon'(\chi_\varepsilon)|\partial_t\chi_\varepsilon|^2+\int_{Q_t}\sigma''(\chi_\varepsilon)|\partial_t\chi_\varepsilon|^2 \\
-\lambda\int_{Q_t}\partial_t u_\varepsilon\partial_t\chi_\varepsilon+\lambda\int_{Q_t}(\ell\circ\varrho)'(u_\varepsilon)\partial_t u_\varepsilon\partial_t\chi_\varepsilon.
\end{aligned}
$$

Therefore, combining (7.10) with (7.9) and recalling that $\beta_\varepsilon'\geqslant 0$ and that σ' is Lipschitz continuous, we obtain

$$
\begin{aligned}
(7.11)\qquad \frac{1}{2}\|\partial_t\chi_\varepsilon(t)\|_*^2+\int_{Q_t}|\nabla\partial_t\chi_\varepsilon|^2\leqslant\frac{\varepsilon}{2}\int_\Omega|\partial_t\chi_\varepsilon(0)|^2+\frac{1}{2}\|\partial_t\chi_\varepsilon(0)\|_*^2 \\
+c\int_{Q_t}|\partial_t\chi_\varepsilon|^2+\lambda\int_{Q_t}\partial_t u_\varepsilon\partial_t\chi_\varepsilon \\
-\lambda\int_{Q_t}(\ell\circ\varrho)'(u_\varepsilon)\partial_t u_\varepsilon\partial_t\chi_\varepsilon.
\end{aligned}
$$

In order to estimate the first two terms on the right-hand side of (7.11), observe that $\partial_t\chi_\varepsilon(0)$ is the unique solution $v\in V$ to the elliptic problem

$$
v+\varepsilon Bv=-B(B\chi_{0\varepsilon}+z^\varepsilon(0)),
$$

which is meaningful since $B\chi_{0\varepsilon} + z^\varepsilon(0) \in V$. More precisely, we clearly have that $B\chi_{0,\varepsilon} + z^\varepsilon(0)$ is bounded in V uniformly with respect to ε, thanks to (2.40) and (7.4). We deduce that the right-hand side of the above equation is bounded in V', and standard arguments (already used in the proof of Lemma 4.2) ensure that $\partial_t\chi_\varepsilon(0) = v$ satisfies the bounds

$$\varepsilon^{1/2}\|\partial_t\chi_\varepsilon(0)\|_H \leqslant c \quad \text{and} \quad \|\partial_t\chi_\varepsilon(0)\|_* \leqslant c.$$

From these considerations, summing up (7.8) and (7.11), integrating by parts the last term in (7.8) and taking into account that $\|f_\varepsilon\|_{W^{1,1}(0,T;V')} \leqslant c$ due to (7.5), we obtain

$$\begin{aligned}(7.12)\quad \int_{Q_t} \varrho'_\varepsilon(u_\varepsilon)|\partial_t u_\varepsilon|^2 + \frac{1}{2}\|u_\varepsilon(t)\|_V^2 + \frac{1}{2}\|\partial_t\chi_\varepsilon(t)\|_*^2 + \int_{Q_t} |\nabla(\partial_t\chi_\varepsilon)|^2 \\ \leqslant c + \frac{1}{2}\|u_0\|_V^2 + \max_{0\leqslant s\leqslant T}\{\|f_\varepsilon(s)\|_{V'}\}(\|u_\varepsilon(t)\|_V + \|u_0\|_V) \\ + \int_0^t \|\partial_t f_\varepsilon\|_{V'}\|u_\varepsilon(s)\|_V + c\int_{Q_t} |\partial_t\chi_\varepsilon|^2 \\ + \int_{Q_t} |(\ell\circ\varrho)'(u_\varepsilon)|\,|\partial_t u_\varepsilon\partial_t\chi_\varepsilon|.\end{aligned}$$

