ON VALUATION OF DERIVATIVE SECURITIES: A LIE GROUP ANALYTICAL APPROACH*

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Abstract. This paper proposes a Lie group analytical approach to tackle the problem of pricing derivative securities. By exploiting the infinitesimal symmetries of the Boundary Value Problem (BVP) satisfied by the price of a derivative security, our method provides an effective algorithm for obtaining its explicit solution.

Keywords: Lie groups, infinitesimal transformations, invariants, pricing of derivative securities, Bessel equations, Bessel functions

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1. Introduction

The option pricing model developed by Black and Scholes (see [1]) enjoys great popularity. Option pricing theory and its applications in many areas of finance as well as actuarial science have enjoyed rapid development during the past 30 years. Many different methods have been employed to tackle the problem of pricing derivative security. Black and Scholes used a non arbitrage principle; by constructing a duplicated portfolio to the derivative, a PDE satisfied by the price of the derivative security was obtained. Another very popular method is the so called martingale measure or the risk-neutral probability method. Gerber and Shiu in [7] introduced an option pricing framework using the Esscher transform. Papers [3] and [4] used backward stochastic differential equation techniques to tackle the problem. In a series of articles by Lo and Hui (see [9] and [10]), a Lie algebraic approach was proposed to deal with the option pricing problem. In this paper, we propose a slightly different approach, the Lie group approach, to tackle the problem.

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When we deal with option pricing problems, if the underlying stock price is a rather general model, or the option is an exotic one, it is usually not easy to obtain an explicit solution. Many researchers have put a lot of effort into this problem. Kunitomo and Ikeda in [8] obtained a closed form solution to a class of barrier options. Geman and Yor obtained an explicit pricing formula to the Asian option (see [6]). Paper [2] considered the option pricing problem under a Constant Elasticity of Variance (CEV) model by using some results in [5]. All the above mentioned works used some specified techniques. In this paper, we provide an effective algorithm for dealing with this kind of problems. Our method is a general one. In this paper, we use the European call option under the CEV model as an example to illustrate the idea. However, the method can be used to many other models. Compared to the work of Lo and Hui, our method can be more easily extended to problems under more general models.

2. Some concepts and results on Lie groups

In this section, we provide some concepts and main results on Lie groups which will be used later. Consider a one-parameter connected local Lie group of transformations acting on an $(\mathbf{x}, u)$-space with an infinitesimal generator

$$X = \sum_i \xi_i(\mathbf{x}, u)\frac{\partial}{\partial x_i} + \eta(\mathbf{x}, u)\frac{\partial}{\partial u}. \tag{1}$$

Explicit formulas for the extended infinitesimals $\eta^{(k)}$ of the corresponding kth extension with an infinitesimal generator

$$\begin{aligned} X^{(k)} = {} & \sum_i \xi_i(\mathbf{x}, u)\frac{\partial}{\partial x_i} + \eta(\mathbf{x}, u)\frac{\partial}{\partial u} + \sum_i \eta_i^{(1)}(\mathbf{x}, u, u_{(1)})\frac{\partial}{\partial u_i} \\ & + \ldots + \sum_{i_1 i_2 \ldots i_k} \eta^{(k)}_{i_1 i_2 \ldots i_k}\frac{\partial}{\partial u_{i_1 i_2 \ldots i_k}}, \quad k = 1, 2, \ldots \end{aligned} \tag{2}$$

are given by

Proposition 1. *We have*

$$(3)\qquad \begin{pmatrix} \eta_1^{(1)} \\ \eta_2^{(1)} \\ \vdots \\ \eta_n^{(1)} \end{pmatrix} = \begin{pmatrix} D_1\eta \\ D_2\eta \\ \vdots \\ D_n\eta \end{pmatrix} - B\cdot \begin{pmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{pmatrix},$$

$$(4)\qquad \begin{pmatrix} \eta^{(k)}_{i_1i_2\dots i_{k-1}1} \\ \eta^{(k)}_{i_1i_2\dots i_{k-1}2} \\ \vdots \\ \eta^{(k)}_{i_1i_2\dots i_{k-1}n} \end{pmatrix} = \begin{pmatrix} D_1\eta^{(k-1)}_{i_1i_2\dots i_{k-1}} \\ D_2\eta^{(k-1)}_{i_1i_2\dots i_{k-1}} \\ \vdots \\ D_n\eta^{(k-1)}_{i_1i_2\dots i_{k-1}} \end{pmatrix} - B\cdot \begin{pmatrix} u_{i_1i_2\dots i_{k-1}1} \\ u_{i_1i_2\dots i_{k-1}2} \\ \vdots \\ u_{i_1i_2\dots i_{k-1}n} \end{pmatrix}$$

where $i_l = 1,2,\dots,n$ *for* $l = 1,2,\dots,k-1$ *with* $k = 2,3,\dots$ *and an* $n\times n$ *matrix* $B = (D_i\xi_j)$*. In particular, we have the infinitesimals, up to order 2, given by*

