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WEAK SOLUTIONS TO A NONLINEAR VARIATIONAL WAVE EQUATION AND SOME RELATED PROBLEMS

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Abstract. In this paper we present some results on the global existence of weak solutions to a nonlinear variational wave equation and some related problems. We first introduce the main tools, the L^p Young measure theory and related compactness results, in the first section. Then we use the L^p Young measure theory to prove the global existence of dissipative weak solutions to the asymptotic equation of the nonlinear wave equation, and comment on its relation to Camassa-Holm equations in the second section. In the third section, we prove the global existence of weak solutions to the original nonlinear wave equation under some restrictions on the wave speed. In the last section, we present global existence of renormalized solutions to two-dimensional model equations of the asymptotic equation, which is also the so-called vortex density equation arising from sup-conductivity.

Keywords: variational wave equation, weak solutions, L^p Young measure, renormalized solutions

MSC 2000: 35D05, 35L05

1. L^p Young measure and some related tools

Young measure for bounded sequence was introduced by Young [33]. This notion was used to prove the existence of weak solutions by Tartar [30] for a single conservation law, and by DiPerna [5] for 2×2 conservation laws. The definition was then extended to sequences in L^p for $1 < p < \infty$ by DiPerna and Majda in [8]. However, in order to use this tool to establish the existence of weak solutions to a nonlinear pde, it is useful to consider the part of the Young measure from [8] located at finite distance. For the convenience of the reader, let us first recall the following theorem from [20]:

Theorem 1.1. *Let U be an open subset of $\mathbb{R}^d$ whose boundary has zero Lebesgue measure. Given a bounded family $\{u^\varepsilon(y)\} \subset L^p(U)$, $p > 1$, of $\mathbb{R}^N$-valued functions, there exist a subsequence $\{\varepsilon_j\}$ and a measurable family of probability measures* *on $\mathbb{R}^N$, $\{\mu_y(\cdot),\, y \in U\}$, such that for all continuous functions $F(\lambda,\tau)$ with $F(\lambda,\tau) = O(|\lambda|^q + |\tau|^q)$ as $|\lambda| + |\tau| \to \infty$ and $q < p$,*

$$\lim_{\varepsilon_j \to 0} \int_U \varphi(y) F(u^{\varepsilon_j}(y), a(y))\,\mathrm{d}y = \int_U\int_{\mathbb{R}^N} \varphi(y) F(\lambda, a(y))\,\mathrm{d}\mu_y(\lambda)\,\mathrm{d}y$$

holds for all $\varphi(y) \in L^r(U)$ with compact support in the closure of U, where $r^{-1} + qp^{-1} = 1$ and $a(y) \in L^p(U)$.

Proof. The proof of the above theorem can be found in [30], [8] and [11], and is omitted here. □

Definition 1.1. A bounded sequence u^ε in $L^p(U)$ is "pure" when no extraction of a subsequence is necessary.

Proposition 1.1. *Let u^ε be a pure bounded sequence in $L^p(U)$ with Young measure μ. Then $\lambda \in L^p(U \times \mathbb{R}^N, \mu)$ and*

$$\iint |\lambda|^p\,\mathrm{d}\mu_y(\lambda)\,\mathrm{d}y \leqslant \underline{\lim}_{\varepsilon_j \to 0} \|u^{\varepsilon_j}(y)\|^p_{L^p}.$$

Proof. Let $\zeta(\lambda) \in C_0^\infty(\mathbb{R}^N)$ be such that $\zeta(\lambda) = 1$ for $|\lambda| \leqslant 1$ and $0 \leqslant \zeta(\lambda) \leqslant 1$. Thanks to Theorem 1.1, one has

$$\begin{aligned}\int_U\int_{\mathbb{R}^N} \zeta\Big(\frac{y}{k}\Big)\zeta\Big(\frac{\lambda}{k}\Big)|\lambda|^p\,\mathrm{d}\mu_y(\lambda)\,\mathrm{d}y &= \lim_{\varepsilon \to 0}\int_U \zeta\Big(\frac{y}{k}\Big)\zeta\Big(\frac{u^\varepsilon(y)}{k}\Big)|u^\varepsilon(y)|^p\,\mathrm{d}y \\ &\leqslant \underline{\lim}_{\varepsilon_j \to 0}\|u^{\varepsilon_j}(y)\|^p_{L^p}.\end{aligned}$$

Fatou's Lemma gives

$$\begin{aligned}\int_U\int_{\mathbb{R}^N} |\lambda|^p\,\mathrm{d}\mu_y(\lambda)\,\mathrm{d}y &\leqslant \lim_{k \to \infty}\int_U\int_{\mathbb{R}^N} \zeta\Big(\frac{y}{k}\Big)\zeta\Big(\frac{\lambda}{k}\Big)|\lambda|^p\,\mathrm{d}\mu_y(\lambda)\,\mathrm{d}y \\ &\leqslant \underline{\lim}_{\varepsilon_j \to 0}\|u^{\varepsilon_j}(y)\|^p_{L^p},\end{aligned}$$

which completes the proof of the proposition. □

Proposition 1.2. *Suppose that u^ε is a pure bounded sequence in $L^p(U)$ with Young measure μ_y and $a \in L^p(U)$. Then $v^\varepsilon := u^\varepsilon - a$ is again a pure sequence in $L^p(U)$, and the associated Young measure ν_y is given by*

$$\int_U \varphi(y)\int_{\mathbb{R}^N} f(\lambda)\,\mathrm{d}\nu_y(\lambda)\,\mathrm{d}y = \int_U \varphi(y)\int_{\mathbb{R}^N} f(\lambda - a(y))\,\mathrm{d}\mu_y(\lambda)\,\mathrm{d}y$$

for all $\varphi(y) \in C_0^\infty(U)$ and $f \in C_0^0(\mathbb{R}^N)$.

P r o o f. Thanks to Theorem 1.1, we have

$$\begin{aligned}\int_U \varphi(y) \int_{\mathbb{R}^N} f(\lambda)\,\mathrm{d}\nu_y(\lambda)\,\mathrm{d}y &= \lim_{\varepsilon\to 0}\int_U \varphi(y) f(u^\varepsilon(y)-a(y))\,\mathrm{d}y \\ &= \int_U \varphi(y)\int_{\mathbb{R}^N} f(\lambda - a(y))\,\mathrm{d}\mu_y(\lambda)\,\mathrm{d}y\end{aligned}$$

from which we conclude the proof of the proposition. □

Proposition 1.3. *Suppose that u^ε is a pure bounded sequence in $L^p(U)$ with Young measure μ_y. Let $\bar u(y) := \int_{\mathbb{R}^N}\lambda\,\mathrm{d}\mu_y(\lambda)$. Then $\mu_y(\lambda) = \delta(\lambda - \bar u(y))$ for a.e. $y\in U$ iff $u^\varepsilon \to \bar u$ strongly in $L^q_{\mathrm{loc}}(U)$ for any $q<p$.*

P r o o f. Again thanks to Theorem 1.1, for any $q<p$ we have

$$\lim_{\varepsilon\to 0}\int_U \varphi(y)|u^\varepsilon(y)-\bar u(y)|^q\,\mathrm{d}y = \int_U\varphi(y)\int_{\mathbb{R}^N}|\lambda-\bar u(y)|^q\,\mathrm{d}\mu_y(\lambda)\,\mathrm{d}y$$

for all $\varphi(y)\in C^0_0(U)$. Then if $u^\varepsilon\to\bar u(y)$ strongly in $L^q_{\mathrm{loc}}(U)$, we obtain

$$\int_U\varphi(y)\int_{\mathbb{R}^N}|\lambda-\bar u(y)|^q\,\mathrm{d}\mu_y(\lambda)\,\mathrm{d}y = 0 \quad \text{for all } \varphi(y)\in C^0_0(U),$$

which implies that $\mu_y(\lambda)=\delta(\lambda-\bar u(y))$ for a.e. $y\in U$. Similarly, when $\mu_y(\lambda) = \delta(\lambda-\bar u(y))$ for a.e. $y\in U$, one concludes that $u^\varepsilon\to\bar u$ strongly in $L^q_{\mathrm{loc}}(U)$ for any $q<p$. □

In order to apply the above mentioned L^p Young measure theory to system of equations, one may need the so-called Div-Curl Lemma from Murat [28], the key insight of which is that if we have enough information concerning various combinations of derivatives, we can sometimes show that certain nonlinear functions are weakly continuous.

Theorem 1.2 (Div-Curl Lemma)**.** *Assume that $\{v_n\}$, $\{w_n\}$ are two bounded sequences in $L^2_{\mathrm{loc}}(U,\mathbb{R}^N)$ such that*

i) *$\{\operatorname{div} v_n\}$ lies in a compact subset of $W^{-1,2}_{\mathrm{loc}}(U)$;*

ii) *$\{\operatorname{curl} w_n\}$ lies in a compact subset of $W^{-1,2}_{\mathrm{loc}}(U;M^{N\times N})$, where*

$$(\operatorname{curl} w_n)_{ij} = \partial_{x_j} w^i_n - \partial_{x_i} w^j_n, \quad (1\leqslant i,\, j\leqslant N).$$

Suppose further that $v_n \rightharpoonup \bar v$, $w_n\rightharpoonup \overline{w}$ in $L^2_{\mathrm{loc}}(U;\mathbb{R}^N)$. Then

$$v_n\cdot w_n \rightharpoonup \bar v\cdot\overline{w}$$

in the sense of distributions.

P r o o f. The proof of the above theorem can be found in [28], [11], and is omitted here. □

Let us conclude this section by recalling the following mollification lemma from [6]:

Proposition 1.4. *Let j_ε be a regularizing kernel:*

$$j_\varepsilon = \frac{1}{\varepsilon^d} j\Big(\frac{\cdot}{\varepsilon}\Big) \quad \text{with } 0 \leqslant j \in \mathcal{D}(\mathbb{R}^d), \ \int_{\mathbb{R}^d} j \,\mathrm{d}x = 1, \ \varepsilon > 0.$$

Let $B \in L^1((0,T); (W^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^d))^d)$, $w \in L^\infty((0,T); L^p_{\mathrm{loc}}(\mathbb{R}^d))$. Then

$$(B \cdot \nabla w) * j_\varepsilon - B \cdot \nabla (w * j_\varepsilon) \to 0 \quad \text{in } L^1((0,T); L^\beta_{\mathrm{loc}}(\mathbb{R}^d)),$$

where β is given by $\beta^{-1} = \alpha^{-1} + p^{-1}$.

2. Asymptotic equation of a nonlinear variational wave equation and Camassa-Holm equation

2.1. Existence of dissipative solutions to the asymptotic equation

In this sub-section, we establish the global existence of admissible dissipative weak solutions to the initial-boundary value problem

$$\begin{cases} \partial_t v + u \partial_x v = -\frac{1}{2} v^2, & x > 0, \ t > 0, \\ \partial_x u = v(t,x), \\ u(t,x)|_{x=0} = 0, \\ v|_{t=0} = v_0(x), \end{cases} \tag{2.1}$$

where $v_0(x) \in L^2(\mathbb{R}^+)$. The two equations in (2.1) are normally written as one equation which is referred to as the asymptotic equation (see [17]). It governs uni-directional and weakly nonlinear solutions of a class of hyperbolic variational equations

$$\partial_t^2 u - c\partial_x(c\partial_x u) = 0, \quad (c = c(u) > 0).$$

We use the notation $\mathbb{R}^+ := (0,\infty)$. We recall that Hunter and Zheng [18] established the global existence of both the dissipative and conservative weak solutions to (2.1) with initial data $v_0(x) \in BV(\mathbb{R}^+)$. We [35] established the global existence of dissipative weak solutions to (2.1) with general initial data $v_0(x) \in L^2(\mathbb{R}^+)$. Due to the space restriction, we only present the part of [35] on the global existence of admissible dissipative weak solutions to (2.1) by applying the L^p Young measure theory of the last section.

Let us first give the definition of admissible dissipative weak solutions. Let $Q_\infty := [0,\infty) \times [0,\infty)$.

Definition 2.1 (Admissible dissipative weak solutions). We call $(v(t,x),u(t,x))$ an admissible dissipative weak solution of (2.1) if

(d1) the functions have the regularity

$$(v,u)(t,x) \in L^\infty_{\rm loc}(\mathbb{R}^+, L^2(\mathbb{R}^+)) \otimes C(Q_\infty);$$

(d2) the functions satisfy in the sense of distributions the equations

$$\partial_t v + \partial_x(uv) = \frac{1}{2}v^2, \quad \partial_x u = v;$$

(d3) the energy $\int_{\mathbb{R}^+} v^2(t,x)\,\mathrm{d}x$ is non-increasing in $t \in [0,\infty)$;

(d4) the function $u(t,x)$ is equal to zero at $x=0$ as a continuous function. The function $v(t,x)$ takes on the initial value $v_0(x)$ in the sense of $C([0,\infty), L^2(\mathbb{R}^+))$;

(d5) the entropy condition holds:

$$v(t,x) \leqslant \frac{2}{t}, \quad \text{a.e. } (t,x) \in Q_\infty.$$

Theorem 2.1 (Dissipative solutions). *Let $v_0(x) \in L^2(\mathbb{R}^+)$ have compact support. Then (2.1) has a unique global admissible dissipative weak solution (v,u) in the sense of Definition 2.1. In addition, the solution satisfies $v \in L^p_{\rm loc}(Q_\infty)$, $u \in W^{1,p}_{\rm loc}(Q_\infty)$ for all $p<3$, $v(t,x) \in C([0,\infty), L^q(\mathbb{R}^+))$ for all $q<2$, and $v(t,x) \in C_+([0,\infty), L^2(\mathbb{R}^+))$ (the right continuity).*

Proof. As the proof is rather too long, we divide the proof into the following main steps:

Step 1. Approximate solutions

As in [18], we solve (2.1) with simple functions as initial data. Without loss of generality, we assume that $\operatorname{supp} v_0 \subset [0,1)$. We approximate $v_0(x)$ by step functions $\{v_0^n(x)\}$ defined by

$$v_0^n(x) = v_i^n := n\int_{\frac{i-1}{n}}^{\frac{i}{n}} v_0(y)\,\mathrm{d}y, \quad x \in \Big[\frac{i-1}{n}, \frac{i}{n}\Big), \;\; i = 1,2\ldots,n.$$

Without loss of generality, we assume that every point of $[0,1]$ is a Lebesgue point of $v_0(x)$, thus

$$\lim_{n\to\infty} \|v_0^n - v_0\|_{L^2([0,1])} = 0 \quad \text{and} \quad \lim_{n\to\infty} v_0^n(x) = v_0(x), \;\; \text{for all } x \in [0,1]. \tag{2.2}$$

From [18] or by directly applying the characteristic method, we obtain admissible dissipative weak solutions

$$v^n(t,x) = \frac{2v_i^n}{2+v_i^n t}, \quad x_{i-1}^n(t) \leqslant x < x_i^n(t) \tag{2.3}$$

where $x_0^n(t) := 0$ and

$$x_i^n(t) := \frac{1}{n}\sum_{j=1}^{i}\Big(1+\frac{1}{2}v_j^n t\Big)^2 1_{\{2+v_j^n t\geqslant 0\}}. \tag{2.4}$$

The associated u^n follows by integrating v^n.