The third term on the right-hand side is estimated by Young's inequality and compensated with the second term on the left-hand side. The fifth term is easily treated using (2.18), i.e. we have

$$(7.13)\qquad c\int_{Q_t} |\partial_t\chi_\varepsilon|^2 \leqslant \frac{1}{4}\int_{Q_t} |\nabla\partial_t\chi_\varepsilon|^2 + c\int_0^t \|\partial_t\chi_\varepsilon(s)\|_*^2.$$

Regarding the last term in (7.12), let us observe that, due to [11, Lemma 3.4], $(\ell\circ\varrho)'$ satisfies

$$(7.14)\qquad |(\ell\circ\varrho)'(s)| \leqslant c\sqrt{\varrho'(s)} \quad \forall\, s\in\mathbb{R}.$$

Hence we obtain

$$\begin{aligned}(7.15)\quad \int_{Q_t} |(\ell\circ\varrho)'(u_\varepsilon)|\,|\partial_t u_\varepsilon\partial_t\chi_\varepsilon| &\leqslant c\int_{Q_t} \sqrt{\varrho'_\varepsilon(u_\varepsilon)}|\partial_t u_\varepsilon\partial_t\chi_\varepsilon| \\ &\leqslant \frac{1}{2}\int_{Q_t} \varrho'_\varepsilon(u_\varepsilon)|\partial_t u_\varepsilon|^2 + c\int_{Q_t} |\partial_t\chi_\varepsilon|^2 \\ &\leqslant \frac{1}{2}\int_{Q_t} \varrho'_\varepsilon(u_\varepsilon)|\partial_t u_\varepsilon|^2 + \frac{1}{4}\int_{Q_t} |\nabla\partial_t\chi_\varepsilon|^2 \\ &\quad + c\int_0^t \|\partial_t\chi_\varepsilon(s)\|_*^2.\end{aligned}$$

Using (7.13) and (7.15) in (7.12) we deduce

$$\frac{1}{2}\int_{Q_t}\varrho_\varepsilon'(u_\varepsilon)|\partial_t u_\varepsilon|^2+\frac{1}{4}\|u_\varepsilon(t)\|_V^2+\frac{1}{2}\|\partial_t\chi_\varepsilon(t)\|_*^2+\frac{1}{2}\int_{Q_t}|\nabla(\partial_t\chi_\varepsilon)|^2$$
$$\leqslant c+\frac{3}{4}\|u_0\|_V^2+\int_0^t\|\partial_t f_\varepsilon\|_{V'}\|u_\varepsilon(s)\|_V+c\int_0^t\|\partial_t\chi_\varepsilon(s)\|_*^2$$

and hence, using the Gronwall lemma, we derive a uniform estimate for the quantities

$$\|u_\varepsilon(t)\|_V,\quad \|\partial_t\chi_\varepsilon(t)\|_*,\quad \int_{Q_t}|\nabla(\partial_t\chi_\varepsilon)|^2.$$

This leads us to the conclusion that

$$\begin{aligned}\partial_t\chi_\varepsilon\to\partial_t\chi &\quad \text{weakly star in } L^\infty(0,T;V'),\\ u_\varepsilon\to u &\quad \text{weakly star in } L^\infty(0,T;V)\end{aligned}$$

and, by virtue of (2.18),

$$\partial_t\chi_\varepsilon\to\partial_t\chi\quad \text{weakly in } L^2(0,T;V),$$

again at least for a subsequence. Clearly, the limit solves (2.29)–(2.34) thanks to the previous section. Hence we get (2.45) and

$$\partial_t\chi\in L^\infty(0,T;V')\cap L^2(0,T;V).$$

Moreover, using (2.31), we find that $\nabla w\in L^\infty(0,T;H)$ so that (2.46) holds. In order to recover that $\chi\in L^\infty(0,T;W)$ (and hence (2.44) holds), observe that we can use equation (2.32) and the same arguments as in [9, p. 8]. Moreover, using (2.29) and the additional regularity of χ and u, we get (2.43).

Finally, (2.47) can be derived by the same arguments as in the proof of [11, (54)], and the proof is complete.

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