$$(5)\quad \eta_1^{(1)} = \frac{\partial\eta}{\partial x_1} + \Big[\frac{\partial\eta}{\partial u} - \frac{\partial\xi_1}{\partial x_1}\Big]u_1 - \frac{\partial\xi_2}{\partial x_1}u_2 - \frac{\partial\xi_1}{\partial u}(u_1)^2 - \frac{\partial\xi_2}{\partial u}u_1u_2,$$

$$(6)\quad \eta_2^{(1)} = \frac{\partial\eta}{\partial x_2} + \Big[\frac{\partial\eta}{\partial u} - \frac{\partial\xi_2}{\partial x_2}\Big]u_2 - \frac{\partial\xi_1}{\partial x_2}u_1 - \frac{\partial\xi_2}{\partial u}(u_2)^2 - \frac{\partial\xi_1}{\partial u}u_1u_2,$$

$$(7)\quad \begin{aligned} \eta_{11}^{(2)} = {} & \frac{\partial^2\eta}{\partial x_1^2} + \Big[2\frac{\partial^2\eta}{\partial x_1\partial u} - \frac{\partial^2\xi_1}{\partial x_1^2}\Big]u_1 - \frac{\partial^2\xi_2}{\partial x_1^2}u_2 + \Big[\frac{\partial\eta}{\partial u} - 2\frac{\partial\xi_1}{\partial x_1}\Big]u_{11} \\ & - 2\frac{\partial\xi_2}{\partial x_1}u_{12} + \Big[\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi_1}{\partial x_1\partial u}\Big](u_1)^2 - 2\frac{\partial^2\xi_2}{\partial x_1\partial u}u_1u_2 \\ & - \frac{\partial^2\xi_1}{\partial u^2}(u_1)^3 - \frac{\partial^2\xi_2}{\partial u^2}(u_1)^2u_2 - 3\frac{\partial\xi_1}{\partial u}u_1u_{11} - \frac{\partial\xi_2}{\partial u}u_2u_{11} - 2\frac{\partial\xi_2}{\partial u}u_1u_{12}, \end{aligned}$$

$$(8)\quad \begin{aligned} \eta_{12}^{(2)} = \eta_{21}^{(2)} = {} & \frac{\partial^2\eta}{\partial x_1\partial x_2} + \Big[\frac{\partial^2\eta}{\partial x_1\partial u} - \frac{\partial^2\xi_2}{\partial x_1\partial x_2}\Big]u_2 + \Big[\frac{\partial^2\eta}{\partial x_2\partial u} - \frac{\partial^2\xi_1}{\partial x_1\partial x_2}\Big]u_1 \\ & - \frac{\partial\xi_2}{\partial x_1}u_{22} + \Big[\frac{\partial\eta}{\partial u} - \frac{\partial\xi_1}{\partial x_1} - \frac{\partial\xi_2}{\partial x_2}\Big]u_{12} - \frac{\partial\xi_1}{\partial x_2}u_{11} - \frac{\partial^2\xi_2}{\partial x_1\partial u}(u_2)^2 \\ & + \Big[\frac{\partial^2\eta}{\partial u^2} - \frac{\partial^2\xi_1}{\partial x_1\partial u} - \frac{\partial^2\xi_2}{\partial x_2\partial u}\Big]u_1u_2 - \frac{\partial^2\xi_1}{\partial x_2\partial u}(u_1)^2 - \frac{\partial\xi_2}{\partial u^2}u_1(u_2)^2 \\ & - \frac{\partial^2\xi_1}{\partial u^2}(u_1)^2u_2 - 2\frac{\partial\xi_2}{\partial u}u_2u_{12} - 2\frac{\partial\xi_1}{\partial u}u_1u_{12} \\ & - \frac{\partial\xi_1}{\partial u}u_2u_{11} - \frac{\partial\xi_2}{\partial u}u_1u_{22}, \end{aligned}$$

$$(9)\quad \begin{aligned} \eta_{22}^{(2)} = {} & \frac{\partial^2\eta}{\partial x_2^2} + \Big[2\frac{\partial^2\eta}{\partial x_2\partial u} - \frac{\partial^2\xi_2}{\partial x_2^2}\Big]u_2 - \frac{\partial^2\xi_1}{\partial x_2^2}u_1 + \Big[\frac{\partial\eta}{\partial u} - 2\frac{\partial\xi_2}{\partial x_2}\Big]u_{22} \\ & - 2\frac{\partial\xi_1}{\partial x_2}u_{12} + \Big[\frac{\partial^2\eta}{\partial u^2} - 2\frac{\partial^2\xi_2}{\partial x_2\partial u}\Big](u_2)^2 - 2\frac{\partial^2\xi_1}{\partial x_2}\partial u\, u_1u_2 \\ & - \frac{\partial^2\xi_2}{\partial u^2}(u_2)^3 - \frac{\partial^2\xi_1}{\partial u^2}u_1(u_2)^2 - 3\frac{\partial\xi_2}{\partial u}u_2u_{22} - \frac{\partial\xi_1}{\partial u}u_1u_{22} - 2\frac{\partial\xi_1}{\partial u}u_2u_{12}. \end{aligned}$$

Proof. For details, see [11]. □

Consider a kth order partial differential equation

$$F(x, u, u_{(1)}, u_{(2)}, \dots, u_{(k)}) = 0. \tag{10}$$

Definition 1. A one-parameter connected local Lie group of the transformations

$$\begin{aligned} \bar{\mathbf{x}} &= f(\mathbf{x}, u, \varepsilon), \\ \bar{u} &= U(\mathbf{x}, u, \varepsilon) \end{aligned} \tag{11}$$

is said to leave the partial differential equation (10) invariant if and only if its kth extension leaves the surface $F = 0$ invariant.

