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ON THE EXISTENCE OF MULTIPLE PERIODIC SOLUTIONS
FOR THE VECTOR p -LAPLACIAN VIA
CRITICAL POINT THEORY*

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Abstract. We study the vector p -Laplacian

$$(*) \quad \begin{cases} -(|u'|^{p-2}u')' = \nabla F(t, u) & \text{a.e. } t \in [0, T], \\ u(0) = u(T), \quad u'(0) = u'(T), & 1 < p < \infty. \end{cases}$$

We prove that there exists a sequence (u_n) of solutions of $(*)$ such that u_n is a critical point of φ and another sequence (u_n^*) of solutions of $(*)$ such that u_n^* is a local minimum point of φ , where φ is a functional defined below.

Keywords: p -Laplacian equation, periodic solution, critical point theory

MSC 2000: 34B15

1. INTRODUCTION AND MAIN RESULTS

Consider the second order system

$$(1.1) \quad \begin{cases} -(|u'|^{p-2}u')' = \nabla F(t, u) & \text{a.e. } t \in [0, T], \\ u(0) = u(T), \quad u'(0) = u'(T), & 1 < p < \infty, \end{cases}$$

where $T > 0$ and $F: [0, T] \times \mathbb{R}^N \rightarrow \mathbb{R}$ satisfies the following assumption:

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(A) $F(t, x)$ is measurable in t for every $x \in \mathbb{R}^N$ and continuously differentiable in x for a.e. $t \in [0, T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0, T; \mathbb{R}^+)$ such that

$$|F(t, x)| \leq a(|x|)b(t), \quad |\nabla F(t, x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$; here $\mathbb{R}^+ = [0, \infty)$.

Recently some papers have appeared discussing scalar periodic problems driven by the one-dimensional p -Laplacian. We refer the reader to the works of Del Pino-Manasevich-Murua [3], Fabry-Fayyad [4], Gao [5] and Dang-Oppenheimer [6]. In all of these works the approach is degree theoretical and the existence of one solution is established. In [2], J. Mawhin generalized the Hartman-Knobloch results to perturbations of a vector p -Laplacian ordinary operator.

For $p = 2$, Mawhin-Willem [1] proved the existence of solutions for the problem (1.1) under the conditions

(M1) there exists $g \in L^1(0, T; \mathbb{R}^+)$ such that $|\nabla F(t, x)| \leq g(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$, and

(M2) $\int_0^T F(t, x) dt \rightarrow +\infty$ as $|x| \rightarrow \infty$ or $\int_0^T F(t, x) dt \rightarrow -\infty$ as $|x| \rightarrow \infty$.

In this paper we study the existence of periodic solutions of a vector p -Laplacian with potential oscillating around the first eigenvalue. Our arguments are based on a result by Habets-Manasevich-Zanolin [8] related to the two-point boundary value problem

$$u'' + u + f(t, u) = 0, \quad u(0) = u(1) = 0.$$

Our main result is the following theorem.

Theorem 1.1. *Suppose*

(H1) *there exists $g \in L^q(0, T; \mathbb{R}^+)$ (here q is the conjugate of p) such that $|\nabla F(t, x)| \leq g(t)$ for all $x \in \mathbb{R}^N$ and a.e. $t \in [0, T]$,*

(H2)

$$(1.2) \quad \limsup_{R \rightarrow +\infty} \inf_{a \in \mathbb{R}^N, |a|=R} \int_0^T F(s, a) ds = +\infty$$

and

$$(1.3) \quad \liminf_{r \rightarrow +\infty} \sup_{b \in \mathbb{R}^N, |b|=r} \int_0^T F(s, b) ds = -\infty$$

hold. Then

- i) *there exists a sequence (u_n) of solutions of (1.1) such that u_n is a critical point of φ and $\lim_{n \rightarrow \infty} \varphi(u_n) = +\infty$;*

ii) *there exists a sequence (u_n^*) of solutions of (1.1) such that u_n^* is a local minimum point of φ and $\lim_{n \rightarrow \infty} \varphi(u_n^*) = -\infty$; here φ is a functional defined below.*

Throughout the paper, for $N \geq 1$ and $I = [0, T]$, we will set $C = C(I, \mathbb{R}^N)$, $L^p = L^p(I, \mathbb{R}^N)$, $W^{1,p} = W^{1,p}(I, \mathbb{R}^N)$ and $X = W_T^{1,p} = \{u \in W^{1,p}; u(0) = u(T)\}$. The norm in C will be defined by $\|u\|_\infty = \max_{t \in [0, T]} |u(t)|$, the norm in L^p by

$$\|u\|_p = \left(\int_0^T |u(t)|^p dt \right)^{1/p},$$

and the norm in X by $\|u\|_X = \|u\|_p + \|u'\|_p$. Note that for each $p > 1$, X is compact embedded in C . Let $B_r = \{x \in \mathbb{R}^N : |x| \leq r\}$.