Step 2. Primitive estimates and pre-compactness

Lemma 2.1 (primitive estimates)**.** *For all $p \in [2,3)$, $T > 0$ and $R > 0$, the approximate solution sequence $\{v^n, u^n\}$ constructed above satisfies the estimates*

(a) $v^n(t,x) \leqslant 2\cdot t^{-1}$,

(b) $\|v^n(t_2,\cdot)\|_{L^2(\mathbb{R}^+)} \leqslant \|v^n(t_1,\cdot)\|_{L^2(\mathbb{R}^+)} \leqslant \|v_0^n\|_{L^2([0,1])}$, $0 < t_1 < t_2$,

(c) $\|v^n\|^p_{L^p([0,T]\times\mathbb{R}^+)} \leqslant C_{T,p}\|v_0^n\|^2_{L^2([0,1])}$.

Moreover, $\{u^n(t,x)\}$ are uniformly bounded in $W^{1,p}_{\mathrm{loc}}(Q_\infty)$.

P r o o f. The proof can be found on p. 329 of [18]. The third inequality (c) can also be deduced from Theorem 3 of [34]. $\square$

Lemma 2.2 (Basic pre-compactness)**.** *There exist $u \in W^{1,p}_{\mathrm{loc}}(Q_\infty)$ for all $p < 3$ and a subsequence of $\{u^n\}$ which we still denote by $\{u^n\}$ such that*

$$u^n(t,x) \longrightarrow u(t,x)$$

uniformly on any compact subset of Q_∞. Moreover,

$$v^n(t,x) = \partial_x u^n(t,x) \rightharpoonup \partial_x u(t,x) =: v(t,x)$$

weakly in $L^p_{\mathrm{loc}}(Q_\infty)$ for all $p < 3$. Further,

$$\|u(t,\cdot)\|_{L^\infty} \leqslant \int_0^1 |v_0(x)|\,\mathrm{d}x + \frac{t}{2}\int_0^1 |v_0|^2\,\mathrm{d}x$$

and

$$\operatorname{supp} v^n(t,\cdot) \subset [0, K(t))$$

for some $K(t) < \infty$.

P r o o f. The basic convergence follows directly from Lemma 2.1, and thanks to the construction of the approximate solutions, we have

$$\begin{aligned}\|u^n(t,\cdot)\|_{L^\infty} &\leqslant \int_0^\infty |v^n(t,x)|\,\mathrm{d}x\\ &= \frac{1}{n}\sum_{i=1}^n \frac{2|v_i^n|}{|2+v_i^n t|}\Big(1+\frac{1}{2}v_i^n t\Big)^2 1_{\{2+v_i^n t\geqslant 0\}}\\ &\leqslant \int_0^1 |v_0^n(x)|\,\mathrm{d}x + \frac{t}{2}\int_0^1 |v_0^n(x)|^2\,\mathrm{d}x \leqslant C(t)\end{aligned}$$

for some locally bounded function $C(t)$. Finally, a similar argument together with (2.4) gives the support properties of $v^n(t,\cdot)$. □

In the sequel, we shall use the notation $Q_T := [0,T]\times[0,K(T)]$. In particular, $Q_\infty = [0,\infty)\times[0,\infty)$ is consistent with our earlier notation.

Step 3. Strong pre-compactness

We prove pre-compactness of the solution sequence $\{v^n(t,x)\}$ in $L^p([0,\,T]\times\mathbb{R}^+)$ for all $T<\infty$, $p<3$, by applying the Young measure theory ([8], [11], [30], [33]), the ideas used by P.-L. Lions [24] in the proof of the global existence of weak solutions to multi-dimensional Navier-Stokes equations, and the ideas used by Joly, Metivier and Rauch [20] in the rigorous justification of the weakly nonlinear geometric optics for semilinear wave equations. Thanks to Theorem 1.1 and Lemma 2.1, we immediately have

Lemma 2.3 (Time-distinguished Young measures)**.** *There exist a subsequence of the solution sequence $\{v^n(t,x)\}$, for convenience still denoted by $\{v^n(t,x)\}$, and a family of Young measures $\mu_{t,x}(\lambda)$ such that for all continuous functions $f(t,x,\lambda) = O(|\lambda|^q)$ we have $\partial_\lambda f(t,x,\lambda) = O(|\lambda|^{q-1})$ as $\lambda\to\infty$ for $q<2$, and for all $\psi(x)\in L_c^r([0,\infty))$ with $r^{-1}+\frac{1}{2}q=1$, we have*

$$\lim_{n\to\infty}\int_{\mathbb{R}^+} f(t,x,v^n(t,x))\psi(x)\,\mathrm{d}x = \int_{\mathbb{R}^+}\overline{f(t,x,v)}\psi(x)\,\mathrm{d}x$$

uniformly in every compact subset of $[0,\infty)$*, where*

$$\overline{f(t,x,v)} := \int_{\lambda\in\mathbb{R}} f(t,x,\lambda)\,\mathrm{d}\mu_{t,x}(\lambda) \in C([0,\infty); L^{\frac{q'}{q}}(\mathbb{R}^+))$$

for all $q' \in (q, 2)$. *Moreover, for all* $T > 0$,

$$\lim_{n\to\infty} \int_0^T \int_{\mathbb{R}+} g(t,x,v^n)\varphi \,\mathrm{d}x\,\mathrm{d}t = \int_0^T \int_{\mathbb{R}+} \overline{g(t,x,v)}\varphi \,\mathrm{d}x\,\mathrm{d}t$$

and

$$\lambda \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^+ \times \mathbb{R}, \mathrm{d}x \otimes \mathrm{d}\mu_{t,x}(\lambda))) \cap L^p_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^+ \times \mathbb{R}, \mathrm{d}t \otimes \mathrm{d}x \otimes \mathrm{d}\mu_{t,x}(\lambda))$$

for all $p < 3$, *where the continuous function* $g(t,x,\lambda) = O(|\lambda|^p)$ *as* $\lambda \to \infty$ *for some* $p < 3$, *and* $\varphi(t,x) \in L^m(Q_T)$ *with* $p^{-1} + m^{-1} = 1$.

Proof. The detailed proof of the above lemma can be found in [34]. $\square$

Comparing the notation in Lemma 2.2 with that of Lemma 2.3, we have $\bar{v} \equiv v$.

Lemma 2.4. *For almost all* $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^+$ *we have* $\mu_{t,x}(\lambda) = \delta(\lambda - \bar{v}(t,x))$.

Proof. *Step a.*

We derive an evolution equation for the variance of the Young measures.

Step a.1. From the construction of $\{v^n\}$, we have

$$\partial_t v^n + u^n \partial_x v^n = -\frac{1}{2}(v^n)^2$$

in the weak sense. Thanks to Proposition 1.4, one has

$$\partial_t v^{n,\varepsilon} + u^n \partial_x v^{n,\varepsilon} = -\frac{1}{2}(v^{n,\varepsilon})^2 + r_n^\varepsilon, \tag{2.5}$$

where $v^{n,\varepsilon}(t,x) := \int_{\mathbb{R}} v^n(t,y) j_\varepsilon(x-y)\,\mathrm{d}y$, $r_n^\varepsilon(t,x) := -(u^n\partial_x v^n) * j_\varepsilon + u^n \partial_x v^{n,\varepsilon} + \frac{1}{2}((v^{n,\varepsilon})^2 - (v^n)^2 * j_\varepsilon)$, with $j_\varepsilon(x)$ the standard Friedrichs' mollifier, and $r_n^\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+, L^1\ (\mathbb{R}^+)) \cap L^{p/2}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^+)$ for all $p < 3$. We remark that across the boundary $x = 0$, we extend the functions v^n and u^n by zero. Let

$$T_R^+(\xi) = \begin{cases} 0, & \xi < 0, \\ \xi, & 0 \leqslant \xi \leqslant R, \\ R, & \xi \geqslant R, \end{cases} \qquad S_R^+(\xi) = \begin{cases} 0, & \xi < 0, \\ \frac{1}{2}\xi^2, & 0 \leqslant \xi \leqslant R, \\ R\xi - \frac{1}{2}R^2, & \xi \geqslant R. \end{cases} \tag{2.6}$$

Now, we multiply both sides of (2.5) by $T_R^+(v^{n,\varepsilon})$ to conclude that

$$\partial_t S_R^+(v^{n,\varepsilon}) + \partial_x(u^n S_R^+(v^{n,\varepsilon})) = v^n S_R^+(v^{n,\varepsilon}) - \frac{1}{2}T_R^+(v^{n,\varepsilon})(v^{n,\varepsilon})^2 + T_R^+(v^{n,\varepsilon}) r_n^\varepsilon.$$

By taking $\varepsilon \to 0$ we find

$$\partial_t S_R^+(v^n) + \partial_x(u^n S_R^+(v^n)) = v^n S_R^+(v^n) - \frac{1}{2}T_R^+(v^n)(v^n)^2.$$

Hence, thanks to Lemmas 2.2–2.3, we obtain

$$\partial_t \overline{S_R^+(v)} + \partial_x (u \overline{S_R^+(v)}) = \overline{v S_R^+(v) - \frac{1}{2} T_R^+(v) v^2} =: F^+. \tag{2.7}$$

Step a.2. On the other hand, applying Lemmas 2.2–2.3 to the equation

$$\partial_t v^n + \partial_x (u^n v^n) = \frac{1}{2}(v^n)^2$$

and passing to the limit $n \to \infty$, we find

$$\partial_t \overline{v} + \partial_x (u \overline{v}) = \frac{1}{2} \overline{v^2},$$

from which we deduce by a similar argument as in the proof of (2.7) that

$$\partial_t S_R^+(\overline{v}) + \partial_x (u S_R^+(\overline{v})) = T_R^+(\overline{v}) \Big(\frac{1}{2} \overline{v^2} - (\overline{v})^2 \Big) + \overline{v} S_R^+(\overline{v}) =: G^+. \tag{2.8}$$

Step a.3. Subtracting (2.8) from (2.7), we obtain

$$\partial_t (\overline{S_R^+(v)} - S_R^+(\overline{v})) + \partial_x (u (\overline{S_R^+(v)} - S_R^+(\overline{v}))) = F^+ - G^+. \tag{2.9}$$

However,

$$\begin{aligned} F^+ &= \int_{\mathbb{R}} \Big(\lambda S_R^+(\lambda) - \frac{1}{2} T_R^+(\lambda) \lambda^2 \Big) \, \mathrm{d}\mu_{t,x}(\lambda) \\ &= \int_{\mathbb{R}} \Big(\frac{1}{2} R \lambda (\lambda - R) 1_{\lambda \geqslant R} \Big) \, \mathrm{d}\mu_{t,x}(\lambda) \end{aligned}$$

and

$$G^+ = \frac{1}{2} T_R^+(\overline{v}) (\overline{v^2} - \overline{v}^2) + \frac{1}{2} R \overline{v} (\overline{v} - R) 1_{\overline{v} \geqslant R},$$

and by the construction of $\{v^n\}$ we find that both $\overline{v}(t,x)$ and $v^n(t,x)$ are less than or equal to $2t^{-1}$. Thus we have $\operatorname{supp} \mu_{t,x}(\cdot) \subset (-\infty, \frac{2}{t}]$ and

$$F^+ = 0, \quad \frac{1}{2} R \overline{v} (\overline{v} - R) 1_{\overline{v} \geqslant R} = 0 \quad \text{for } t > \frac{2}{R}.$$

Summing up the above, we find

$$\partial_t (\overline{S_R^+(v)} - S_R^+(\overline{v})) + \partial_x (u (\overline{S_R^+(v)} - S_R^+(\overline{v}))) \leqslant 0, \quad t > \frac{2}{R}. \tag{2.10}$$

Step a.4. Similarly to the proof of (2.10), we can also prove that

$$\partial_t(\overline{S_R^-(v)} - S_R^-(\bar{v})) + \partial_x(u(\overline{S_R^-(v)} - S_R^-(\bar{v}))) = F^- - G^-,$$

where

$$T_R^-(\xi) = \begin{cases} 0, & \xi > 0, \\ \xi, & -R \leqslant \xi \leqslant 0, \\ -R, & \xi \leqslant -R, \end{cases} \qquad S_R^-(\xi) = \begin{cases} 0, & \xi > 0, \\ \frac{1}{2}\xi^2, & -R \leqslant \xi \leqslant 0, \\ -R\xi - \frac{1}{2}R^2, & \xi \leqslant -R, \end{cases}$$

and

$$\begin{aligned} F^- &:= \overline{vS_R^-(v) - \frac{1}{2}T_R^-(v)v^2} = \int_{\mathbb{R}} \Big(-\frac{1}{2}R\lambda(\lambda + R)1_{\lambda \leqslant -R}\Big)\, \mathrm{d}\mu_{t,x}(\lambda), \\ G^- &:= \bar{v}S_R^-(\bar{v}) + T_R^-(\bar{v})\Big(\frac{1}{2}\overline{v^2} - (\bar{v})^2\Big) \\ &= \frac{1}{2}T_R^-(\bar{v})(\overline{v^2} - \bar{v}^2) - \frac{1}{2}R\bar{v}(\bar{v} + R)1_{\bar{v} \leqslant -R}. \end{aligned}$$

Therefore,

$$F^- - G^- = -\frac{1}{2}RJ_R - \frac{1}{2}T_R^-(\bar{v})(\overline{v^2} - \bar{v}^2),$$

where

$$J_R := \int_{\mathbb{R}} \lambda(\lambda + R)1_{\lambda \leqslant -R}\, \mathrm{d}\mu_{t,x}(\lambda) - \bar{v}(\bar{v} + R)1_{\bar{v} \leqslant -R}.$$

To handle $(\overline{v^2} - \bar{v}^2)$, we use the splitting

$$\overline{v^2} = \overline{(v_+)^2} + \overline{(v_-)^2}, \quad \bar{v}^2 = ((\bar{v})_+)^2 + ((\bar{v})_-)^2$$

and the identity

$$\frac{1}{2}(\overline{(v_-)^2} - ((\bar{v})_-)^2) = \overline{S_R^-(v)} - S_R^-(\bar{v}) + \frac{1}{2}J_R + \frac{1}{2}RH_R,$$

where we have introduced the notation

$$\begin{gathered} w_\pm = \pm\max\{0, \pm w\}; \quad (w = v, \lambda, \text{ or } \bar{v}) \\ \overline{(v_\pm)^2} = \int_{\mathbb{R}} (\lambda_\pm)^2\, \mathrm{d}\mu_{t,x}(\lambda); \\ H_R := \int (\lambda + R)1_{\lambda \leqslant -R}\, \mathrm{d}\mu_{t,x}(\lambda) - (\bar{v} + R)1_{\bar{v} \leqslant -R}. \end{gathered}$$

We then have

$$F^- - G^- = -\frac{1}{2}RJ_R - \frac{1}{2}T_R^-(\overline{v})(J_R + RH_R)$$
$$- T_R^-(\overline{v})(\overline{S_R^-(v)} - S_R^-(\overline{v})) - \frac{1}{2}T_R^-(\overline{v})(\overline{(v_+)^2} - ((\overline{v})_+)^2).$$

And since $R + T_R^-(\overline{v}) \geqslant 0$, $-T_R^-(\overline{v}) \geqslant 0$, and $\lambda(\lambda + R)1_{\lambda \leqslant -R}$ is a convex function while $(\lambda + R)1_{\lambda \leqslant -R}$ is a concave function, we obtain

$$-\frac{1}{2}(R + T_R^-(\overline{v}))J_R \leqslant 0; \quad -\frac{1}{2}T_R^-(\overline{v})H_R \leqslant 0.$$

Thus, we arrive at

$$\begin{aligned}\partial_t(\overline{S_R^-(v)} - S_R^-(\overline{v})) + \partial_x(u(\overline{S_R^-(v)} - S_R^-(\overline{v})))\\ \leqslant R\Big\{[\overline{S_R^-(v)} - S_R^-(\overline{v})] + \frac{1}{2}[\overline{(v_+)^2} - ((\overline{v})_+)^2]\Big\}.\end{aligned} \tag{2.11}$$

Step b. We show that the family of Young measures are Dirac masses.