Proposition 2. *Let a one-parameter connected local Lie group of transformations be given having*

$$X = \sum_i \xi_i(\mathbf{x}, u) \frac{\partial}{\partial x_i} + \eta(\mathbf{x}, u) \frac{\partial}{\partial u} \tag{12}$$

as its infinitesimal generator with

$$\begin{aligned} X^{(k)} = {} & \sum_i \xi_i(\mathbf{x}, u) \frac{\partial}{\partial x_i} + \eta(\mathbf{x}, u) \frac{\partial}{\partial u} + \sum_i \eta_i^{(1)}(\mathbf{x}, u, u_{(1)}) \frac{\partial}{\partial x_i} \\ & + \dots + \sum_{i_1, \dots, i_k} \eta^{(k)}_{i_1, i_2, \dots, i_k}(\mathbf{x}, u, u_{(1)}, \dots, u_{(k)}) \frac{\partial}{\partial u_{i, i_2, \dots, i_k}} \end{aligned} \tag{13}$$

as the kth extended infinitesimal generator. The Lie group leaves equation (10) *invariant if and only if $X^{(k)}F = 0$ whenever $F = 0$.*

Proof. For details, see [11]. □

Definition 2. $u = \Theta(\mathbf{x})$ is called an invariant solution of $F = 0$ corresponding to a one-parameter connected local Lie group of transformations admitted by this equation if and only if

(i) $u = \Theta(\mathbf{x})$ is an invariant manifold of the Lie group,

(ii) $u = \Theta(\mathbf{x})$ solves $F = 0$.

Proposition 3. *Suppose that f is a function not depending on $u_{i_1\ldots i}$. A kth order partial differential equation ($k \geqslant 2$)*

$$u_{i_1\ldots i_l} = f(\mathbf{x}, u, u_{(1)}, u_{(2)}, \ldots, u_{(k)}) \tag{14}$$

admits an infinitesimal generator

$$X = \sum_i \xi_i(\mathbf{x}, u)\frac{\partial}{\partial x_i} + \eta(\mathbf{x}, u)\frac{\partial}{\partial u} \tag{15}$$

if and only if

$$\eta^{(l)}_{i_1\ldots i_l} = \sum \xi_j \frac{\partial f}{\partial x_j} + \eta\frac{\partial f}{\partial u} + \sum_j \eta^{(1)}_j \frac{\partial}{\partial u_j} + \ldots + \sum_{j_1,\ldots,j_k} \eta^{(k)}_{j_1\ldots j_k} \frac{\partial f}{\partial u_{j_1\ldots j_k}} \tag{16}$$

whenever $u_{i_1\ldots i_k} = f$. In addition,

(i) *$\eta^{(p)}_{j_1\ldots j_p}$ is linear in the components of $u_{(p)}$ if $p \geqslant 2$;*

(ii) *$\eta^{(p)}_{j_1\ldots j_p}$ is a polynomial in the components of $u_{(1)}, \ldots, u_{(p)}$ whose coefficients are linear homogeneous in ξ_i and η and in their partial derivatives with respect to $(\mathbf{x}, u)$ of orders up to p.*

P r o o f. For details, see [11]. □

If f is a polynomial in the components of $u_{(1)}, \ldots, u_{(k)}$, then the equation (16) is a polynomial equation in $u_{(1)}, \ldots, u_{(k)}$ whose coefficients are linear homogeneous in ξ_i, η and in their partial derivatives up to the kth order. Clearly, at any point $\mathbf{x}$, one can assign an arbitrary value to each u, $u_{(1)}, \ldots, u_{(k)}$, provided the partial differential equation $u_{i_1\ldots i_l} = f$ is satisfied; in other words, one can assign any values to u, $u_{(1)}, \ldots, u_{(k)}$ except to the coordinates $u_{i_1\ldots i_l}$. Therefore, after replacing $u_{i_1\ldots,i_l}$, the resulting polynomial equation must hold for arbitrary values of $u_{(1)}, \ldots, u_{(k)}$. Consequently, the coefficients of the polynomial must vanish separately, resulting in a system of linear homogeneous partial differential equations for $n+1$ functions ξ_i and η. This resulting system is called the set of determining equations for the infinitesimal generator X admitted by $u_{i_1\ldots i_l} = f$. In general, there are usually more than $n+1$ determining equations, hence the set of determining equations is an overdetermined system. For, when f is a non-polynomial function, one can still break up the equation

$$\eta^{(l)}_{i_1\ldots i_l} = \sum \xi_j \frac{\partial f}{\partial x_i} + \eta\frac{\partial f}{\partial u} + \ldots + \sum_{j_1\ldots j_k} \eta^{(k)}_{j_1\ldots j_k} \frac{\partial f}{\partial u_{j_1\ldots j_k}} \tag{17}$$

into a system of linear homogeneous partial differential equations for ξ_i and η by using similar arguments.