Each $u \in L^1$ can be written as $u(t) = \bar{u} + \tilde{u}(t)$ with

$$\bar{u} := \frac{1}{T} \int_0^T u(t) dt, \quad \int_0^T \tilde{u}(t) dt = 0.$$

We will use the Sobolev inequality

$$\|\tilde{u}\|_\infty \leq T^{1/q} \|u'\|_p \quad \text{for each } u \in X \text{ (here } q \text{ is the conjugate of } p)$$

and Wirtinger's inequality

$$\|\tilde{u}\|_p^p \leq T^p \|\tilde{u}'\|_p^p \quad \text{for each } u \in X.$$

The proof of the theorem is given in Section 3. In Section 2 we present some preliminary results on the variational setting of p -Laplacian equations in X and the related Palais-Smale compactness.

2. PRELIMINARIES

We first recall some facts about the eigenvalue problem for the p -Laplacian, see [9]. A $\lambda \in \mathbb{R}$ is said to be an "eigenvalue" of the p -Laplacian with periodic boundary conditions, if the problem

$$\begin{cases} (|u'|^{p-2}u')' + \lambda|u|^{p-2}u = 0 & \text{a.e. on } [0, T], \\ u(0) - u(T) = u'(0) - u'(T) = 0 \end{cases}$$

has a nontrivial solution $u \in C^1(I, \mathbb{R}^N)$, known as the corresponding one to λ "eigenfunction". Let S denote the set of these eigenvalues. Evidently $0 \in S$, each element

of S is nonnegative and 0 is the smallest (first) eigenvalue. If $N = 1$ (scalar case), by direct integration of the equation we see that all the eigenvalues are

$$\lambda_n = \left(\frac{2\pi_p n}{T} \right)^p, \quad n = 0, 1, 2, 3, \dots$$

where

$$\pi_p = 2(p-1)^{1/p} \int_0^1 (1-t^p)^{-1/p} dt.$$

In the case $N > 1$ (vector case), $\{\lambda_n\}_{n \geq 1} \subseteq S$ but S contains more elements.

It follows from assumption (A) that the functional φ on X given by

$$\varphi(u) = \frac{1}{p} \int_0^T |u'(t)|^p dt - \int_0^T F(t, u(t)) dt$$

is continuously differentiable and weakly lower semicontinuous on X (the proof is similar to the case $p = 2$, see [1]). Moreover, one has

$$\langle \varphi'(u), v \rangle = \int_0^T |u'(t)|^{p-2} (u'(t), v'(t)) dt - \int_0^T (\nabla F(t, u(t)), v(t)) dt$$

for all $u, v \in X$. It is easy to see that the solutions of problem (1.1) correspond to the critical points of φ .

Let us write

$$I(u) = \frac{1}{p} \int_0^T |u'|^p dt, \quad G(u) = \int_0^T F(t, u(t)) dt, \quad \forall u \in X.$$

Proposition 2.1. *The mapping $I': X \rightarrow X^*$ is of type (S_+) (see [7]), i.e. any sequence $\{u_n\}$ in X satisfying $u_n \rightharpoonup u$ in X and*

$$\overline{\lim}_{n \rightarrow \infty} \langle I'(u_n), u_n - u \rangle \leq 0$$

contains a convergent subsequence.