Step b.1. Let us first claim the right continuity

$$\lim_{t\to 0+} \int \overline{v}^2(t,x)\,\mathrm{d}x = \int_{\mathbb{R}} v_0^2(x)\,\mathrm{d}x. \tag{2.12}$$

In fact, by the equation satisfied by $\overline{v}(t,x)$, we have

$$\overline{v}(t,x) \rightharpoonup v_0(x) \quad \text{in } L^2(\mathbb{R}) \text{ as } t \to 0.$$

Hence, by Theorem 1 on p. 4 of [11], we obtain

$$\int v_0^2(x)\,\mathrm{d}x \leqslant \underline{\lim}_{t\to 0_+} \int \overline{v}^2(t,x)\,\mathrm{d}x.$$

But by Lemma 2.1

$$\int_{\mathbb{R}} \overline{v}^2(t,x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}} v_0^2(x)\,\mathrm{d}x, \quad \text{for all } t > 0,$$

from which we conclude (2.12).

Step b.2. We then claim that

$$\int_{\mathbb{R}} (\overline{(v_+)^2} - (\overline{v}_+)^2)(t,x)\,\mathrm{d}x = 0, \quad \text{for all } t \in \mathbb{R}^+. \tag{2.13}$$

In fact, by Lemma 2.3 and (2.10) we have for $t > 2/R$

$$\int_{\mathbb{R}} (\overline{S_R^+(v)} - S_R^+(\overline{v}))(t,x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}} (\overline{S_R^+(v)} - S_R^+(\overline{v}))\Big(\frac{2}{R}, x\Big)\,\mathrm{d}x.$$

On the other hand, thanks to Lemma 2.3 and the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{R\to\infty}\int_{\mathbb{R}}(\overline{S_R^+(v)}-S_R^+(\overline{v}))(t,x)\,\mathrm{d}x=\frac{1}{2}\int_{\mathbb{R}}(\overline{(v_+)^2}-(\overline{v}_+)^2)(t,x)\,\mathrm{d}x.$$

Further, by the definition of $S_R^+(\xi)$, we find

$$\frac{1}{2}(\overline{(v_+)^2}-((\overline{v})_+)^2)=\overline{S_R^+(v)}-S_R^+(\overline{v})+\frac{1}{2}I_R,$$

where

$$I_R:=\int(\lambda-R)^2 1_{\lambda\geqslant R}\,\mu(t,x,d\lambda)-(\overline{v}-R)^2 1_{\overline{v}\geqslant R}\geqslant 0$$

due to the convexity, which implies

$$\int_{\mathbb{R}}(\overline{S_R^+(v)}-S_R^+(\overline{v}))\Big(\frac{2}{R},x\Big)\,\mathrm{d}x\leqslant\frac{1}{2}\int_{\mathbb{R}}(\overline{v^2}-\overline{v}^2)\Big(\frac{2}{R},x\Big)\,\mathrm{d}x,$$

and therefore

$$\lim_{R\to\infty}\int_{\mathbb{R}}(\overline{v^2}-\overline{v}^2)\Big(\frac{2}{R},x\Big)\,\mathrm{d}x\leqslant\int_{\mathbb{R}}v_0^2(x)\,\mathrm{d}x-\lim_{R\to\infty}\int_{\mathbb{R}}\overline{v}^2\Big(\frac{2}{R},x\Big)\,\mathrm{d}x=0.$$

Step b.3. Integrating (2.11) with respect to x and using (2.13), we have

$$\partial_t\int_{\mathbb{R}}(\overline{S_R^-(v)}-S_R^-(\overline{v}))\,\mathrm{d}x\leqslant R\int_{\mathbb{R}}(\overline{S_R^-(v)}-S_R^-(\overline{v}))\,\mathrm{d}x.$$

Thus, by Lemma 2.3 and Gronwall's inequality, we immediately obtain

$$\int_{\mathbb{R}}(\overline{S_R^-(v)}-S_R^-(\overline{v}))(t,x)\,\mathrm{d}x=0,\quad\text{for all }\ t\in\mathbb{R}^+.$$

Then the Lebesgue Dominated Convergence Theorem gives

$$\frac{1}{2}\int_{\mathbb{R}}(\overline{(v_-)^2}-(\overline{v}_-)^2)(t,x)\,\mathrm{d}x=\lim_{R\to\infty}\int_{\mathbb{R}}(\overline{S_R^-(v)}-S_R^-(\overline{v}))(t,x)\,\mathrm{d}x=0.$$

Thus,

$$\int_{\mathbb{R}}(\overline{v^2}-\overline{v}^2)(t,x)\,\mathrm{d}x=\iint_{\mathbb{R}^2}|\lambda-\overline{v}(t,x)|^2\,\mathrm{d}\mu_{t,x}(\lambda)\,\mathrm{d}x=0.$$

This shows that $\mu_{t,x}(\lambda)=\delta(\lambda-\overline{v}(t,x))$ for almost all $(t,x)\in\mathbb{R}^+\times\mathbb{R}$. This completes the proof of the lemma. $\square$

C o n c l u s i o n o f t h e p r o o f of Theorem 2.1. By Lemmas 2.1 and 2.4, we have that $v^n(t,y) \to v(t,y)$ in $L^p_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^+)$ for all $p<3$ and $v(t,y) \leqslant 2t^{-1}$. Thus by Lemma 2.2, we can take the limit $n\to\infty$ in the equation

$$\partial_t v^n + \partial_x(u^n v^n) = \frac{1}{2}(v^n)^2.$$

This proves that $v(t,x)$ is indeed a global weak solution of (2.1). The uniqueness part is omitted here, one may check [35] for more details. $\square$

2.2. Relation to the Camassa-Holm equation

Physically, by approximating directly the Hamiltonian for Euler equations in the shallow water regime, Camassa and Holm [2] obtained the equation

$$\partial_t u - \partial_x^2\partial_t u + 3u\partial_x u = 2\partial_x u \partial_x^2 u + u\partial_x^3 u, \quad t>0,\ x\in\mathbb{R}. \tag{2.14}$$

Mathematically, (2.14) is obtained and proved to be formally integrable by Fuchssteiner and Fokas [13] as a bi-Hamiltonian generalization of the KDV equation. (2.14) has several important features that distinguish it from the well-known KDV equation. First, Camassa and Holm discovered that (2.14) possesses peaked solutions with a corner at their crest, which is in sharp contrast to the smooth solitary waves for KDV. Second, physical water waves often break down, which cannot be predicted by the solutions to the KDV equation.

Formally (2.14) is equivalent to

$$\begin{cases} \partial_t u + u\partial_x u + \partial_x P = 0, \quad t>0,\ x\in\mathbb{R}, \\ P(t,x) = \dfrac{1}{2}\displaystyle\int_{-\infty}^{\infty} \mathrm{e}^{-|x-y|}\Big(u^2 + \frac{1}{2}(u_y)^2\Big)\,\mathrm{d}y, \\ u|_{t=0} = u_0. \end{cases} \tag{2.15}$$

In [4], [27], the authors proved the finite time break-down of the smooth solution to (2.15). In particular, Mckean gave a necessary and sufficient condition on the initial data for the finite-time formulation of singularities in a smooth solution to (2.15). Furthermore, Mckean described the blowup process by showing the formation of cusps instead of shocks for compressible fluids. Motivated by the Young measure approach of our paper [35], Xin and Zhang in [32] proved the global existence of a weak solution to (2.15) with initial data in $H^1(\mathbb{R})$. More precisely, they obtained the following result.

Theorem 2.2. *Assume that* $u_0 \in H^1(\mathbb{R}^1)$. *Then the Cauchy problem* (2.15) *has a weak solution* $u = u(t,x)$. *Furthermore, this weak solution* $u(t,x)$ *possesses the following properties:*

(a) (Non-increasing of energy).

$$\int_{\mathbb{R}} (u^2 + (u_x)^2)(t,x)\,\mathrm{d}x \leqslant \int_{\mathbb{R}} (u_0^2 + (\partial_x u_0)^2)(x)\,\mathrm{d}x, \quad t > 0.$$

(b) (One-sided super-norm estimate). *There exists a positive constant*

$$C = C(\|u_0\|_{H^1(\mathbb{R}^1)})$$

such that the following one-sided L^∞*-norm estimate of the first order spatial derivative holds in the sense of distributions:*

$$\partial_x u(t,x) \leqslant C\Big(1 + \frac{1}{t}\Big), \quad \text{for all } t > 0.$$

(c) (Space-time higher integrability estimate). $P(t,x) \in L^\infty(\mathbb{R}^1_+, W^{1,\infty}(\mathbb{R}^1))$ *and* $\partial_x u(t,x) \in L^p_{\mathrm{loc}}(\mathbb{R}^1_+ \times \mathbb{R}^1)$ *for any* $p < 3$, *i.e. for any* $0 < R$, $T < +\infty$, *there exists a positive constant* $C_1 = C_1(R, T, p, \|u_0\|_{H^1})$ *such that*

$$\int_0^T \int_{|x| \leqslant R} |\partial_x u(t,x)|^p \,\mathrm{d}x\,\mathrm{d}t \leqslant C_1 \quad \text{for all } p < 3.$$

(d) (Large time behavior). *If, in addition,* $u_0 \in L^1(\mathbb{R}^1)$, *then the admissible weak solution* $u = u(t,x)$ *approaches zero pointwise as time goes to infinity, i.e.,*

$$\lim_{t \to +\infty} u(t,x) = 0 \quad \text{for all } x \in \mathbb{R}^1,$$

provided that $u(t,x)$ *is of one sign.*

The main idea of the proof lies in the following observation: setting $v = \partial_x u$, by taking ∂_x to the first equation of (2.15) we get

$$\partial_t v + u \partial_x v = -\frac{1}{2} v^2 - P + u^2,$$

which can be viewed as the first-order approximation to (2.1). However, due to the global properties of (2.15), the proof of Theorem 3.1 is much more involved that the proof of Theorem 2.1. One may check [32] for more details.

3. Weak solutions to a nonlinear variational wave equation

In this section we study the existence and regularity properties of weak solutions to the nonlinear wave equation

$$\begin{cases} \partial_t^2 u - c(u)\partial_x(c(u)\partial_x u) = 0, \\ u|_{t=0} = u_0, \quad \partial_t u|_{t=0} = u_1, \end{cases} \tag{3.1}$$

where $c(\cdot)$ is a given smooth, bounded, and positive function, $u_0(x) \in \mathrm{Lip}(\mathbb{R})$, and $u_1(x) \in L^\infty(\mathbb{R})$.

Equation (3.1) is the Euler-Lagrange equation of the action principle

$$\frac{\partial}{\partial u} \iint \{(\partial_t u)^2 - c^2(u)(\partial_x u)^2\} \,\mathrm{d}x \,\mathrm{d}t = 0,$$

from which the name variational wave equation comes. We are attracted to study the equation first of all by its simplicity and closeness to the linear wave equation with constant wave speeds. What makes it more interesting is that equation (3.1) arises in a number of various physical contexts. For example, it describes, to the first order, the motion of long waves on a dipole chain in the continuum limit, see Zorski and Infeld [40], Grundland and Infeld [16], or Glassey, Hunter, and Zheng [14]. For another example, it is the simplest representative of a large class of variational wave equations in the classical field theories and general relativity, see Glassey, Hunter, and Zheng [14]. And most importantly, it was derived by Saxton [29] and [17] for the director field in a simplified situation of a nematic liquid crystal in the regime where inertia of the director field dominates dissipation.

The general problem of global existence and uniqueness of solutions to the Cauchy problem of equation (3.1) is open. The global existence and uniqueness of solutions to the Cauchy problem of the asymptotic equation describing weakly nonlinear waves of (3.1) is fairly complete, see Hunter and Saxton [17], Hunter and Zheng [18] and the author [35]. Glassey, Hunter, and Zheng [15] have shown that singularities can form from smooth data for equation (3.1). Some partial existence results are given in [36], where the authors prove the global existence of weak rarefactive solutions to (3.1) under the conditions $c'(\cdot) \geqslant 0, R_0 \leqslant 0$, $S_0 \leqslant 0$, $(R_0, S_0) \in L^p(\mathbb{R})$, $p > 3$. The notation here is that R and S are the Riemann invariants, see below. If the condition $c'(\cdot) \geqslant 0$ is strengthened to

$$c'(\cdot) > 0,$$

then the condition $p > 3$ can be relaxed to $p = 2$. If, in addition, the initial data are in the regularity class $u_0 \in H^{k+1}(\mathbb{R})$, $u_1 \in H^k(\mathbb{R})$ for some integer $k \geqslant 1$, then the solutions are in the same regularity class. In [37], the condition $R_0 \leqslant 0$ is removed. Finally, in [38], both the restrictions $R_0 \leqslant 0$ and $S_0 \leqslant 0$ are removed.