Proposition 4. *Suppose $u_{i_1\dots i_l} = f$ is a linear partial differential equation of order $k \geqslant 2$ which admits an infinitesimal generator*

$$X = \sum_i \xi_i \frac{\partial}{\partial x_i} + \eta \frac{\partial}{\partial u}. \tag{18}$$

Then

$$\frac{\partial \xi_i}{\partial u} = 0 \quad \text{for } i = 1, 2, \dots, n, \qquad \frac{\partial^2 \eta}{\partial u^2} = 0, \tag{19}$$

hence, for $n = 2$, the infinitesimal generator is of the form

$$X = \xi_1(x_1, x_2)\frac{\partial}{\partial x_1} + \xi_2(x_1, x_2)\frac{\partial}{\partial x_2} + [f(x_1, x_2)u + g(x_1, x_2)]\frac{\partial}{\partial u}. \tag{20}$$

According to Proposition 1*, we get*

$$\eta_1^{(1)} = \frac{\partial g}{\partial x_1} + \frac{\partial f}{\partial x_1}u + \Big[f - \frac{\partial \xi_1}{\partial x_1}\Big]u_1 - \frac{\partial \xi_2}{\partial x_1}u_2, \tag{21}$$

$$\eta_2^{(1)} = \frac{\partial g}{\partial x_2} + \frac{\partial f}{\partial x_2}u - \frac{\partial \xi_1}{\partial x_2}u_1, + \Big[f - \frac{\partial \xi_2}{\partial x_2}\Big]u_2, \tag{22}$$

$$\begin{aligned}\eta_{11}^{(2)} = {} & \frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1^2}u + \Big[2\frac{\partial f}{\partial x_1} - \frac{\partial^2 \xi_1}{\partial x_1^2}\Big]u_1 - \frac{\partial^2 \xi_2}{\partial x_1^2}u_2 \\ & + \Big[f - 2\frac{\partial \xi_1}{\partial x_1}\Big]u_{11} - 2\frac{\partial \xi_2}{\partial x_1}u_{12},\end{aligned} \tag{23}$$

$$\begin{aligned}\eta_{12}^{(2)} = \eta_{21}^{(2)} = {} & \frac{\partial^2 g}{\partial x_1 \partial x_2} + \frac{\partial^2 f}{\partial x_1 \partial x_2}u + \Big[\frac{\partial f}{\partial x_2} - \frac{\partial^2 \xi_1}{\partial x_1 \partial x_2}\Big]u_1 \\ & + \Big[\frac{\partial f}{\partial x_1} - \frac{\partial^2 \xi_2}{\partial x_1 \partial x_2}\Big]u_2 - 2\frac{\partial \xi_1}{\partial x_2}u_{11} + \Big[f - \frac{\partial \xi_1}{\partial x_1} - \frac{\partial \xi_2}{\partial x_2}\Big]u_{12} - \frac{\partial \xi_2}{\partial x_1}u_{22},\end{aligned} \tag{24}$$

$$\begin{aligned}\eta_{22}^{(2)} = {} & \frac{\partial^2 g}{\partial x_2^2} + \frac{\partial^2 f}{\partial x_2^2}u - \frac{\partial^2 \xi_1}{\partial x_2^2}u_1 + \Big[2\frac{\partial f}{\partial x_2} - \frac{\partial^2 \xi_2}{\partial x_2^2}\Big]u_2 \\ & - 2\frac{\partial \xi_1}{\partial x_2}u_{12} + \Big[f - 2\frac{\partial \xi_2}{\partial x_2}\Big]u_{22}.\end{aligned} \tag{25}$$

P r o o f. For details, see [12]. □

Consider a boundary value problem for a kth order partial differential equation in the form $F(\mathbf{x}, u, u_{(1)}, \dots, u_{(k)}) = 0$ defined on a domain $\Omega_{\mathbf{x}}$ in the $\mathbf{x}$-space with boundary conditions

$$B_\alpha(\mathbf{x}, u, u_{(1)}, \dots, u_{(k-1)}) = 0 \tag{26}$$

prescribed on the boundary surfaces

$$\omega_\alpha(\mathbf{x}) = 0, \tag{27}$$

where $\alpha = 1, 2, \dots, s$. From now on, we only deal with boundary values problems having unique solutions. Therefore, the invariant solution is precisely the unique solution.