Proof. Assume that $u_n \rightharpoonup u$ in X and $\overline{\lim}_{n \rightarrow \infty} \langle I'(u_n), u_n - u \rangle \leq 0$. Then we get

$$\overline{\lim}_{n \rightarrow \infty} \langle I'(u_n) - I'(u), u_n - u \rangle \leq 0,$$

and together with the monotonicity property of I' this implies

$$\lim_{n \rightarrow \infty} \langle I'(u_n) - I'(u), u_n - u \rangle = 0,$$

i.e.,

$$(2.1) \quad \int_0^T \langle |u'_n|^{p-2} u'_n - |u'|^{p-2} u', u'_n - u' \rangle dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Recall that for all $x, y \in \mathbb{R}^N$ the following inequalities hold:

$$\langle |x|^{p-2} x - |y|^{p-2} y, x - y \rangle \geq \left(\frac{1}{2}\right)^{p-1} |x - y|^p \quad \text{when } p \geq 2,$$

and

$$(|x| + |y|)^{2-p} \langle |x|^{p-2} x - |y|^{p-2} y, x - y \rangle \geq (p-1) |x - y|^2 \quad \text{when } 1 < p \leq 2.$$

Hence, by (2.1), one has that u'_n converges to u' in $[0, T]$ in measure. Thus there exists a subsequence (without loss of generality assume it to be the whole sequence) with

$$u'_n(t) \rightarrow u'(t) \quad \text{for a.e. } t \in [0, T], \quad n \rightarrow \infty,$$

and so

$$(2.2) \quad \frac{1}{p} |u'_n(t) - u'(t)|^p \rightarrow 0 \quad \text{for a.e. } t \in [0, T], \quad n \rightarrow \infty.$$

One also has

$$\langle I'(u_n), u_n - u \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Moreover,

$$\begin{aligned} \langle I'(u_n), u_n - u \rangle &= \int_0^T |u'_n|^p dt - \int_0^T |u'_n|^{p-2} u'_n u' dt \\ &\geq \int_0^T |u'_n|^p dt - \int_0^T |u'_n|^{p-1} |u'| dt \\ &\geq \int_0^T |u'_n|^p dt - \int_0^T \left[\frac{p-1}{p} |u'_n|^p + \frac{1}{p} |u'|^p \right] dt \\ &= \frac{1}{p} \int_0^T (|u'_n|^p - |u'|^p) dt. \end{aligned}$$

Now the lower semi-continuity of I yields

$$\liminf_{n \rightarrow \infty} \frac{1}{p} \int_0^T |u'_n|^p dt \geq \frac{1}{p} \int_0^T |u'|^p dt.$$

Thus

$$\frac{1}{p} \int_0^T |u'_n|^p dt \rightarrow \frac{1}{p} \int_0^T |u'|^p dt \quad \text{as } n \rightarrow \infty.$$

Consequently, the sequence $\{p^{-1} \int_0^t |u'_n|^p dt\}$ is equi-absolutely continuous on $[0, T]$. From the inequality

$$\frac{1}{p} |u'_n(t) - u'(t)|^p \leq \frac{2^p}{p} (|u'_n|^p + |u'|^p)$$

we obtain that

$$(2.3) \quad \left\{ \frac{1}{p} \int_0^t |u'_n(t) - u'(t)|^p dt \right\} \text{ is equi-absolutely continuous on } [0, T].$$

Now (2.2), (2.3) and the convergence theorem of Vitali (see [7], p. 1017) guarantee that

$$\int_0^T \frac{1}{p} |u'_n(t) - u'(t)|^p dt \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Thus $u'_n \rightarrow u'$ in L^p .

Also $u_n \rightarrow u$ in X , which implies that $u_n \rightarrow u$ in L^p . Hence $u_n \rightarrow u$ in X , and the proof is complete. \square

Since the sum of a mapping of type (S_+) with a weakly-strongly continuous mapping is still a mapping of type (S_+) , we obtain the following result.

Proposition 2.2. $\varphi' : X \rightarrow X^*$ is a mapping of type (S_+) .

Proposition 2.3. Suppose that if a sequence $\{u_n\}$ in X is such that

$$\{\varphi(u_n)\} \text{ is bounded and } \varphi'(u_n) \rightarrow 0$$

as $n \rightarrow \infty$, then $\{u_n\}$ has a bounded subsequence. Then φ satisfies the PS-condition.