Before we present our main result from [38], let us first introduce the following definition. Our notation is $\mathbb{R}^+ = (0, \infty)$, H^k are Sobolev spaces, Lip stands for Lipschitz. We use

$$R := \partial_t u + c(u)\partial_x u, \quad S := \partial_t u - c(u)\partial_x u, \quad \tilde{c}(\cdot) := \frac{1}{4}\ln c(\cdot),$$

so that $\tilde{c}'(u) = c'(u)/(4c(u))$. We use $R_0(x) = R(0,x)$ and $S_0(x) = S(0,x)$.

Definition 3.1. We call $u(t,x)$ a weak solution of (3.1) if

1) $u(t,x) \in L^\infty(\mathbb{R}^+, H^1_{\mathrm{loc}}(\mathbb{R})) \cap \mathrm{Lip}(\mathbb{R}^+, L^2(\mathbb{R}))$ and

$$\int_{\mathbb{R}} |\partial_t u|^2 + |c(u)\partial_x u|^2 \,\mathrm{d}x \leqslant \int_{\mathbb{R}} |u_1|^2 + |c(u_0)\partial_x u_0|^2 \,\mathrm{d}x;$$

2) for all test functions $\varphi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$:

$$\iint_{\mathbb{R}^+\times\mathbb{R}} (\partial_t\varphi\partial_t u - c^2(u)\partial_x\varphi\partial_x u - c'(u)c(u)\varphi(\partial_x u)^2)\,\mathrm{d}x\,\mathrm{d}t = 0;$$

3) $u(t,x) \to u_0(x)$ in $C([0,\infty), L^2(\mathbb{R}))$ and $\partial_t u(t,x) \to u_1(x)$ in the sense of distributions as $t \to 0+$.

In the sequel, we always assume that there exist two positive constants C_1, C_2 such that

$$0 < C_1 \leqslant c(\cdot) \leqslant C_2, \quad \text{and} \quad |c^{(l)}(\cdot)| \leqslant M_l, \;\; l \geqslant 1 \tag{3.2}$$

for some positive constants M_l.

Theorem 3.1 (Global weak solutions). *Assume $c' \geqslant 0$ and $(R_0, S_0) \in L^\infty(\mathbb{R}) \cap L^2(\mathbb{R})$. Then (3.1) has a global weak solution u in the sense of Definition (3.1). Moreover, $(R,S) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$ and $c'(u)|\partial_x u|^{2+\alpha} \in L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$ hold for any $\alpha \in (0,1)$. Furthermore, the characteristics exist; i.e., the following ordinary differential equations have global solutions $\Phi_t^\pm(x) \in C([0,\infty)\times\mathbb{R})$ with $\partial_x\Phi_t^\pm(x) \in L^\infty([0,\infty)\times\mathbb{R})$:*

$$\begin{cases} \dfrac{\mathrm{d}\Phi_t^\pm(x)}{\mathrm{d}t} = \pm c(u(t,\Phi_t^\pm(x))), \\ \Phi_0^\pm(x) = x. \end{cases}$$

In particular, if $S_0 \leqslant 0$, then $S(t,x) \leqslant 0$, $S(t,x) \in L^\infty(\mathbb{R}^+\times\mathbb{R})$, and for any $T > 0$ there exist two positive constants $M_1(T)$ and $M_2(T)$ such that

$$M_1(T) \leqslant \partial_x\Phi_t^+(x) \leqslant M_2(T) \quad (0 \leqslant t \leqslant T).$$

O u t l i n e o f t h e p r o o f. Similarly to the proof of Theorem 2.1, we divide the proof of this theorem into several steps:

Step 1. Approximate solutions

Let us define for $\varepsilon > 0$

$$Q_\varepsilon(\xi) := \begin{cases} \dfrac{1}{\varepsilon}\Big(\xi - \dfrac{1}{2\varepsilon}\Big), & \xi \geqslant \dfrac{1}{\varepsilon}, \\ \dfrac{1}{2}\xi^2, & -\infty < \xi < \dfrac{1}{\varepsilon}. \end{cases}$$

Let us now define the approximate solution sequence by the equations

$$\begin{cases} \partial_t R^\varepsilon - c(u^\varepsilon)\partial_x R^\varepsilon = \tilde{c}'(u^\varepsilon)(2Q_\varepsilon(R^\varepsilon) - (S^\varepsilon)^2), \\ \partial_t S^\varepsilon + c(u^\varepsilon)\partial_x S^\varepsilon = \tilde{c}'(u^\varepsilon)(2Q_\varepsilon(S^\varepsilon) - (R^\varepsilon)^2), \\ \partial_x u^\varepsilon = \dfrac{R^\varepsilon - S^\varepsilon}{2c(u^\varepsilon)}, \\ \lim\limits_{x\to-\infty} u^\varepsilon(t,x) = 0, \quad (R^\varepsilon, S^\varepsilon)|_{t=0} = (R_0, S_0)(x). \end{cases} \tag{3.3}$$

For convenience, we sometimes omit the superscript ε in the approximate solution sequence $\{(R^\varepsilon, S^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$.

Lemma 3.1 (Solution of (3.3) with smooth data)**.** *Let* $(R_0, S_0)(x) \in C_c^\infty(\mathbb{R})$. *Then problem* (3.3) *has a global smooth solution* $(R, S)(t, x) \in L^\infty(\mathbb{R}^+, W^{2,\infty}(\mathbb{R}))$, $u(t, x) \in L^\infty(\mathbb{R}^+, W^{3,\infty}(\mathbb{R}))$, *which satisfies the energy inequalities*

$$\int (R^2 + S^2)(t, x)\,\mathrm{d}x \leqslant \int (R_0^2 + S_0^2)(x)\,\mathrm{d}x$$

and

$$\int_0^\infty \int_{\mathbb{R}} c'(u^\varepsilon) G_\varepsilon \,\mathrm{d}x\,\mathrm{d}t \leqslant \int (R_0^2 + S_0^2)(x)\,\mathrm{d}x,$$

where

$$G_\varepsilon := R(R^2 - 2Q_\varepsilon(R)) + S(S^2 - 2Q_\varepsilon(S)) \geqslant 0.$$

Moreover, if we introduce the plus and minus characteristics $\Phi_t^\pm(b)$ *as*

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\Phi_t^\pm = \pm c(u(t, \Phi_t^\pm)), \\ \Phi_t^\pm|_{t=0} = b, \end{cases}$$

then we have the energy inequality in a characteristic cone

$$\int_a^d R^2(t_a^+(y), y)\,\mathrm{d}y + \int_d^b S^2(t_b^-(y), y)\,\mathrm{d}y \leqslant \frac{1}{2}\int_a^b (R_0^2 + S_0^2)(x)\,\mathrm{d}x,$$

where $a < b$, and d is where the two characteristics $\Phi_t^+(a)$ and $\Phi_t^-(b)$ meet at some positive time, and $t = t_a^+(y)$ is the inverse of $y = \Phi_t^+(a)$, etc. Besides, we have

$$R(t,x) \geqslant -M, \quad S(t,x) \geqslant -M \quad (t \geqslant 0),$$

where $M = \|(R_0, S_0)\|_{L^\infty} + \frac{1}{8} M C_1^{-1} \|(R_0, S_0)\|_{L^2}^2$. Finally, $S(t,x) \leqslant 0$ holds provided that $S_0(x) \leqslant 0$.

Proof. The proof of the above lemma is standard, and one can check [38] for more details. ☐

Step 2. Further estimate

In order to cancel the possible concentrations in the approximate solution sequence, we need the following lemma:

Lemma 3.2 ($L^{2+\alpha}$ estimate)**.** *Let $c'(\cdot) \geqslant 0$, $(R_0, S_0) \in L^\infty \cap L^2$, $\alpha \in (0,1)$, $T > 0$, $a < b$. Then the solutions $\{(R^\varepsilon, S^\varepsilon, u^\varepsilon)\}_{\varepsilon>0}$ of (3.3) satisfy the estimate*

$$\int_0^T \int_a^b c'(u^\varepsilon)|\partial_x u^\varepsilon|^{2+\alpha} \, \mathrm{d}x \, \mathrm{d}t \leqslant C_{\alpha,T,a,b}, \tag{3.4}$$

where the constant $C_{\alpha,T,a,b}$ depends only on the $L^\infty(\mathbb{R})$ and $L^2(\mathbb{R})$ norms of (R_0, S_0), and on the listed variables.

Proof. We assume that $\alpha = d_2/d_1 \in (0,1)$ where d_2 is an even positive integer and d_1 an odd positive integer. We then multiply the first equation of (3.3) by $R^\alpha(t,x)$ to obtain

$$\begin{aligned}\frac{1}{1+\alpha}\{\partial_t R^{1+\alpha} - \partial_x(c(u)R^{1+\alpha})\} &+ \frac{2}{1+\alpha}\tilde{c}'(u)(R-S)R^{1+\alpha} \\ &= \tilde{c}'(u)(2R^\alpha Q_\varepsilon(R) - R^\alpha S^2).\end{aligned}$$

We note that

$$\frac{2}{1+\alpha}R^{2+\alpha} - 2R^\alpha Q_\varepsilon(R) = \frac{1-\alpha}{1+\alpha}R^{2+\alpha} + R^\alpha\Big(R - \frac{1}{\varepsilon}\Big)^2\Big|_{R\geqslant\frac{1}{\varepsilon}}.$$

Therefore,

$$\begin{aligned}&\frac{1-\alpha}{1+\alpha}\tilde{c}'(u)(R-S)R^{1+\alpha} + \tilde{c}'(u)(R^\alpha S^2 - SR^{1+\alpha}) \\ &= -\frac{1}{1+\alpha}\partial_t R^{1+\alpha} + \frac{1}{1+\alpha}\partial_x(c(u)R^{1+\alpha}) - \tilde{c}'(u)R^\alpha\Big(R - \frac{1}{\varepsilon}\Big)^2 1_{R\geqslant 1/\varepsilon}.\end{aligned} \tag{3.5}$$

Similarly to the proof of (3.5), for S we can obtain the equation

$$\begin{aligned}(3.6)\qquad &\frac{1-\alpha}{1+\alpha}\tilde{c}'(u)(S-R)S^{1+\alpha}+\tilde{c}'(u)(S^\alpha R^2-RS^{1+\alpha})\\ &=-\frac{1}{1+\alpha}\partial_t S^{1+\alpha}-\frac{1}{1+\alpha}\partial_x(c(u)S^{1+\alpha})-\tilde{c}'(u)S^\alpha\Big(S-\frac{1}{\varepsilon}\Big)^2 1_{S\geqslant 1/\varepsilon}.\end{aligned}$$

For the specific choice of α we have

$$R^\alpha S^2-SR^{1+\alpha}+S^\alpha R^2-RS^{1+\alpha}=R^\alpha S^\alpha(R-S)(R^{1-\alpha}-S^{1-\alpha})\geqslant 0.$$

Summing up (3.5)–(3.6), we obtain

$$\begin{aligned}(3.7)\qquad &\frac{1-\alpha}{1+\alpha}\tilde{c}'(u)(R-S)(R^{1+\alpha}-S^{1+\alpha})+\tilde{c}'(u)R^\alpha S^\alpha(R-S)(R^{1-\alpha}-S^{1-\alpha})\\ &\leqslant\frac{1}{1+\alpha}\{-\partial_t(R^{1+\alpha}+S^{1+\alpha})+\partial_x(c(u)(R^{1+\alpha}-S^{1+\alpha}))\}.\end{aligned}$$

Note that our solutions of (3.3) have finite speed of propagation due to condition (3.2). We can cut off our initial data (R_0,S_0) to make them compactly supported without changing the solutions in the domain $[0,T]\times[a,b]$. So we assume that $\operatorname{supp} R_0,\operatorname{supp} S_0\subset[a,b]$. By (3.2) and Lemma 3.1, we have that $\operatorname{supp} R(t,\cdot)$ and $\operatorname{supp} S(t,\cdot)$ are contained in $[a-C_2T,b+C_2T]$ for $t\leqslant T$. Thus, integrating (3.7) over $[0,T]\times\mathbb{R}$, we find

$$\begin{aligned}\int_0^T\int_{\mathbb{R}}\Big\{&\frac{1-\alpha}{1+\alpha}\tilde{c}'(u)(R-S)(R^{1+\alpha}-S^{1+\alpha})\\ &+\tilde{c}'(u)R^\alpha S^\alpha(R-S)(R^{1-\alpha}-S^{1-\alpha})\Big\}\,\mathrm{d}x\,\mathrm{d}s\leqslant C_{\alpha,T,a,b},\end{aligned}$$

from which we immediately obtain

$$\frac{1-\alpha}{1+\alpha}\int_0^T\int_a^b\tilde{c}'(u)(R-S)^2(R^\alpha+S^\alpha)\,\mathrm{d}x\,\mathrm{d}t\leqslant C_{\alpha,T,a,b}.$$

This implies (3.4), which completes the proof of the lemma. □

We note that the constant $C_{\alpha,T,a,b}$ in Lemma 3.2 tends to infinity as $\alpha\to 1$.

Step 3. Pre-compactness

Let $(R_0,S_0)\in L^\infty\cap L^2(\mathbb{R})$. Let $j_\varepsilon(x)$ be the standard Friedrichs' mollifier. We denote $R_0^\varepsilon=R_0*j_\varepsilon$ and $S_0^\varepsilon=S_0*j_\varepsilon$. Then by Lemma 3.1, problem (3.3) has a global smooth solution $(R^\varepsilon, S^\varepsilon, u^\varepsilon)$ with the initial data $(R_0^\varepsilon, S_0^\varepsilon)$. Moreover, we have

$$\int ((R^\varepsilon)^2 + (S^\varepsilon)^2)(t,x)\,\mathrm{d}x \leqslant \int (R_0^2 + S_0^2)(x)\,\mathrm{d}x \tag{3.8}$$

and

$$R^\varepsilon(t,x) \geqslant -M, \quad S^\varepsilon(t,x) \geqslant -M \quad (t \geqslant 0). \tag{3.9}$$

Now, we establish the precompactness of $\{(R^\varepsilon, S^\varepsilon, u^\varepsilon)(t,x)\}$.

Firstly, by copying the proof of Lemma 3 of [37], we can prove up to a subsequence of ε_j that

$$u^{\varepsilon_j}(t,x) \quad \text{converges uniformly to a continuous function} \;\; u(t,x) \tag{3.10}$$

on every compact subset of $[0,\infty) \times \mathbb{R}$.