Definition 3. An infinitesimal generator X is said to be admitted by the boundary value problem (26)–(27) if and only if

(i) $X^{(k)}F = 0$ whenever $F = 0$,
(ii) $X\omega_\alpha = 0$ whenever $\omega_\alpha = 0$ for $\alpha = 1, 2, \dots, s$,
(iii) $X^{(k-1)}B_\alpha = 0$ whenever $B_\alpha = 0$ on $\omega_\alpha = 0$ for $\alpha = 1, 2, \dots, s$.

Proposition 5. *Suppose that the boundary value problem* (26)–(27) *admits a one-parameter connected local Lie group of transformations. Let* $\Phi = (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \dots, \varphi_{n-1}(\mathbf{x}))$ *be* $n-1$ *independent group invariants of the Lie group depending only on* $\mathbf{x}$. *Let* $\nu(\mathbf{x}, u)$ *be a group invariant of the Lie group such that* $\partial\nu/\partial u \neq 0$. *Then* (26)–(27) *reduces to*

$$G(\Phi, \nu, \nu_{(1)}, \dots, \nu_{(k)}) = 0 \tag{28}$$

defined on some domain Ω_Φ *in the* Φ*-space with boundary conditions*

$$C_\alpha(\Phi, \nu, \nu_{(1)}, \dots, \nu_{(k-1)}) = 0 \tag{29}$$

prescribed on the boundary surfaces

$$\theta_\alpha(\Phi) = 0 \tag{30}$$

for some G, C_α, θ_α *for* $\alpha = 1, 2, \dots, s$. *In particular, if the infinitesimal generator is of the form*

$$X = \sum_i \xi_i(\mathbf{x})\frac{\partial}{\partial x_i} + f(\mathbf{x})u\frac{\partial}{\partial u}, \tag{31}$$

then $\nu = u/g(\mathbf{x})$ *for a known function* g *and hence an invariant solution arising from* X *is of the separated form*

$$u = g(\mathbf{x})\psi(\Phi) \tag{32}$$

for an arbitrary function ψ *of* $\Phi = (\varphi_1(\mathbf{x}), \varphi_2(\mathbf{x}), \dots, \varphi_{n-1}(\mathbf{x}))$.

Proof. For details, see [12]. □

3. European option pricing under CEV model

The Constant Elasticity of Variance (CEV) model with time-dependent model parameters for a standard European call option is described by the boundary value problem

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2(t)S^\theta\frac{\partial^2 V}{\partial S^2} + (r(t) - d(t))S\frac{\partial V}{\partial S} - r(t)V = 0 \tag{33}$$

with a boundary condition

$$V(S,T) = \delta(S - S_0) \tag{34}$$

prescribed on the boundary surface

$$t \leqslant T, \quad S \geqslant 0 \tag{35}$$

where δ is the Dirac δ-function, T is the expiry date and S_0 is the strike price.

First, for ease of calculation, we transform the boundary value problem to the standard form by incorporating the transformation

$$\overline{V} = V\mathrm{e}^{\beta(t)}, \quad \overline{S} = S\mathrm{e}^{\alpha(t)}, \quad \bar{t} = \gamma(t) \tag{36}$$

where α, β and γ are determined as follows:

$$\frac{\partial V}{\partial t} = \Big\{\Big(\dot{\gamma}\frac{\partial \overline{V}}{\partial \bar{t}} + \overline{S}\dot{\alpha}\frac{\partial \overline{V}}{\partial \overline{S}}\Big) - \dot{\beta}\overline{V}\Big\}\mathrm{e}^{-\beta(t)}, \tag{37}$$

$$\frac{\partial V}{\partial S} = \frac{\partial \overline{V}}{\partial \overline{S}}\mathrm{e}^{\alpha(t)}\mathrm{e}^{-\beta(t)}, \tag{38}$$

$$\frac{\partial^2 V}{\partial S^2} = \frac{\partial^2 \overline{V}}{\partial \overline{S}^2}(\mathrm{e}^{\alpha(t)})^2\mathrm{e}^{-\beta(t)}. \tag{39}$$

Substituting (37)–(39) into (33), we have

$$\dot{\gamma}\frac{\partial \overline{V}}{\partial \bar{t}} + \frac{1}{2}\sigma^2(\bar{t})\mathrm{e}^{(2-\theta)\alpha(\bar{t})}\overline{S}^\theta\frac{\partial^2 \overline{V}}{\partial \overline{S}^2} + (r - d + \dot{\alpha})\overline{S}\frac{\partial \overline{V}}{\partial \overline{S}} - (\dot{\beta} + r)\overline{V} = 0. \tag{40}$$

Choosing α, β and γ such that

$$\dot{\alpha} = -(r - d), \qquad \alpha = \int_t^T (r - d)\,\mathrm{d}t',$$

$$\dot{\beta} = -r, \qquad \beta = \int_t^T r\,\mathrm{d}t',$$

$$\dot{\gamma} = -\frac{1}{2}\sigma^2(t)\mathrm{e}^{(2-\theta)\alpha(t)}, \qquad \gamma = \int_t^T \frac{1}{2}\sigma^2(t')\mathrm{e}^{(2-\theta)\alpha(t')}\,\mathrm{d}t'.$$

Equation (40) can now be reduced to

$$\frac{\partial \overline{V}}{\partial \bar{t}} = \overline{S}^{\theta} \frac{\partial^2 \overline{V}}{\partial \overline{S}^2}. \tag{41}$$

In addition, the original boundary condition and the surface are now transformed to

$$\overline{V}(\overline{S}, 0) = \delta(\overline{S} - S_0) \tag{42}$$

and

$$\bar{t} \geqslant 0, \quad \overline{S} \geqslant 0, \tag{43}$$

respectively.