Proof. Assume that $\{u_n\} \subset X$, $\{\varphi(u_n)\}$ is bounded and $\varphi'(u_n) \rightarrow 0$. Now we know that $\{u_n\}$ contains a bounded subsequence and for simplicity we denote it again by $\{u_n\}$. Since X is reflexive we can extract a subsequence (again denoting it by $\{u_n\}$) such that $u_n \rightarrow u$ in X . Since $\varphi'(u_n) \rightarrow 0$, one has

$$\langle \varphi'(u_n), u_n - u \rangle \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Since φ' is a mapping of type (S_+) we have that $u_n \rightarrow u$ in X . Hence φ satisfies the PS-condition. \square

3. PROOFS OF THEOREMS

We shall prove Theorem 1.1 from a sequence of claims.

Consider the direct sum decomposition $X = \mathbb{R}^N \oplus V$ with $V = \{v \in X : \int_0^T v(t) dt = 0\}$. So for all $u \in X$ we can write $u = \bar{u} + \tilde{u}$ with $\bar{u} \in \mathbb{R}^N$ and $\tilde{u} \in V$.

Claim 1. φ is coercive on V .

Proof. Combining hypotheses (H1) with the Mean Value Theorem, we see that for almost all $t \in I$ and all $x \in \mathbb{R}^N$, $|F(t, x)| \leq g(t)|x|$. If $\tilde{u} \in V$, then

$$\varphi(\tilde{u}) \geq \frac{1}{p} \|\tilde{u}'\|_p^p - \int_0^T g(t)|\tilde{u}(t)| dt \geq \frac{1}{p} \|\tilde{u}'\|_p^p - \|g\|_q \|\tilde{u}\|_p \geq \frac{1}{p} \|\tilde{u}'\|_p^p - T \|g\|_q \|\tilde{u}'\|_p$$

by Wirtinger's inequality (here q is the conjugate of p). The above inequality and $\tilde{u} \in V$ imply that $\varphi(\tilde{u}) \rightarrow \infty$ as $\|\tilde{u}'\|_p \rightarrow \infty$. Now it follows that $\varphi(\tilde{u}) \rightarrow \infty$, as $\|\tilde{u}\|_X \rightarrow 0$. \square

Claim 2. There exists a sequence (R_n) such that

$$\lim_{n \rightarrow \infty} R_n = +\infty, \quad \lim_{n \rightarrow \infty} \left[\sup_{a \in \mathbb{R}^N, |a|=R_n} \varphi(a) \right] = -\infty.$$

Proof. This follows from (1.2) since

$$\varphi(a) = - \int_0^T F(t, a) dt.$$

\square

Claim 3. There exists a sequence (r_m) such that

$$\lim_{m \rightarrow \infty} r_m = +\infty, \quad \text{and} \quad \lim_{m \rightarrow \infty} \left[\inf_{b \in \mathbb{R}^N, |b|=r_m, \tilde{u} \in V} \varphi(b + \tilde{u}) \right] = +\infty.$$

Proof. For any $b \in \mathbb{R}^N$, $|b| = r_m$, and $\tilde{u} \in V$, let $u = b + \tilde{u}$. Note that

$$\begin{aligned} \varphi(u) &= \frac{1}{p} \|u'\|_p^p - \int_0^T F(t, u) dt \\ &= \frac{1}{p} \|\tilde{u}'\|_p^p - \int_0^T [F(t, u) - F(t, b)] dt - \int_0^T F(t, b) dt \\ &= \frac{1}{p} \|\tilde{u}'\|_p^p - \int_0^T \int_0^1 (\nabla F(t, b + s\tilde{u}(t)), \tilde{u}(t)) ds dt - \int_0^T F(t, b) dt \\ &\geq \frac{1}{p} \|\tilde{u}'\|_p^p - \left(\int_0^T g(t) dt \right) \|\tilde{u}\|_\infty - \int_0^T F(t, b) dt \\ &\geq \frac{1}{p} \|\tilde{u}'\|_p^p - C_1 \|\tilde{u}'\|_p - \int_0^T F(t, b) dt \end{aligned}$$

by Sobolev's inequality; here $C_1 = (\int_0^T g(t) dt)T^{1/q}$. Thus

$$\inf_{b \in \mathbb{R}^N, |b|=r_m, \tilde{u} \in V} \varphi(b + \tilde{u}) \geq \inf_{\tilde{u} \in V} \left(\frac{1}{p} \|\tilde{u}'\|_p^p - C_1 \|\tilde{u}'\|_p \right) + \inf_{b \in \mathbb{R}^N, |b|=r_m} \left(- \int_0^T F(t, b) dt \right).$$