Thanks to Theorem 1.1 and a similar proof of Lemma 2.3, we immediately have

Lemma 3.3 (Time-distinguished Young measures). *There exist a subsequence of the solution sequence $\{R^\varepsilon(t,x), S^\varepsilon(t,x)\}$, for convenience still denoted by $\{R^\varepsilon(t,x), S^\varepsilon(t,x)\}$, and three families of Young measures $\nu^1_{t,x}(\xi)$ and $\nu^2_{t,x}(\eta)$ on $\mathbb{R}$ and $\mu_{t,x}(\xi,\eta)$ on $\mathbb{R}^2$ such that for all continuous functions $f(\lambda) \in C_c^\infty(\mathbb{R})$, $\psi(x) \in C_c^\infty(\mathbb{R})$, $g(\xi,\eta) \in C_c^\infty(\mathbb{R}^2)$ and $\varphi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ we have*

$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} f(R^\varepsilon(t,x))\psi(x)\,\mathrm{d}x = \iint_{\mathbb{R}\times\mathbb{R}} f(\xi)\psi(x)\,\mathrm{d}\nu^1_{t,x}(\xi)\,\mathrm{d}x,$$
$$\lim_{\varepsilon\to 0} \int_{\mathbb{R}} f(S^\varepsilon(t,x))\psi(x)\,\mathrm{d}x = \iint_{\mathbb{R}\times\mathbb{R}} f(\eta)\psi(x)\,\mathrm{d}\nu^2_{t,x}(\eta)\,\mathrm{d}x$$

uniformly in every compact subset of $[0,\infty)$, and

$$\begin{aligned}&\lim_{\varepsilon\to 0} \int_0^\infty \int_{\mathbb{R}} g(R^\varepsilon(t,x), S^\varepsilon(t,x))\varphi(t,x)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_0^\infty \int_{\mathbb{R}} \iint_{\mathbb{R}\times\mathbb{R}} g(\xi,\eta)\varphi(t,x)\,\mathrm{d}\mu_{t,x}(\xi,\eta)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

Moreover,

$$t \in [0,\infty) \longmapsto \iint_{\mathbb{R}\times\mathbb{R}} f(\xi)\psi(x)\,\mathrm{d}\nu^1_{t,x}(\xi)\,\mathrm{d}x \quad \textit{is continuous;}$$
$$t \in [0,\infty) \longmapsto \iint_{\mathbb{R}\times\mathbb{R}} f(\eta)\psi(x)\,\mathrm{d}\nu^2_{t,x}(\eta)\,\mathrm{d}x \quad \textit{is continuous.}$$

In the sequel, we use the notation

$$\overline{g(R,S)} = \int_{\mathbb{R}} g(\xi,\eta)\,\mathrm{d}\mu_{t,x}(\xi,\eta).$$

With Theorem 1.2, (3.10) and the above lemma, we can prove the decoupling of the Young measure $\mu_{t,x}(\xi,\eta)$ into the tensor product of the Young measures $\nu^1_{t,x}(\xi)$ and $\nu^2_{t,x}(\eta)$.

Lemma 3.4 (Decoupling of the Young measure). *Let* $\{R^\varepsilon, S^\varepsilon\}$ *be solutions to* (3.3). *Then the Young measures* $\nu^1_{t,x}(\xi)$, $\nu^2_{t,x}(\eta)$ *and* $\mu_{t,x}(\xi,\eta)$ *satisfy the equality*

$$\mu_{t,x}(\xi,\eta) = \nu^1_{t,x}(\xi) \otimes \nu^2_{t,x}(\eta).$$

Proof. Take any $f \in C_c^\infty(\mathbb{R})$ and $g \in C_c^\infty(\mathbb{R})$. By (3.3) and a trivial calculation, we find that

$$\partial_t f(R^\varepsilon) - \partial_x(c(u^\varepsilon)f(R^\varepsilon)) = T^\varepsilon,$$

where

$$T^\varepsilon := 2\tilde{c}'(u^\varepsilon)(S^\varepsilon - R^\varepsilon)f(R^\varepsilon) + \tilde{c}'(u_\varepsilon)(2Q_\varepsilon(R^\varepsilon) - (S^\varepsilon)^2)f'(R^\varepsilon).$$

By (3.8), we find that $\{T^\varepsilon\}$ is uniformly bounded in $L^1_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$. Since $f(R_\varepsilon)$ and $c(u_\varepsilon)$ are uniformly bounded in $L^\infty(\mathbb{R}^+ \times \mathbb{R})$, we obtain by Murat Lemma [28] (or Corollary 1 on p. 8 of [11]) that $\{T^\varepsilon\}$ is also a pre-compact subset of $H^{-1}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R})$. Summing up the above, we have proved the pre-compactness

$$\{\partial_t f(R^\varepsilon) - \partial_x(c(u^\varepsilon)f(R^\varepsilon))\} \subset\subset H^{-1}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}). \tag{3.11}$$

Exactly as in the proof of (3.11), we can also prove that

$$\{\partial_t g(S^\varepsilon) + \partial_x(c(u^\varepsilon)g(S^\varepsilon))\} \subset\subset H^{-1}_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}). \tag{3.12}$$

Hence, by Theorem 1.2, (3.10), (3.11) and (3.12), we find that

$$c(u^\varepsilon)f(R^\varepsilon)g(S^\varepsilon) \rightharpoonup c(u)\overline{f(R)} \cdot \overline{g(S)} \quad \text{as } \varepsilon \to 0,$$

where $(\overline{f(R)}, \overline{g(S)})$ is the weak limit of $(f(R^\varepsilon), g(S^\varepsilon))$. Thus, by Lemma 3.3, we have proved that for any $\varphi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$,

$$\begin{aligned}\iiiint \varphi(t,x) f(\xi) g(\eta)\, \mathrm{d}\mu_{t,x}(\xi,\eta)\, \mathrm{d}x\, \mathrm{d}t \\ &= \lim_{\varepsilon\to 0} \iint \varphi(t,x) f(R_\varepsilon) g(S_\varepsilon)\, \mathrm{d}x\, \mathrm{d}t \\ &= \iint \varphi(t,x) \overline{f(R)} \cdot \overline{g(S)}\, \mathrm{d}x\, \mathrm{d}t \\ &= \iint \varphi(t,x) \iint f(\xi) g(\eta)\, \mathrm{d}\nu^1_{t,x}(\xi) \otimes \nu^2_{t,x}(\eta)\, \mathrm{d}x\, \mathrm{d}t.\end{aligned}$$

Since the above equality holds for any $\varphi(t,x) \in C_c^\infty(\mathbb{R}^+ \times \mathbb{R})$ and $f(\xi), g(\eta) \in C_c^\infty(\mathbb{R})$, the proof of Lemma 3.4 is complete. □

Motivated by the proof of Lemma 2.4, we can now prove the pre-compactness of $\{R^\varepsilon, S^\varepsilon\}$.

Lemma 3.5 (Precompactness of $\{(R^\varepsilon, S^\varepsilon)\}$). *Assume $c' \geqslant 0$ and $(R_0(x), S_0(x)) \in L^\infty \cap L^2(\mathbb{R})$. Then $\nu^1_{t,x}(\xi) = \delta_{\overline{R}(t,x)}(\xi)$ and $\nu^2_{t,x}(\eta) = \delta_{\overline{S}(t,x)}(\eta)$.*

Outline of the proof. The idea is to derive an evolution equation (inequality) for the quantity $\overline{R^2} - \overline{R}^2$, so that it is zero for all positive times if it is zero at time zero, which is true in our case. In the derivation of the evolution equation we need to cut off the desired multipliers and mollify various equations that are true only in the weak sense.

Since the proof of $\nu^1_{t,x}(\xi) = \delta_{\overline{R}(t,x)}(\xi)$ is the same as that of $\nu^2_{t,x}(\eta) = \delta_{\overline{S}(t,x)}(\eta)$, we present only the proof for the former.

Step a. Derivation of the equation for $\overline{R}$.

$$\partial_t \overline{R} - \partial_x(c(u)\overline{R}) = -\tilde{c}'(u)(\overline{R^2} - 2\overline{R}\,\overline{S} + \overline{S^2}). \tag{3.13}$$

Step b. Cut-off of $(R^\varepsilon)^2$.

$$\begin{aligned}&\partial_t \overline{S^+_\lambda(R)} - \partial_x(c(u)\overline{S^+_\lambda(R)}) \\ &= \tilde{c}'(u)\{-2\overline{RS^+_\lambda(R)} + \overline{T^+_\lambda(R)R^2} + 2\overline{S}\,\overline{S^+_\lambda(R)} - \overline{T^+_\lambda(R)}\,\overline{S^2}\}.\end{aligned} \tag{3.14}$$

Step c. Cut-off of $\overline{R}^2$.

$$\begin{aligned}&\partial_t S^+_\lambda(\overline{R}) - \partial_x(c(u)S^+_\lambda(\overline{R})) \\ &= \tilde{c}'(u)\{-2\overline{R}S^+_\lambda(\overline{R}) + T^+_\lambda(\overline{R})\overline{R}^2 + 2\overline{S}S^+_\lambda(\overline{R}) - T^+_\lambda(\overline{R})\overline{S^2} - T^+_\lambda(\overline{R})(\overline{R^2} - \overline{R}^2)\}.\end{aligned} \tag{3.15}$$

Step d. Evolution equation for "$\overline{R^2} - \overline{R}^2$".

$$\begin{aligned}(3.16)\quad &\partial_t(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})) - \partial_x(c(u)(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})))\\ &= \tilde{c}'(u)\big\{-(\lambda - T_\lambda^+(\overline{R}))[\overline{(R-\lambda)^2 1_{R\geqslant\lambda}} - (\overline{R}-\lambda)^2 1_{\overline{R}\geqslant\lambda}]\\ &\quad - \lambda^2[T_\lambda^+(\overline{R}) - \overline{T_\lambda^+(R)}] - (\overline{T_\lambda^+(R)} - T_\lambda^+(\overline{R}))\overline{S^2}\\ &\quad + 2(\overline{S} + T_\lambda^+(\overline{R}))(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R}))\big\}\\ &\leqslant \tilde{c}'(u)\big\{(T_\lambda^+(\overline{R}) - \overline{T_\lambda^+(R)})\overline{S^2} + 2(\overline{S} + T_\lambda^+(\overline{R}))(\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R}))\big\},\end{aligned}$$

since $T_\lambda^+(\xi)$ is concave and $(\xi-\lambda)^2 1_{\xi\geqslant\lambda}$ is convex in ξ. We note in passing that we could save $-\lambda^2$ to reduce the term $\overline{S^2}$ by $-\lambda^2$, as we have done in paper [37]. But that is not enough when S is unbounded, so we choose a new path—renormalization.

Step e. Renormalization. We set $f_\lambda(t,x) := \overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R})$, then

$$(3.17)\qquad \partial_t f_\lambda - \partial_x(c(u)f_\lambda) \leqslant \tilde{c}'(u)\big\{2(\overline{S} + T_\lambda^+(\overline{R}))f_\lambda + \overline{S^2}(T_\lambda^+(\overline{R}) - \overline{T_\lambda^+(R)})\big\}.$$

Next, we claim that

$$(3.18)\qquad \frac{1}{2}(\overline{T_\lambda^+(R)} - T_\lambda^+(\overline{R}))^2 \leqslant \overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R}).$$

In fact, by the Cauchy inequality, we have

$$\overline{T_\lambda^+(R)}^2 = \left(\int T_\lambda^+(\xi)\,\mathrm{d}\nu_{tx}^1(\xi)\right)^2 \leqslant \overline{(T_\lambda^+(R))^2}.$$

Using the identities

$$\xi = T_\lambda^+(\xi) + (\xi-\lambda)1_{\xi\geqslant\lambda}, \quad \overline{R} = T_\lambda^+(\overline{R}) + (\overline{R}-\lambda)1_{\overline{R}\geqslant\lambda},$$

we obtain

$$\begin{aligned}T_\lambda^+(\overline{R})\overline{T_\lambda^+(R)} &= T_\lambda^+(\overline{R})\overline{R} - T_\lambda^+(\overline{R})\overline{(R-\lambda)1_{R\geqslant\lambda}}\\ &= (T_\lambda^+(\overline{R}))^2 - T_\lambda^+(\overline{R})(\overline{(R-\lambda)1_{R\geqslant\lambda}} - (\overline{R}-\lambda)1_{\overline{R}\geqslant\lambda}).\end{aligned}$$

Thus

$$\begin{aligned}&(\overline{T_\lambda^+(R)} - T_\lambda^+(\overline{R}))^2\\ &\quad= \overline{T_\lambda^+(R)}^2 + (T_\lambda^+(\overline{R}))^2 - 2T_\lambda^+(\overline{R})\overline{T_\lambda^+(R)}\\ &\quad\leqslant \overline{(T_\lambda^+(R))^2} - (T_\lambda^+(\overline{R}))^2 + 2T_\lambda^+(\overline{R})(\overline{(R-\lambda)1_{R\geqslant\lambda}} - (\overline{R}-\lambda)1_{\overline{R}\geqslant\lambda}).\end{aligned}$$

Using

$$S_\lambda^+(\xi) - \frac{1}{2}(T_\lambda^+(\xi))^2 = \lambda(\xi-\lambda)1_{\xi\geqslant\lambda},$$

we then have

$$\begin{aligned}
&\overline{S_\lambda^+(R)} - S_\lambda^+(\overline{R}) \\
&\quad= \frac{1}{2}(\overline{(T_\lambda^+(R))^2} - (T_\lambda^+(\overline{R}))^2) + \lambda(\overline{(R-\lambda)1_{R\geqslant\lambda}} - (\overline{R}-\lambda)1_{\overline{R}\geqslant\lambda}) \\
&\quad\geqslant \frac{1}{2}(T_\lambda^+(\overline{R}) - \overline{T_\lambda^+(R)})^2 + (\lambda - T_\lambda^+(\overline{R}))(\overline{(R-\lambda)1_{R\geqslant\lambda}} - (\overline{R}-\lambda)1_{\overline{R}\geqslant\lambda}) \\
&\quad\geqslant \frac{1}{2}(T_\lambda^+(\overline{R}) - \overline{T_\lambda^+(R)})^2.
\end{aligned}$$

This proves (3.18).