For the sake of reference, we replace $\overline{V}$ by u, $\overline{S}$ by x_1 and $\bar{t}$ by x_2 in (41), i.e.

$$\frac{\partial u}{\partial x_2} = x_1^{\theta} \frac{\partial^2 u}{\partial x_1^2}, \quad \text{or} \quad u_2 = x_1^{\theta} u_{11}. \tag{44}$$

According to Proposition 3, the system of determining equations can be found from

$$\eta_2^{(1)} = \theta x_1^{\theta-1} \xi_1 u_{11} + x_1^{\theta} \eta_{11}^{(2)}. \tag{45}$$

According to Proposition 4, $\eta_2^{(1)}$, $\eta_{11}^{(2)}$ and the infinitesimal generator L are given by

$$\eta_2^{(1)} = \frac{\partial g}{\partial x_2} + \frac{\partial f}{\partial x_2} u - \frac{\partial \xi_1}{\partial x_2} u_1 + \left[f - \frac{\partial \xi_2}{\partial x_2}\right] u_2, \tag{46}$$

$$\begin{aligned} \eta_{11}^{(2)} = {} & \frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1^2} u + \left[2\frac{\partial f}{\partial x_1} - \frac{\partial^2 \xi_1}{\partial x_1^2}\right] u_1 - \frac{\partial^2 \xi_2}{\partial^2 x_1^2} u_2 \\ & + \left[f - 2\frac{\partial \xi_1}{\partial x_1}\right] u_{11} - 2\frac{\partial \xi_2}{\partial x_1} u_{12}, \end{aligned} \tag{47}$$

$$L = \xi_1 \frac{\partial}{\partial x_1} + \xi_2 \frac{\partial}{\partial x_2} + (f \cdot u + g) \frac{\partial}{\partial u}. \tag{48}$$

Substituting (46)–(47) into (45), we get

$$\begin{aligned}
&\frac{\partial g}{\partial x_2} + \frac{\partial f}{\partial x_2}u - \frac{\partial \xi_1}{\partial x_2}u_1 + \Big[f - \frac{\partial \xi_2}{\partial x_2}\Big]u_2 \\
&= \theta x_1^{\theta-1}\xi_1 u_{11} + x_1^\theta\Big\{\frac{\partial^2 g}{\partial x_1^2} + \frac{\partial^2 f}{\partial x_1^2}u + \Big[2\frac{\partial f}{\partial x_1} - \frac{\partial^2 \xi_1}{\partial x_1^2}\Big]u_1 - \frac{\partial^2 \xi_2}{\partial x_1^2}u_2 \\
&\quad + \Big[f - 2\frac{\partial \xi_1}{\partial x_1}\Big]u_{11} - 2\frac{\partial \xi_2}{\partial x_1}u_{12}\Big\},\\
0 &= \Big[x_1^\theta\frac{\partial^2 g}{\partial x_1^2} - \frac{\partial g}{\partial x_2}\Big] + \Big[x_1^\theta\frac{\partial^2 f}{\partial x_1^2} - \frac{\partial f}{\partial x_2}\Big]u \\
&\quad + \Big[2x_1^\theta\frac{\partial f}{\partial x_1} - x_1^\theta\frac{\partial^2 \xi_1}{\partial x_1^2} + \frac{\partial \xi_1}{\partial x_2}\Big]u_1 + \Big[-x_1^\theta\frac{\partial^2 \xi_2}{\partial x_1^2} - \Big[f - \frac{\partial \xi_2}{\partial x_2}\Big]\Big]u_2 \\
&\quad + \Big[\theta x_1^{\theta-1}\xi_1 + x_1^\theta f - 2x_1^\theta\frac{\partial \xi_1}{\partial x_1}\Big]u_{11} + \Big[-2x_1^\theta\frac{\partial \xi_2}{\partial x_1}\Big]u_{12}.
\end{aligned}$$