On the other hand, there exists $\beta \in \mathbb{R}$ such that $\inf_{\tilde{u} \in V} (p^{-1} \|\tilde{u}'\|_p^p - C_1 \|\tilde{u}'\|_p) \geq \beta$. The claim now follows from (1.3). \square

Consider now the set

$$S_n = \{ \gamma \in C(B_{R_n}, X), \gamma|_{\partial B_{R_n}} = i|_{\partial B_{R_n}} \}$$

and define

$$(3.1) \quad c_n = \inf_{\gamma \in S_n} \left[\max_{x \in B_{R_n}} \varphi(\gamma(x)) \right].$$

We prove that each γ intersects the hyperplane V . Let $\pi: X \rightarrow \mathbb{R}^N$ be the (continuous) projection of X onto \mathbb{R}^N , defined by

$$\pi(u) = \frac{1}{T} \int_0^T u dt \quad \text{for } u \in X.$$

Let γ be any continuous map such that $\gamma|_{\partial B_{R_n}} = i|_{\partial B_{R_n}}$. We have to show that $0 \in \pi(\gamma(B_{R_n}))$.

For $t \in [0, 1]$, $u \in \mathbb{R}^N$ define

$$\gamma_t(u) = t\pi(\gamma(u)) + (1-t)u.$$

Note that $\gamma_t \in C^0(\mathbb{R}^N; \mathbb{R}^N)$ defines a homotopy of $\gamma_0 = \text{id}$ with $\gamma_1 = \pi \circ \gamma$. Moreover, $\gamma_t|_{\partial B_{R_n}} = \text{id}$ for all t . By homotopy invariance and normalization of the degree (see for instance Deimling [10, Theorem 1.3.1]) we have

$$\deg(\pi \circ \gamma, B_{R_n}, 0) = \deg(\text{id}, B_{R_n}, 0) = 1.$$

Hence $0 \in \pi(\gamma(B_{R_n}))$. Thus each γ intersects the hyperplane V .

Now since φ is coercive on V , there is a constant K such that

$$\max_{x \in B_{R_n}} \varphi(\gamma(x)) \geq \inf_{\tilde{u} \in V} \varphi(\tilde{u}) \geq K.$$

Hence $c_n \geq K$ and, for all large values of n ,

$$c_n > \sup_{a \in \mathbb{R}^N, |a|=R_n} \varphi(a).$$

Let us now fix such n and apply Theorem 4.3 in [1]. This proves the next claim (see also [1, Corollary 4.3]).

Claim 4. *If n is large enough there exist sequences (γ_k) in S_n and (v_k) in X such that*

$$\max_{x \in B_{R_n}} \varphi(\gamma_k(x)) \rightarrow c_n \quad (k \rightarrow \infty).$$

Then there exists a sequence (v_k) in X such that

$$\varphi(v_k) \rightarrow c_n, \quad \text{dist}(v_k, \gamma_k(B_{R_n})) \rightarrow 0, \quad |\varphi'(v_k)| \rightarrow 0$$

when $k \rightarrow \infty$.

Claim 5. *The sequence (v_k) is bounded in X .*

Proof. For any k large enough,

$$c_n \leq \max_{x \in B_{R_n}} \varphi(\gamma_k(x)) \leq c_n + 1$$

and we can find $w_k \in \gamma_k(B_{R_n})$ such that

$$(3.2) \quad \|v_k - w_k\|_X \leq 1.$$

Using Claim 3, for a fixed $n \in \mathbb{N}$ there exists $m \in \mathbb{N}$ such that $r_m > R_n$. As in the proof above, let $\pi: X \rightarrow \mathbb{R}^N$ be the (continuous) projection of X onto \mathbb{R}^N . Then $|\pi(\gamma_k(B_{R_n}))| \leq R_n$. Let $H_{r_m} = \{b \in \mathbb{R}^N: |b| = r_m\} + V$. Then $|\pi(H_{r_m})| = r_m$, so $\gamma_k(B_{R_n})$ cannot intersect the hyperplanes H_{r_m} . Hence, if we write $w_k = \bar{w}_k + \tilde{w}_k$, where $\bar{w}_k \in \gamma_k(B_{R_n})$ and $\tilde{w}_k \in V$, we have