Notice that $f_\lambda(t,x) \in L^\infty(R^+, L^2(\mathbb{R}))$ for any fixed λ. Thus Proposition 1.4 and the Lebesgue Dominated Convergence Theorem in the time direction again yield

$$\partial_t f_\lambda^\varepsilon - \partial_x(c(u)f_\lambda^\varepsilon) \leqslant \tilde{c}'(u)\big\{2(\overline{S} + T_\lambda^+(\overline{R}))f_\lambda^\varepsilon + \overline{S^2}[T_\lambda^+(\overline{R}) - \overline{T_\lambda^+(R)}]\big\} + \gamma_\varepsilon,$$

where $f_\lambda^\varepsilon(t,x) := \int_{\mathbb{R}} f_\lambda(t,y) j_\varepsilon(x-y)\,\mathrm{d}y$ and $\gamma_\varepsilon \to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R})$. For any $\eta > 0$, we multiply the above equation by $\frac{1}{2}(f_\lambda^\varepsilon + \eta)^{-1/2}$ to obtain

$$\begin{aligned}
&\partial_t(f_\lambda^\varepsilon + \eta)^{1/2} - \partial_x(c(u)(f_\lambda^\varepsilon+\eta)^{1/2}) \\
&\quad\leqslant \tilde{c}'(u)(\overline{R} + T_\lambda^+(\overline{R}))f_\lambda^\varepsilon(f_\lambda^\varepsilon+\eta)^{-1/2} - 2\tilde{c}'(u)(\overline{R} - \overline{S})(f_\lambda^\varepsilon+\eta)^{1/2} \\
&\qquad + \frac{1}{2}\tilde{c}'(u)\overline{S^2}(f_\lambda^\varepsilon+\eta)^{-1/2}(T_\lambda^+(\overline{R}) - \overline{T_\lambda^+(R)}) + \frac{1}{2}(f_\lambda^\varepsilon+\eta)^{-1/2}\gamma_\varepsilon.
\end{aligned}$$

By taking $\varepsilon \to 0$ we find

$$\begin{aligned}
\text{(3.19)}\quad &\partial_t(f_\lambda + \eta)^{1/2} - \partial_x(c(u)(f_\lambda+\eta)^{1/2}) \\
&\quad\leqslant \tilde{c}'(u)(\overline{R} + T_\lambda^+(\overline{R}))f_\lambda(f_\lambda+\eta)^{-1/2} - 2\tilde{c}'(u)(\overline{R} - \overline{S})(f_\lambda+\eta)^{1/2} \\
&\qquad + \frac{1}{2}\tilde{c}'(u)\overline{S^2}(f_\lambda+\eta)^{-1/2}(T_\lambda^+(\overline{R}) - \overline{T_\lambda^+(R)}).
\end{aligned}$$

Moreover, by (3.18), we find that

$$\overline{S^2}(f_\lambda+\eta)^{-1/2}(T_\lambda^+(\overline{R}) - \overline{T_\lambda^+(R)}) \leqslant 2\overline{S^2}.$$

To establish the almost everywhere convergence, we first have by the Cauchy-Schwarz inequality that

$$|\overline{R} - \overline{T_\lambda(R)}| = \int (\xi-\lambda)1_{\xi\geqslant\lambda}\,\mathrm{d}\nu_{t,x}^1(\xi) \leqslant \frac{1}{\lambda}\int \xi^2\,\mathrm{d}\nu_{t,x}^1(\xi),$$

hence

$$\lim_{\lambda\to\infty}\left\|\overline{R}-\overline{T_\lambda^+(R)}\right\|_{L^1([0,T]\times\mathbb{R})}=0,\quad \text{for all } T<\infty.$$

Similarly, we can prove that $\lim\limits_{\lambda\to\infty}\|\overline{R}-T_\lambda^+(\overline{R})\|_{L^1([0,T]\times\mathbb{R})}=0$. Then by the triangle inequality we obtain

$$\lim_{\lambda\to\infty}\left\|\overline{T_\lambda^+(R)}-T_\lambda^+(\overline{R})\right\|_{L^1([0,T]\times\mathbb{R})}=0,\quad \text{for all } T<\infty.$$

Thus, by the Lebesgue Dominated Convergence Theorem, we find for any $T>0$ that

$$\lim_{\lambda\to\infty}\left\|\overline{S^2}(f_\lambda+\eta)^{-1/2}(T_\lambda^+(\overline{R})-\overline{T_\lambda^+(R)})\right\|_{L^1([0,T]\times\mathbb{R})}=0. \tag{3.20}$$

Trivially, by the Lebesgue Dominated Convergence Theorem, we have

$$\lim_{\lambda\to\infty}f_\lambda(t,x)=\frac{1}{2}(\overline{R^2}-\overline{R}^2)=:f(t,x).$$

Summing up, we obtain

$$\partial_t(f+\eta)^{1/2}-\partial_x(c(u)(f+\eta)^{1/2})\leqslant 2\tilde{c}'(u)(\overline{R}f(f+\eta)^{-1/2}-(\overline{R}-\overline{S})(f+\eta)^{1/2}).$$

We set $\sqrt{f(t,x)}=:g$. Then, by taking $\eta\to 0$ in the above equation, we find

$$\partial_t g-\partial_x(c(u)g)\leqslant 2\tilde{c}'(u)\overline{S}g. \tag{3.21}$$

Step f. The proof of the precompactness (Re-renormalization). Notice that $g(t,x)\in L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))$. Thus Proposition 1.4 implies

$$\partial_t g^\varepsilon-\partial_x(c(u)g^\varepsilon)\leqslant 2\tilde{c}'(u)\overline{S}g^\varepsilon+\gamma_\varepsilon, \tag{3.22}$$

where $0\leqslant g^\varepsilon(t,x):=\int g(t,y)j_\varepsilon(x-y)\,\mathrm{d}y$ and $\gamma_\varepsilon(t,x)\to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^+\times\mathbb{R})$.

On the other hand, parallelling the proof of (3.13), we can prove that

$$\partial_t\overline{S}+\partial_x(c(u)\overline{S})=-\tilde{c}'(u)\overline{(R-S)^2}. \tag{3.23}$$

Moreover, the third equation of (3.3) yields

$$2c(u)u_x=\overline{R}-\overline{S}. \tag{3.24}$$

Subtracting (3.23) from (3.13), we obtain

$$\partial_t(\overline{R}-\overline{S})-\partial_x(c(u)(\overline{R}+\overline{S}))=0.$$

Substituting (3.24) into the above equation, we find

$$\partial_x(c(u)(2u_t - (\overline{R} + \overline{S}))) = 0,$$

that is

$$u_t = \frac{1}{2}(\overline{R} + \overline{S}). \tag{3.25}$$

Dividing (3.22) by $c(u)$, we obtain

$$\partial_t\Big(\frac{g^\varepsilon}{c(u)}\Big) - \partial_x(g^\varepsilon) \leqslant -2\tilde{c}'(u)\overline{S}\frac{g^\varepsilon}{c(u)} + \frac{\gamma_\varepsilon}{c(u)}.$$

Taking $\varepsilon \to 0$ in the above, we conclude that

$$\partial_t\Big(\frac{g}{c(u)}\Big) - \partial_x g \leqslant -2\tilde{c}'(u)\overline{S}\frac{g}{c(u)}. \tag{3.26}$$

We next claim that

$$g(t,x) \in L^\infty_{\mathrm{loc}}(\mathbb{R}^+, L^1(\mathbb{R})). \tag{3.27}$$

First, by the definition of $g(t,x)$ we have $g(t,x) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}))$. Now, let us take $\varphi(x) \in C_c^\infty(\mathbb{R})$ with $\varphi(x) = 1$ for $|x| \leqslant 1$, $\operatorname{supp}\varphi \subset \{x\colon |x| \leqslant 2\}$. Then multiplying (3.26) by $\varphi(\frac{x}{n})$ and integrating it over $[0,t]\times\mathbb{R}$ we obtain

$$\begin{aligned}\frac{1}{C_2}\int g(t,x)\varphi\Big(\frac{x}{n}\Big)\,\mathrm{d}x &\leqslant \int \frac{g(0,x)}{c(u)}\varphi\Big(\frac{x}{n}\Big)\,\mathrm{d}x + \frac{1}{n}\int_0^t\int g\Big|\varphi'\Big(\frac{x}{n}\Big)\Big|\,\mathrm{d}x\,\mathrm{d}s\\ &\quad + \int_0^t\int 2\tilde{c}'(u)\overline{S}\frac{g}{c(u)}\,\mathrm{d}x\,\mathrm{d}s\\ &\leqslant \frac{t}{\sqrt{n}}\|g\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}\|\partial_x\varphi\|_{L^2}\\ &\quad + Ct\|\overline{S}\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}\|g\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}.\end{aligned}$$

Thus, Fatou's Lemma yields that

$$\int g(t,x)\,\mathrm{d}x \leqslant Ct\|\overline{S}\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}\|g\|_{L^\infty(\mathbb{R}^+,L^2(\mathbb{R}))}.$$

This proves claim (3.27)

On the other hand, it follows from (3.9) that there is a constant C such that $-\overline{S} \leqslant C$. Thus due to (3.27), we can integrate (3.26) over $\mathbb{R}$ to obtain

$$\int_{\mathbb{R}} \frac{g}{c(u)}(t,x)\,\mathrm{d}x \leqslant C\int_0^t\int_{\mathbb{R}} \frac{g}{c(u)}(s,x)\,\mathrm{d}x\,\mathrm{d}s.$$

Applying Gronwall's inequality, we have

$$g(t,x)=0 \quad \text{a.e. } (t,x)\in\mathbb{R}^+\times\mathbb{R}.$$

Hence, $f(t,x)=0$ a.e. $(t,x)\in\mathbb{R}^+\times\mathbb{R}$ and therefore $\nu^1_{t,x}(\xi)=\delta_{\overline{R}(t,x)}(\xi)$. Similarly, we can prove that $\nu^2_{t,x}(\eta)=\delta_{\overline{S}(t,x)}(\eta)$. This completes the proof of the lemma. □

Lemma 3.6 (Flow regularity)**.** *Let u be the limit of $\{u^{\varepsilon_j}\}$. Then the flows $\Phi^{\pm}_t(x)$*

$$\begin{cases}\dfrac{\mathrm{d}\Phi^{\pm}_t(x)}{\mathrm{d}t}=\pm c(u(t,\Phi^{\pm}_t(x))),\\ \Phi^{\pm}_t(x)|_{t=0}=x\end{cases}$$

are Lipschitz continuous with respect to x. Moreover, if $S_0\leqslant 0$, then for any $T>0$ there exist two positive constants $M_1(T)$ and $M_2(T)$ such that

$$M_2(T)\leqslant \partial_x\Phi^+_t(x)\leqslant M_1(T)\quad (0\leqslant t\leqslant T).$$

P r o o f. For simplicity, we deal only with positive characteristics. Consider the approximate solutions u^ε and their flows $\Phi^\varepsilon_t(x)$:

$$\begin{cases}\dfrac{\mathrm{d}\Phi^{\varepsilon}_t(x)}{\mathrm{d}t}=c(u^\varepsilon(t,\Phi^{\varepsilon}_t(x))),\\ \Phi^{\varepsilon}_t(x)|_{t=0}=x.\end{cases}$$

Taking ∂_x on both sides of the above equation, we find

$$\begin{aligned}\partial_x\Phi^\varepsilon_t(x)&=\exp\Big[\int_0^t(c'(u^\varepsilon)\partial_y u^\varepsilon)(s,\Phi^\varepsilon_s(x))\,\mathrm{d}s\Big]\\&=\exp\Big[\int_0^t c'(u^\varepsilon)\Big(\frac{R^\varepsilon-S^\varepsilon}{2c(u^\varepsilon)}\Big)(s,\Phi^\varepsilon_s(x))\,\mathrm{d}s\Big].\end{aligned}\tag{3.28}$$

Now, $-S^\varepsilon$ has a uniform upper bound and R^ε is uniformly bounded in L^2 together with the characteristics from Lemma 3.1, therefore the exponent in (3.28) is bounded from above. This shows that $\Phi^+_t(x)$ is Lipschitz continuous with respect to x.

Furthermore, if $S_0\leqslant 0$, then S^ε is uniformly bounded, hence the last assertion of the lemma follows directly from (3.28). This completes the proof of Lemma 3.6. □

Now, we prove Theorem 3.1.

C o n c l u s i o n o f t h e p r o o f of Theorem 3.1. First, by (3.24) and (3.25), we find

$$\overline{R}=\partial_t u+c(u)\partial_x u,\quad \overline{S}=\partial_t u-c(u)\partial_x u.$$

Second, by (3.13), (3.23), Lemma 3.2 and Lemma 3.5, we find that

$$\begin{aligned} \partial_t \overline{R} - \partial_x (c(u)\overline{R}) &= -\tilde{c}'(u)(\overline{R}-\overline{S})^2, \\ \partial_t \overline{S} + \partial_x (c(u)\overline{S}) &= -\tilde{c}'(u)(\overline{R}-\overline{S})^2 \end{aligned}$$

hold in the sense of distributions. Summing up the above equations, we find that u solves the nonlinear wave equation in the sense of distributions. The other assertions follow directly from (3.8) and Lemma 3.6. This completes the proof of the theorem. □

4. Renormalized solutions to the vortex density equations arising from sup-conductivity

In this section we shall establish the global existence of renormalized solutions to

$$\begin{cases} \partial_t \varrho + \operatorname{div}(u\varrho) = 0, \quad (t,x) \in (0,\infty)\times\mathbb{R}^2, \\ u = \nabla\Delta^{-1}\varrho, \\ \varrho|_{t=0} = \varrho_0, \end{cases} \tag{4.1}$$

with initial data $\varrho_0 \in L^1(\mathbb{R}^2)$.

Our main motivation for studying this problem comes from the type-II superconductivity. It is generally accepted that, when effects due to thermal or field fluctuations are taken into account, the Abrikosov vortex lattice obtained from the mean-field theory can melt and form a vortex liquid. Then one of the important issues that one wishes to understand is the intrinsic nonlinear effects in the dynamics of such a liquid, where the vortex density satisfies (4.1). The rigorous finite gradient vortex dynamics was studied in [22] (see also [19]). The formal derivation of (4.1) from the finite vortex dynamics was carried out in [10] (see also [3]). Under the assumption that ϱ_0 is a positive Randon measure, the authors in [23] mathematically justified the formal derivation. One can check more physical explanation to (4.1) from [10], [3] and [23].

First of all, let us introduce the definition of renormalized solutions to (4.1):

Definition 4.1. We call $(\varrho(t,x), u(t,x))$ a renormalized solution of (4.1) if for any $\beta(\tau) \in C^1(\mathbb{R})$ with $\beta(0) = 0$ and $\beta'(\tau) = O(|\tau|^{\alpha-1})$ for some $0 < \alpha < 1$ and τ large, we have

$$\partial_t \beta(\varrho) + \operatorname{div}(u\beta(\varrho)) = \varrho\beta(\varrho) - \varrho^2\beta'(\varrho) \tag{4.2}$$

and

$$u = \nabla \Delta^{-1} \varrho \tag{4.3}$$

in the sense of distributions.