Since $u_2 = x_1^\theta u_{11}$, equation (45) is equivalent to

$$\begin{aligned}
(49)\quad 0 &= \Big[x_1^\theta\frac{\partial^2 g}{\partial x_1^2} - \frac{\partial g}{\partial x_2}\Big] + \Big[x_1^\theta\frac{\partial^2 f}{\partial x_1^2} - \frac{\partial f}{\partial x_2}\Big]u + \Big[2x_1^\theta\frac{\partial f}{\partial x_1} - x_1^\theta\frac{\partial^2 \xi_1}{\partial x_1^2} + \frac{\partial \xi_1}{\partial x_2}\Big]u_1 \\
&\quad + \Big[-x_1^{2\theta}\frac{\partial^2 \xi_2}{\partial x_1^2} + x_1^\theta\frac{\partial \xi_2}{\partial x_2} + \theta x_1^{\theta-1}\xi_1 - 2x_1^\theta\frac{\partial \xi_1}{\partial x_1}\Big]u_{11} + \Big[-2x_1^\theta\frac{\partial \xi_2}{\partial x_1}\Big]u_{12}.
\end{aligned}$$

Equating the coefficients of u and the derivatives of u to zero, we get the system of determining equations

(i) $\dfrac{\partial g}{\partial x_2} = x_1^\theta\dfrac{\partial^2 g}{\partial x_1^2}$,

(ii) $\dfrac{\partial f}{\partial x_2} = x_1^\theta\dfrac{\partial^2 f}{\partial x_1^2}$,

(iii) $2x_1^\theta\dfrac{\partial f}{\partial x_1} = x_1^\theta\dfrac{\partial^2 \xi_1}{\partial x_1^2} - \dfrac{\partial \xi_1}{\partial x_2}$,

(iv) $\theta x_1^{\theta-1}\xi_1 - 2x_1^\theta\dfrac{\partial \xi_1}{\partial x_1} = x_1^{2\theta}\dfrac{\partial^2 \xi_2}{\partial x_1^2} - x_1^\theta\dfrac{\partial \xi_2}{\partial x_2}$,

(v) $\dfrac{\partial \xi_2}{\partial x_1} = 0$.

Solving this system, we get

$$(50)\qquad \xi_1 = \frac{1}{2-\theta}(c_1x_2 + c_2)x_1, \qquad \xi_2 = c_1\frac{x_2^2}{2} + c_2x_2 + c_3,$$

$$(51)\qquad f = -\frac{1}{2}\Big(\frac{1}{2-\theta}\Big)^2 c_1x_1^{2-\theta} - \frac{1}{2}\Big(\frac{1-\theta}{2-\theta}\Big)c_1x_2 + c_4, \qquad g = 0,$$

where c_1, c_2, c_3 and c_4 are undetermined constants.

The invariance of the boundary condition and the surfaces imposes further restriction on the constants c_i's:

(i) The condition $x_1 > 0$ implies $\xi_1(0, x_2) = 0 \Rightarrow$ no restriction.

(ii) The condition $x_2 > 0$ implies $\xi_2(x_1, 0) = 0 \Rightarrow c_3 = 0$.

(iii) The condition $u(x_1, 0) = \delta(x_1 - \hat{x}_1)$, where $0 < \hat{x}_1 < \infty$ implies

$$
\begin{aligned}
f(x_1,0)u(x_1,0) &= \xi_1(x_1,0)\delta'(x_1-\hat{x}_1) \\
\Rightarrow f(x_1,0)\delta(x_1-\hat{x}_1) &= \xi_1(x_1,0)\delta'(x_1-\hat{x}_1).
\end{aligned} \tag{52}
$$

Equation (52) is satisfied if

(i) $\xi_1(\hat{x}_1, 0) = 0$ implies $(2-\theta)^{-1} c_2 \hat{x}_1 = 0$, i.e. $c_2 = 0$;

(ii) $f(\hat{x}_1, 0) = -\dfrac{\partial \xi_1}{\partial x_1}(\hat{x}_1, 0) = -\dfrac{1}{2-\theta} c_1(0) = 0$, therefore

$$
-\frac{1}{2}\Big(\frac{1}{2-\theta}\Big)^2 c_1 \hat{x}_1^{2-\theta} + c_4 = 0, \quad \text{i.e. } c_4 = \frac{1}{2}\Big(\frac{1}{2-\theta}\Big)^2 c_1 \hat{x}_1^{2-\theta}.
$$

Hence, we have

$$
\xi_1 = \frac{1}{2-\theta}(c_1 x_1 x_2), \quad \xi_2 = c_1 \frac{x_2^2}{2}, \tag{53}
$$

$$
f = \bigg(\frac{1}{2}\Big(\frac{1}{2-\theta}\Big)^2 \hat{x}_1^{2-\theta} - \frac{1}{2}\frac{1-\theta}{2-\theta} x_2 - \frac{1}{2}\Big(\frac{1}{2-\theta}\Big)^2 x_1^{2-\theta}\bigg) c_1 \tag{54}
$$

and the infinitesimal generator

$$
\begin{aligned}
L = {} & \frac{1}{2-\theta} x_1 x_2 \frac{\partial}{\partial x_1} + \frac{x_2^2}{2}\frac{\partial}{\partial x_2} \\
& + \bigg(\frac{1}{2}\Big(\frac{1}{2-\theta}\Big)^2 \hat{x}_1^{2-\theta} - \frac{1}{2}\Big(\frac{1-\theta}{2-\theta}\Big) x_2 - \frac{1}{2}\Big(\frac{1}{2-\theta}\Big)^2 x_1^{2-\theta}\bigg) u \frac{\partial}{\partial u}.
\end{aligned} \tag{55}
$$