$$|\bar{w}_k| < r_m.$$

We also have

$$\begin{aligned}
c_n + 1 &\geq \varphi(w_k) = \frac{1}{p} \|(\bar{w}_k + \tilde{w}_k)'\|_p^p - \int_0^T F(t, w_k(t)) \, dt \\
&= \frac{1}{p} \|\tilde{w}_k'\|_p^p - \int_0^T [F(t, w_k(t)) - F(t, \bar{w}_k)] \, dt - \int_0^T F(t, \bar{w}_k) \, dt \\
&= \frac{1}{p} \|\tilde{w}_k'\|_p^p - \int_0^T \int_0^1 (\nabla F(t, \bar{w}_k + s\tilde{w}_k(t)), \tilde{w}_k(t)) \, ds \, dt - \int_0^T F(t, \bar{w}_k) \, dt \\
&\geq \frac{1}{p} \|\tilde{w}_k'\|_p^p - \left(\int_0^T g(t) \, dt \right) \|\tilde{w}_k\|_\infty - \int_0^T F(t, \bar{w}_k) \, dt \\
&\geq \frac{1}{p} \|\tilde{w}_k'\|_p^p - C_2 \|\tilde{w}_k'\|_p - \int_0^T F(t, \bar{w}_k) \, dt
\end{aligned}$$

by Sobolev's inequality; here $C_2 = \left(\int_0^T g(t) \, dt\right) T^{1/q}$. Notice also that since $|\bar{w}_k| < r_m$, the integral $\int_0^T F(t, \bar{w}_k) \, dt$ is bounded. Consequently, \tilde{w}_k is bounded in X . Thus the sequence w_k is bounded and the claim follows from 3.2. \square

Claim 6. c_n is a critical value.

Proof. From Proposition 2.3 and the last claim (recall also Claim 4) it follows that (v_k) contains a convergent subsequence, which we rename as (v_k) . Let $u_n = \lim_{k \rightarrow \infty} v_k$, then (see Claim 4)

$$\varphi'(u_n) = \lim_{k \rightarrow \infty} \varphi'(v_k) = 0 \quad \text{and} \quad \varphi(u_n) = \lim_{k \rightarrow \infty} \varphi(v_k) = c_n.$$

\square

Proof of Theorem 1.1. (a) Claim 6 proves that, for each n large enough, there exists at least one solution u_n of (1.1) such that $\varphi(u_n) = c_n$, where c_n is given by (3.1). If $0 < r_k \leq R_n$ and $H_{r_k} = \{b \in \mathbb{R}^N : |b| = r_k\} + V$, then for any $\gamma \in S_n$ we have that γ intersects the hyperplane H_{r_k} (the proof is similar to the above). Then

$$\max_{x \in B_{R_n}} \varphi(\gamma(x)) \geq \inf_{b \in \mathbb{R}^N, |b|=r_k, \tilde{u} \in V} \varphi(b + \tilde{u}).$$

Thus, using Claim 3, we obtain that

$$\lim_{n \rightarrow \infty} c_n = +\infty,$$

and (a) follows.

(b) For $n \in \{1, 2, \dots\}$, define a subset P_n of X by

$$P_n = \{u \in X : u = \bar{u} + \tilde{u}, \bar{u} \in \mathbb{R}^N, |\bar{u}| \leq r_n, \tilde{u} \in V\}.$$

We note that for $u \in P_n$ we have, proceeding as in Claim 3,

$$(3.3) \quad \begin{aligned} \varphi(u) &= \frac{1}{p} \|u'\|_p^p - \int_0^T F(t, u) dt \\ &\geq \frac{1}{p} \|\tilde{u}'\|_p^p - C_1 \|\tilde{u}'\|_p - \int_0^T F(t, \bar{u}) dt \end{aligned}$$

by Sobolev's inequality; here $C_1 = (\int_0^T g(t) dt) T^{1/q}$. Notice also that $\int_0^T F(t, \bar{u}) dt$ is bounded. Thus φ is bounded below on P_n .