Then the main result in this section can be formulated as the following theorem:

Theorem 4.1. *Let $\varrho_0 \in L^1(\mathbb{R}^2)$. Then* (4.1) *has a global renormalized solution (ϱ, u) in the sense of Definition* 4.1. *Furthermore, $\varrho(t,x) \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^q_{\rm loc}(\mathbb{R}^+ \times \mathbb{R}^2)$ for any $q < 2$, and $u(t,x) \in L^{p_1}_{\rm loc}(\mathbb{R}^+, W^{1,p_2}_{\rm loc}(\mathbb{R}^2))$ with exponents p_1, p_2 given on the line next to* (4.14). *Furthermore, for $t > 0$, we have*

$$\varrho(t,x) < \frac{1}{t} \quad \text{a.e. } x \in \mathbb{R}^2. \tag{4.4}$$

Remark 4.1. Note that by (4.4) there are only concentrations on the negative part of $\varrho(t,x)$. Therefore, we actually only need to renormalize the negative part of ϱ in Definition 4.1.

Remark 4.2. In the one space dimension case, (4.1) is reduced to

$$\partial_t \varrho + \partial_x (u\varrho) = 0, \quad u(t,x) = \int_0^x \varrho(t,y)\,\mathrm{d}y. \tag{4.5}$$

It is easy to observe that when $\varrho(0,x)$ takes negative values, a smooth solution to (4.5) will blow up in finite time. In fact, we have the following explicit solution (ϱ, u) to (4.5):

$$\varrho(t,x) = \begin{cases} 0, & t \geqslant 1, \\ -\dfrac{1}{1-t}\chi_{[t-1,1-t]}(x), & 0 < t \leqslant 1, \\ -\chi_{[-1,1]}(x), & t = 0, \end{cases}$$

$$u(t,x) = \begin{cases} 0, & t \geqslant 1 \\ 1, & x \leqslant t-1 \\ -\dfrac{x}{1-t}, & t-1 \leqslant x \leqslant 1-t, \\ -1, & x \geqslant 1-t. \end{cases}$$

On can easily check that (ϱ, u) thus defined is a renormalized solution but not a distributional weak solution to (4.5).

Similarly to the proofs of Theorems 2.1 and 3.1, we divide the proof of Theorem 4.1 into the following steps:

Step 1. The construction of the approximate solutions

Note that given sign-changing smooth initial data ϱ_0 it is easy to observe that the smooth solution to (4.1) will blow-up in finite time. Therefore to construct approximate solutions to (4.1), we first introduce the cut-off function

$$\mathcal{T}_\varepsilon(\xi) := \begin{cases} \xi, & \xi \geqslant -\dfrac{1}{\varepsilon}, \\ -\dfrac{1}{\varepsilon}, & \xi \leqslant -\dfrac{1}{\varepsilon}, \end{cases}$$

and mollify the initial data ϱ_0 by $\varrho_{0,\varepsilon} = (\varrho_0 \chi_\varepsilon) * j_\varepsilon$, where $\chi_\varepsilon(x) = \chi(\varepsilon x)$, $\chi \in C_c^\infty(\mathbb{R}^2)$, $\chi(x) = \begin{cases} 1, & |x| \leqslant 1, \\ 0, & |x| \geqslant 2, \end{cases}$ and j_ε is the standard Friedrich's mollifier with $\operatorname{supp} j_\varepsilon \subset B_\varepsilon(0)$, namely $j_\varepsilon(x) = 1/\varepsilon^2\, j(x/\varepsilon)$, $j \in C_c^\infty(\mathbb{R}^2)$, and $\int j(x)\,\mathrm{d}x = 1$. We consider

$$\begin{cases} \partial_t \varrho_\varepsilon + u_\varepsilon \cdot \nabla \varrho_\varepsilon = -\mathcal{T}_\varepsilon(\varrho_\varepsilon)\varrho_\varepsilon, & (t,x) \in (0,\infty) \times \mathbb{R}^2, \\ u_\varepsilon = \nabla \Delta^{-1} \varrho_\varepsilon, \\ \varrho_\varepsilon|_{t=0} = \varrho_{0,\varepsilon}. \end{cases} \tag{4.6}$$

Then by [9], (4.6) has a unique global smooth solution $(\varrho_\varepsilon, u_\varepsilon)$ for any fixed ε. Moreover, combining Lemma 2.1 and Lemma 2.2 of [9], we have

Lemma 4.1 (Solution of (4.6) with smooth data)**.** *Let $\varrho_0 \in L^1(\mathbb{R}^2)$. Then, for any fixed ε, there exists a unique strong solution $(\varrho_\varepsilon, u_\varepsilon)$ to (4.6) such that $\varrho_\varepsilon \in L^\infty([0,T], W^{1,p}(\mathbb{R}^2))$, $\nabla u_\varepsilon \in L^\infty([0,T], W^{1,p}(\mathbb{R}^2))$ for any $1 < p < \infty$, $T < \infty$, and*

$$\|\varrho_\varepsilon(t,\cdot)\|_{L^1} \leqslant \|\varrho_0\|_{L^1}, \quad \varrho_\varepsilon(t,x) \leqslant \frac{1}{t} \quad \text{for } t > 0. \tag{4.7}$$

Furthermore, for any $\alpha \in (0,1)$, $T, L > 0$, there exists a positive constant $C_{\alpha,T,L}$ which depends only on the L^1 norm of ϱ_0 and the listed variables, such that

$$\int_0^T \int_{|x| \leqslant L} |\varrho_\varepsilon|^{1+\alpha}\,\mathrm{d}x\,\mathrm{d}t \leqslant C_{\alpha,T,L}. \tag{4.8}$$

P r o o f. For completeness, we outline the main idea of the proof. One can check the proof of Lemma 2.2 of [9] for more details.

Step a. Let $\alpha = d_2/d_1 \in (0, \frac{1}{2})$ with d_1 and d_2 being odd positive integers, and $\zeta(x) \in C_c^\infty(\mathbb{R}^2)$, $\zeta \geqslant 0$ with $\zeta = 1$ on $\{x\colon |x| \leqslant R\}$ and $\operatorname{supp} \zeta \subset \{x\colon |x| \leqslant R+1\}$. Set $\eta(\xi) = \alpha \int_0^\xi \max(1, |s|)^{\alpha-1}\,\mathrm{d}s$ for $\xi \in \mathbb{R}^1$ such that $\eta'(\xi) = \alpha \max(1, \xi)^{\alpha-1}$. We now multiply the first equation of (4.6) by $\zeta(x)\eta'(\varrho_\varepsilon)$, integrate the resulting identity over $[0, T] \times \mathbb{R}^2$, and perform integration by parts several times to obtain:

$$\begin{aligned}(4.9)\qquad &\int_0^T \int_{\mathbb{R}^2} \zeta(\varrho_\varepsilon \eta(\varrho_\varepsilon) - \varrho_\varepsilon T_\varepsilon(\varrho_\varepsilon)\eta'(\varrho_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t \\ &\qquad = \int_{\mathbb{R}^2} \zeta\eta(\varrho_\varepsilon)\,\mathrm{d}x|_0^T - \int_0^T \int_{\mathbb{R}^2} \nabla\zeta u_\varepsilon \eta(\varrho_\varepsilon)\,\mathrm{d}x\,\mathrm{d}s.\end{aligned}$$

Taking into account the definition of α and η, we have

$$\int_0^T \int_{\mathbb{R}^2} \zeta(\varrho_\varepsilon \eta(\varrho_\varepsilon) - \varrho_\varepsilon T_\varepsilon(\varrho_\varepsilon)\eta'(\varrho_\varepsilon))\,\mathrm{d}x\,\mathrm{d}t \geqslant \int_0^T \int_{\mathbb{R}^2} 1_{|\varrho_\varepsilon| \geqslant 1}\zeta((1-\alpha)\varrho_\varepsilon^{1+\alpha} + \alpha\varrho_\varepsilon)\,\mathrm{d}x,$$

which together with the first part of (4.7), (4.9) and some classical estimates for u_ε leads to

$$(4.10)\qquad \int_0^T \int_{|\varrho_\varepsilon| \geqslant 1} \zeta\varrho_\varepsilon^{1+\alpha}\,\mathrm{d}x\,\mathrm{d}t \leqslant \frac{1}{1-\alpha}\Big(\alpha \int_{\mathbb{R}^2} \zeta|\varrho_\varepsilon|\,\mathrm{d}x + C_1 + C_2\Big)$$

for all $\alpha = d_2/d_1 \in (0, \frac{1}{2})$.

Step b. In (4.10) we take $\alpha = d_2/d_1 \in (0, \frac{5}{6})$, and repeat the argument from (4.9)–(4.10) to get

$$(4.11)\qquad \int_0^T \int_{|x| \leqslant R+1} |\varrho_\varepsilon|^{p_2}\,\mathrm{d}x\,\mathrm{d}t \leqslant C(\alpha, R, T, \|\varrho_{0,\varepsilon}\|_{L^1}), \quad \text{for all } p_2 < \frac{11}{6}.$$

Step c. Inductively, we can prove that

$$(4.12)\quad \int_0^T \int_{|x| \leqslant R+1} |\varrho_\varepsilon|^{p_{n+1}}\,\mathrm{d}x\,\mathrm{d}t \leqslant C(\alpha, R, T, \|\varrho_{0,\varepsilon}\|_{L^1}), \quad \text{for all } p_{n+1} < 1 + \alpha_{n+1},$$

where $p_{n+1} = 1 + \alpha_n$ and α_n is defined by the inductive formula $\alpha_{n+1} = \frac{1}{2}(1 + 3\alpha_n)(1 + \alpha_n)^{-1}$. Noting that $\lim\limits_{n\to\infty} \alpha_n = 1$, we complete the proof of (4.8). □

Step 2. Pre-compactness of the approximate solution sequence

By virtue of (4.7) and (4.8) there is a subsequence of $\{\varrho_\varepsilon\}$, which we denote by $\{\varrho_{\varepsilon_j}\}$, and a function $\bar\varrho(t, x) \in L^\infty(\mathbb{R}^+, L^1(\mathbb{R}^2)) \cap L^p_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ for any $1 < p < 2$, such that

$$(4.13)\qquad \varrho_{\varepsilon_j} \rightharpoonup \bar\varrho \quad \text{weakly in } L^p_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$$

as $\varepsilon_j \to 0$. Moreover, by (4.7), (4.8) and a trivial interpolation we find that

$$\varrho_\varepsilon \text{ is uniformly bounded in } L^{p_1}_{\mathrm{loc}}(\mathbb{R}^+, L^{p_2}_{\mathrm{loc}}(\mathbb{R}^2)) \tag{4.14}$$

with $p_1^{-1} = \beta q^{-1}$, $p_2^{-1} = 1 - \beta + \beta q^{-1}$ for all $0 < \beta < 1$, $1 < q < 2$. Therefore

$$\{u_\varepsilon\} \text{ is uniformly bounded in } L^{p_1}_{\mathrm{loc}}(\mathbb{R}^+, W^{1,p_2}_{\mathrm{loc}}(\mathbb{R}^2)). \tag{4.15}$$

On the other hand, by (2.51) of [9], $\{\partial_t \varrho_\varepsilon\}$ is uniformly bounded in

$$L^{p_1}_{\mathrm{loc}}(\mathbb{R}^+, W^{-1,2/(5-2q)}_{\mathrm{loc}}(\mathbb{R}^2) + L^1(\mathbb{R}^2)).$$

Then Lions-Aubin's Lemma implies that there is a subsequence of $\{u_\varepsilon\}$, which we denote by $\{u_{\varepsilon_j}\}$, such that

$$u_{\varepsilon_j} \to u \overset{\mathrm{def}}{=} \nabla\Delta^{-1}\bar{\varrho} \quad \text{strongly in } L^{p_1}_{\mathrm{loc}}(\mathbb{R}^+, L^s_{\mathrm{loc}}(\mathbb{R}^2)) \tag{4.16}$$

as $\varepsilon_j \to 0$ and $s < p_3$ with ${p_3}^{-1} = {p_2}^{-1} - \frac{1}{2}$.

To prove that $(\bar{\varrho}, u)$ thus obtained is indeed a renormalized solution to (4.1), we need first to prove that there is no oscillation in the approximate solutions sequence. In order to do, we use again the L^p Young measure method. Actually, thanks to Theorem 1.1 and Lemma 4.1, there is a family of Young measures $\mu_{t,x}(\lambda)$ such that for all continuous functions $F(\lambda)$ with $F(\lambda) = O(|\lambda|^q)$ as $|\lambda| \to \infty$ and $q < 2$, one has

$$\lim_{\varepsilon_j \to 0} \int_{\mathbb{R}^+\times\mathbb{R}^2} \varphi(t,x) F(\varrho_{\varepsilon_j})\,\mathrm{d}x\,\mathrm{d}t = \int_{\mathbb{R}^+\times\mathbb{R}^2}\int_{\mathbb{R}} \varphi(t,x)F(\lambda)\,\mathrm{d}\mu_{t,x}(\lambda)\,\mathrm{d}x\,\mathrm{d}t \tag{4.17}$$

for all test functions $\varphi(t,x) \in C_c^\infty([0,\infty)\times\mathbb{R}^2)$. In particular, from (4.13) we have

$$\bar{\varrho}(t,x) = \int_{\mathbb{R}} \lambda\,\mathrm{d}\mu_{t,x}(\lambda).$$

Lemma 4.2 (pre-compactness of $\{\varrho_\varepsilon\}$). *Let $\varrho_0 \in L^1(\mathbb{R}^2)$. Then $\mu_{t,x}(\lambda) = \delta_{\bar{\varrho}(t,x)}(\lambda)$.*

Proof. The proof is based on an argument of the propagation of pre-compactness (see [24] and [25] for some similar arguments). As in the proof of Lemma 2.4, we separate the analysis of the pre-compactness of the solution sequence into the pre-compactness of the positive part and of the negative one respectively. Therefore, we decompose ϱ_ε into

$$\varrho_\varepsilon = \varrho_\varepsilon 1_{\varrho_\varepsilon \geqslant 0} + \varrho_\varepsilon 1_{\varrho_\varepsilon \leqslant 0} \overset{\mathrm{def}}{=} \varrho_{+,\varepsilon} - \varrho_{-,\varepsilon}, \tag{4.18}$$

where $1_{\varrho_\varepsilon \geqslant 0}$ denotes the characteristic function on the set $\{(t,x)\colon\ \varrho_\varepsilon(t,x) \geqslant 0\}$, and so for $1_{\varrho_\varepsilon \leqslant 0}$.