According to Proposition 5, the corresponding invariant solution is

$$
u = \frac{1}{x_2^{(1-\theta)/(2-\theta)}} \exp\bigg[-\Big(\frac{1}{2-\theta}\Big)^2 \frac{1}{x_2}\big(\hat{x}_1^{2-\theta} + x_1^{2-\theta}\big)\bigg] F\Big(\frac{x_1^{(2-\theta)/2}}{x_2}\Big). \tag{56}
$$

Denote $x_1^{(2-\theta)/2}/x_2$ by z. Now, the partial derivatives of u can be rewritten as

$$\begin{aligned}
\frac{\partial u}{\partial x_2} &= \frac{1}{x_2^{(1-\theta)/(2-\theta)}}\exp\Big[-\Big(\frac{1}{2-\theta}\Big)^2\frac{1}{x_2}(\hat{x}_1^{2-\theta}+x_1^{2-\theta})\Big] \\
&\quad\times\Big\{-\frac{1-\theta}{2-\theta}\frac{1}{x_2}F+\Big(\frac{1}{2-\theta}\Big)^2\Big(\frac{\hat{x}_1^{2-\theta}+x_1^{2-\theta}}{x_2^2}\Big)F-\frac{x_1^{(2-\theta)/2}}{x_2^2}F'\Big\},
\end{aligned} \tag{57}$$

$$\begin{aligned}
\frac{\partial u}{\partial x_1} &= -\frac{1}{2-\theta}\frac{1}{x_2^{(1-\theta)/(2-\theta)+1}}x_1^{1-\theta}\exp\Big[-\Big(\frac{1}{2-\theta}\Big)^2\frac{1}{x_2}(\hat{x}_1^{2-\theta}+x_1^{2-\theta})\Big]\cdot F \\
&\quad+\frac{2-\theta}{2}\frac{1}{x_2^{(1-\theta)/(2-\theta)+1}}x_2^{(2-\theta)/2-1} \\
&\quad\times\exp\Big[-\Big(\frac{1}{2-\theta}\Big)^2\frac{1}{x_2}(\hat{x}_1^{2-\theta}+x_1^{2-\theta})\Big]\cdot F',
\end{aligned} \tag{58}$$

$$\begin{aligned}
x_1^\theta\frac{\partial^2 u}{\partial x_1^2} &= \frac{1}{x_2^{(1-\theta)/(2-\theta)}}\exp\Big[-\Big(\frac{1}{2-\theta}\Big)^2\frac{1}{x_2}(\hat{x}_1^{2-\theta}+x_1^{2-\theta})\Big] \\
&\quad\times\Big\{-\frac{1-\theta}{2-\theta}\frac{1}{x_2}F+\Big(\frac{1}{2-\theta}\Big)^2\frac{1}{x_2^2}x_1^{2-\theta}F-\frac{1}{x_2^2}x_1^{(2-\theta)/2}F' \\
&\qquad+\frac{2-\theta}{2}\Big(\frac{2-\theta}{2}-1\Big)\frac{1}{x_2}x_1^{-(2-\theta)/2}F'+\Big(\frac{2\cdot\theta}{2}\Big)^2\frac{1}{x_2^2}F''\Big\}.
\end{aligned} \tag{59}$$

Substituting (58) and (59) into (44), we get

$$F''+\Big(1-\frac{1}{2-\theta}\Big)\frac{1}{z}F'-\hat{x}_1^{2-\theta}\cdot F=0, \tag{60}$$

which is a modified Bessel equation of the second type. Its solution can be readily found in any standard table of Bessel functions. For a general discussion on Bessel equations, see [13]. Therefore, the explicit pricing formula for a European call option is

$$\begin{aligned}
P_c(S,t) &= S\mathrm{e}^{-\int_t^T d(t')\,\mathrm{d}t'}\sum_{n=0}^{\infty}\frac{z^n\mathrm{e}^{-z}}{\Gamma(n+1)}G\Big(n+1+\frac{1}{2-\theta},\omega\Big) \\
&\quad-S_0\mathrm{e}^{-\int_t^T r(t')\,\mathrm{d}t'}\sum_{n=0}^{\infty}\frac{z^{n+(2-\theta)^{-1}}\mathrm{e}^{-z}}{\Gamma\big(n+1+(2-\theta)^{-1}\big)}G(n+1,\omega)
\end{aligned} \tag{61}$$

where

$$z=\frac{S^{2-\theta}\mathrm{e}^{(2-\theta)\alpha}}{(2-\theta)^2\gamma},\quad \omega=\frac{S_0^{(2-\theta)}}{(2-\theta)^2\gamma},\quad G(a,\omega)=\frac{1}{\Gamma(a)}\int_\omega^\infty\zeta^{a-1}\mathrm{e}^{-\zeta}\,\mathrm{d}\zeta. \tag{62}$$

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