Let us set

$$(3.4) \quad \mu_n = \inf_{u \in P_n} \varphi(u)$$

and let (u_k) be a sequence in P_n such that

$$(3.5) \quad \varphi(u_k) \rightarrow \mu_n \quad \text{as } k \rightarrow \infty.$$

We have

$$u_k = \bar{u}_k + \tilde{u}_k, \quad \bar{u}_k \in \mathbb{R}^N \quad \text{and} \quad |\bar{u}_k| \leq r_n.$$

Without loss of generality we assume that $\bar{u}_k \rightarrow \bar{u} \in B_{r_n}$. From (3.3) and (3.5) we obtain that (u_k) is a bounded sequence in X and thus passing to a subsequence, which we rename (u_k) , we have that

$$u_k \rightharpoonup u_n^* \quad \text{in } X.$$

Since P_n is a convex closed subset of X , $u_n^* \in P_n$.

Note that φ is weakly lower semi-continuous, so

$$\mu_n = \lim_{k \rightarrow \infty} \varphi(u_k) \geq \varphi(u_n^*)$$

and, since $u_n^* \in P_n$, we must have

$$\mu_n = \varphi(u_n^*).$$

Next we want to show that $u_n^* \in \text{Int } P_n$ for large n , where

$$\text{Int } P_n := \{u \in X : u = \bar{u} + \tilde{u}, |\bar{u}| < r_n\}.$$

Indeed, taking

$$0 < R_n < r_n,$$

it follows from Claim 2 and Claim 3 that $u_n^* \notin \{b + \tilde{u}, b \in Y, |b| = r_n\}$ for large n and hence $u_n^* \in \text{Int } P_n$. This fact and (3.4) imply that

$$\varphi'(u_n^*) = 0$$

and u_n^* is a solution of (1.1).

Finally, from the fact that

$$\varphi(u_n^*) \leq \inf_{a \in \mathbb{R}^N, |a|=R_n} \varphi(a)$$

and from Claim 2 we obtain that

$$\lim_{n \rightarrow \infty} \varphi(u_n^*) = -\infty.$$

□

Remark 1. Going through the above proof, it is easy to see that the existence of critical points u_n still holds if assumption (1.2) is weakened to

$$\limsup_{R \rightarrow +\infty} \left[\inf_{a \in \mathbb{R}^N, |a|=R} \int_0^T F(s, a) \, ds \right] \text{ is bounded from below.}$$

In that case, however, the existence of the minima u_n^* is no longer guaranteed.

Example 1. Let us consider the scalar problem

$$(3.6) \quad \begin{aligned} -(|u'|^{p-2}u')' &= a \sin(\ln(u^2 + 1)) + \frac{2au^2}{1+u^2} \cos(\ln(u^2 + 1)) + e(t), \\ u(0) - u(T) &= u'(0) - u'(T) = 0, \end{aligned}$$

where $1 < p < \infty$, $a > 0$, $e \in L^1(0, T)$ and $\int_0^T e(t) \, dt = 0$. In this case

$$F(t, u) = au \sin(\ln(u^2 + 1)) + e(t)u$$

and hence

$$\int_0^T F(t, x) \, dt = ax \sin(\ln(x^2 + 1)).$$

Let $R_k = \sqrt{e^{2k\pi+1/2\pi} - 1}$, $r_k = R_k = \sqrt{e^{2k\pi+3/2\pi} - 1}$ for $k = 1, 2, \dots$. Then

$$\begin{aligned} \lim_{k \rightarrow +\infty} \int_0^T F(t, R_k) dt &= \lim_{k \rightarrow +\infty} aR_k \sin(\ln(R_k^2 + 1)) \\ &= \lim_{k \rightarrow +\infty} a\sqrt{e^{2k\pi+1/2\pi} - 1} = +\infty \end{aligned}$$

and

$$\begin{aligned} \lim_{k \rightarrow +\infty} \int_0^T F(t, r_k) dt &= \lim_{k \rightarrow +\infty} ar_k \sin(\ln(r_k^2 + 1)) \\ &= \lim_{k \rightarrow +\infty} -a\sqrt{e^{2k\pi+3/2\pi} - 1} = -\infty. \end{aligned}$$

As a result, (1.2) and (1.3) are satisfied. Moreover, $|F'_u(t, u)|$ is clearly bounded by $3a + |e(t)|$ and so we have the following result:

- (i) there exists a sequence (u_n) of solutions of (3.6) such that u_n is a critical point of φ and $\lim_{n \rightarrow \infty} \varphi(u_n) = +\infty$;
- (ii) there exists a sequence (u_n^*) of solutions of (3.6) such that u_n^* is a local minimum point of φ and $\lim_{n \rightarrow \infty} \varphi(u_n^*) = -\infty$.

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