Step a. The propagation of the precompactness of the positive part of ϱ_ε.

Let us denote $\omega_\varepsilon = \sqrt{\varrho_{+,\varepsilon}}$. Then by (4.7) and (4.8), $\{\omega_\varepsilon\}$ is actually uniformly bounded in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^{q_1}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ for any $q_1 < 4$. Therefore thanks to Theorem 1.1, there is a subsequence of $\{\omega_\varepsilon\}$ which we denote by $\{\omega_{\varepsilon_j}\}$, a function $\overline{\omega}(t,x) \in L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^{q_1}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$, and a family of Young measures $\nu^1_{t,x}(\lambda)$ such that

$$(4.19)\quad \omega_{\varepsilon_j} \rightharpoonup \overline{\omega} = \int_0^\infty \lambda \,\mathrm{d}\nu^1_{t,x}(\lambda) \quad *\text{-weakly in } L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2)) \cap L^{q_1}_{\text{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$$

as $\varepsilon_j \to 0$, and a similar equality as (4.17) holds with ϱ_{ε_j} and $\mu_{t,x}(\lambda)$ being replaced by ω_{ε_j} and $\nu^1_{t,x}(\lambda)$; $F(\lambda)$ grows at infinity like $O(|\lambda|^q)$ for any $q < 4$.

Next we are going to prove that $\nu^1_{t,x}(\lambda) = \delta_{\overline{\omega}(t,x)}(\lambda)$. Note that ω_ε is only uniformly bounded in $L^\infty(\mathbb{R}^+, L^2(\mathbb{R}^2))$. To study the propagation of the precompactness of ω_ε, we cannot take $F(\lambda)$ growing like $(O|\lambda|^2)$ at infinity. To overcome this technical difficulty, we are going to use the cut-off functions defined in (2.6). Noticing that $\varrho_\varepsilon\omega_\varepsilon = \mathcal{T}_\varepsilon(\varrho_\varepsilon)\omega_\varepsilon = \omega_\varepsilon^3$, by virtue of (4.6) it is easy to observe that

$$(4.20)\qquad \partial_t\omega_\varepsilon + \operatorname{div}(u_\varepsilon\omega_\varepsilon) = \frac{1}{2}\omega_\varepsilon^3,$$

holds in the sense of distributions.

From (4.20) and an argument similar to that in the proof of Lemma 2.4, one arrives at

$$(4.21)\qquad \partial_t\big(\overline{S_R^+(\omega)} - S_R^+(\overline{\omega})\big) + \operatorname{div}\big(u(\overline{S_R^+(\omega)} - S_R^+(\overline{\omega}))\big) \leqslant 0$$

for $t \geqslant R^{-2}$.

Let us denote $g =: \frac{1}{2}(\overline{\omega^2} - \overline{\omega}^2)$. Then

$$(4.22)\qquad (\overline{S_R^+(\omega)} - S_R^+(\overline{\omega}))(t,x) = g(t,x)$$

for a.e. $(t,x) \in (R^{-2}, \infty) \times \mathbb{R}^2$. Furthermore,

$$(4.23)\qquad \begin{aligned} \|g(t,\cdot)\|_{L^1} &\leqslant \frac{1}{2}\int_{\mathbb{R}^2} \overline{\omega^2(t,x)}\,\mathrm{d}x \leqslant \frac{1}{2}\int_{\mathbb{R}^2} |\varrho(t,x)|\,\mathrm{d}x, \\ \|g(t,\cdot)\|_{L^\infty} &\leqslant \frac{1}{2}\overline{\omega^2} \leqslant \frac{1}{2t}. \end{aligned}$$

On the other hand, from (4.21) it is easy to observe that

$$\partial_t\omega_\varepsilon^2 + \operatorname{div}(u_\varepsilon\omega_\varepsilon^2) = 0$$

holds in the sense of distributions, from which and an argument similar to that in the proof of (2.12), we find

$$\lim_{t\to 0_+}\int_{\mathbb{R}^2}(\overline{\omega^2}-\varpi^2)(t,x)\,\mathrm{d}x = 0. \tag{4.24}$$

Furthermore, motivated by [39], let us take $\varphi(x)\in C_c^\infty(\mathbb{R}^2)$ with $\varphi(x)=1$ for $|x|\leqslant 1$ and $\varphi(x)=0$ for $|x|>2$, $\delta=6R^{-2}$, and choose $\bar t>20R^{-2}$ to be one of the Lebesgue points of $\int_{\mathbb{R}^2} g(t,x)\varphi(x/n)\,\mathrm{d}x$, and take $\psi^\delta(t)\in C_c^\infty(R^{-2},\infty)$ such that

$$\psi^\delta(t)=\begin{cases}0, & t\leqslant \frac{1}{2}\delta \ \text{ or } \ t\geqslant \bar t+\delta,\\ 1, & \delta\leqslant t\leqslant \bar t-\delta,\end{cases}$$
$$0\leqslant \partial_t\psi^\varepsilon(t)\leqslant C/\delta,\ \ t\in[0,\delta],\qquad -\partial_t\psi^\varepsilon(t)\leqslant C/\delta,\ \ t\in[\bar t-\delta,\bar t+\delta].$$

Let us multiply (4.21) by $\psi^\delta(t)\varphi(\frac{x}{n})$ and integrate the resulting inequality over $(\frac{1}{4}\delta,\infty)\times\mathbb{R}^2$. This yields

$$\begin{aligned}
&\frac{C}{\delta}\int_{\bar t-\delta}^{\bar t+\delta}\int_{\mathbb{R}^2} g\varphi\Big(\frac{x}{n}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant -\int_{\bar t-\delta}^{\bar t+\delta}\int_{\mathbb{R}^2}\partial_t\psi^\delta g\varphi\Big(\frac{x}{n}\Big)\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant \int_{\frac{\delta}{4}}^{\delta}\int_{\mathbb{R}^2}\partial_t\psi^\delta g\varphi\Big(\frac{x}{n}\Big)\,\mathrm{d}x\,\mathrm{d}t+\frac{1}{n}\int_{\frac{\delta}{4}}^{\bar t+\delta}\int_{\mathbb{R}^2}\psi^\delta\nabla\varphi\Big(\frac{x}{n}\Big)ug\,\mathrm{d}x\,\mathrm{d}t\\
&\quad\leqslant \frac{C}{\delta}\int_{\frac{\delta}{4}}^{\delta}\int_{\mathbb{R}^2} g\varphi\Big(\frac{x}{n}\Big)\,\mathrm{d}x\,\mathrm{d}t-\frac{C}{n}\int_{\frac{\delta}{4}}^{\bar t+\delta}\int_{\mathbb{R}^2}\psi^\delta\varrho\nabla\Delta^{-1}\Big(\nabla\varphi\Big(\frac{x}{n}\Big)g\Big)\,\mathrm{d}x\,\mathrm{d}t,
\end{aligned}\tag{4.25}$$

where in the last step we have used integration by parts and the fact that $u=\nabla\Delta^{-1}\varrho$. To proceed further, note that by the standard inequality in 2 space dimensions,

$$\|\nabla\Delta^{-1}h\|_{L^\infty}\leqslant C\|h\|_{L^\infty}^{\frac{1}{2}}\|h\|_{L^1}^{\frac{1}{2}},$$

and by (4.23) we find

$$\Big\|\nabla\Delta^{-1}\Big(\nabla\varphi\Big(\frac{x}{n}\Big)g\Big)\Big\|_{L^\infty}\leqslant C\|\nabla\varphi\|_{L^\infty}\|g\|_{L^1}^{\frac{1}{2}}\|g\|_{L^\infty}^{\frac{1}{2}}\leqslant C\|\nabla\varphi\|_{L^\infty}t^{-\frac{1}{2}}.$$

Therefore, taking $R\to\infty$ in (4.25), we find by (4.24) that

$$C\int_{\mathbb{R}^2} g(\bar t,x)\varphi\Big(\frac{x}{n}\Big)\,\mathrm{d}x\leqslant\frac{C}{n}\int_0^{\bar t}\int_{\mathbb{R}^2}|\bar\varrho(t,x)|\|\nabla\varphi\|_{L^\infty}t^{-\frac{1}{2}}\,\mathrm{d}x\,\mathrm{d}t\leqslant\frac{C\sqrt{\bar t}\|\nabla\varphi\|_{L^\infty}}{n},$$

which together with Fatou's Lemma yields that

$$\int_{\mathbb{R}^2} g(\bar{t}, x)\,\mathrm{d}x = 0.$$

Note that $\int_{\mathbb{R}^2} g(t,x)\varphi(\frac{x}{n})\,\mathrm{d}x \in L^\infty(\mathbb{R}^+)$, therefore almost every $t \in \mathbb{R}^+$ is a Lebesgue point of $\int_{\mathbb{R}^2} g(t,x)\varphi(\frac{x}{n})\,\mathrm{d}x$. Due to the arbitrariness of $\bar{t}$, we obtain

$$g(t,x) = 0 \quad \text{a.e. } (t,x) \in \mathbb{R}^+ \times \mathbb{R}^2.$$

Hence for a.e. $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^2$, there holds

$$\int_{\mathbb{R}^2} \int_0^\infty |\lambda - \overline{\omega}|^2 \,\mathrm{d}\nu^1_{t,x}(\lambda)\,\mathrm{d}x = \int_{\mathbb{R}^2} (\overline{\omega^2} - \overline{\omega}^2)(t,x)\,\mathrm{d}x = 0,$$

which implies that

$$\nu^1_{t,x}(\lambda) = \delta_{\overline{\omega}(t,x)}(\lambda) \tag{4.26}$$

for a.e. $(t,x) \in \mathbb{R}^+ \times \mathbb{R}^2$.

Step b. The proof of the precompactness for the negative part of ϱ_ε.

To prove the pre-compactness of the negative part of the solutions sequence $\{\varrho_{-,\varepsilon}\}$, we will use a different renormalization procedure to the approximate solutions sequence. First, by (4.6), $\varrho_{-,\varepsilon}$ satisfies

$$\partial_t \varrho_{-,\varepsilon} + u_\varepsilon \cdot \nabla \varrho_{-,\varepsilon} = -\mathcal{T}_\varepsilon(\varrho_\varepsilon)\varrho_{-,\varepsilon} \tag{4.27}$$

in the sense of distributions. We denote $(\varrho_{-,\varepsilon})^{\frac{1}{4}}$ by η_ε; then, by Lemma 4.1, $\{\eta_\varepsilon\}$ is uniformly bounded in $L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2)) \cap L^r_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ for any $r < 8$. Therefore, by Theorem 1.1, there exists a subsequence of $\{\eta_\varepsilon\}$, $\{\eta_{\varepsilon_j}\}$, and its associate Young measure $\nu^2_{t,x}(\lambda)$. Furthermore,

$$\partial_t \eta_\varepsilon + \operatorname{div}(u_\varepsilon \eta_\varepsilon) = -\eta_\varepsilon^5 + \frac{1}{4}\mathcal{T}_\varepsilon(\eta_\varepsilon^4)\eta_\varepsilon. \tag{4.28}$$

From (4.28) and proceeding similarly to the proof of Lemma 2.4 again, we obtain

$$\partial_t(\overline{\eta^2} - \overline{\eta}^2) + \operatorname{div}[u(\overline{\eta^2} - \overline{\eta}^2)] = -\frac{1}{2}\overline{\eta^6} - \overline{\eta^4}\,\overline{\eta}^2 + \frac{3}{2}\overline{\eta^5}\,\overline{\eta} + \overline{\varrho_+}\,\overline{\eta}^2. \tag{4.29}$$

Notice that due to (4.8) and (4.26) we can take a subsequence of $\{\varrho_{+,\varepsilon}\}$, $\{\varrho_{+,\varepsilon_j}\}$ such that $\varrho_{+,\varepsilon_j} \to \overline{\varrho_+}$ in $L^p_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ for any $p < 2$. Therefore,

$$\varrho_{+,\varepsilon_j}\eta_{\varepsilon_j} \rightharpoonup \overline{\varrho_+}\,\overline{\eta} \quad \text{weakly in } L^s_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$$

for any $s < \frac{8}{5}$. But by their definitions, $\varrho_{+,\varepsilon_j}\eta_{\varepsilon_j} = 0$, hence

$$\overline{\varrho_+}\,\overline{\eta} = 0.$$

Hence, the right-hand side of (4.29) equals

$$\begin{aligned}
(4.30) \qquad & -\frac{1}{2}\overline{\eta^6} - \overline{\eta^4}\,\overline{\eta}^2 + \frac{3}{2}\overline{\eta^5}\,\overline{\eta} \\
&= -\Big(\frac{1}{2}(\overline{\eta^6} + \overline{\eta^4}\,\overline{\eta}^2) - \overline{\eta^5}\,\overline{\eta}\Big) + \frac{1}{2}(\overline{\eta^5}\,\overline{\eta} - \overline{\eta^4}\,\overline{\eta}^2) \\
&= -\frac{1}{2}\int_{\mathbb{R}} \lambda^4(\lambda - \overline{\eta})^2 \,\mathrm{d}\nu^2_{t,x}(\lambda) + \frac{1}{2}\int_{\mathbb{R}} \lambda^4\overline{\eta}(\lambda - \overline{\eta}) \,\mathrm{d}\nu^2_{t,x}(\lambda).
\end{aligned}$$

Note that $\int_{\mathbb{R}} \overline{\eta}^5(\lambda - \overline{\eta})\,\mathrm{d}\nu^2_{t,x}(\lambda) = 0$, from which we obtain

$$\begin{aligned}
(4.31) \quad & -\frac{1}{2}\overline{\eta^6} - \overline{\eta^4}\,\overline{\eta}^2 + \frac{3}{2}\overline{\eta^5}\,\overline{\eta} \\
&= -\frac{1}{2}\int_{\mathbb{R}} \lambda^4(\lambda - \overline{\eta})^2 \,\mathrm{d}\nu^2_{t,x}(\lambda) + \frac{1}{2}\int_{\mathbb{R}} (\lambda^4 - \overline{\eta}^4)\overline{\eta}(\lambda - \overline{\eta}) \,\mathrm{d}\nu^2_{t,x}(\lambda) \\
&= \frac{1}{2}\int_{\mathbb{R}} (-\lambda^4 + \lambda^3\overline{\eta} + \lambda^2\overline{\eta}^2 + \lambda\overline{\eta}^3 + \overline{\eta}^4)(\lambda - \overline{\eta})^2 \,\mathrm{d}\nu^2_{t,x}(\lambda) \\
&\leqslant C\overline{\eta}^4 \int_{\mathbb{R}} (\lambda - \overline{\eta})^2 \,\mathrm{d}\nu^2_{t,x}(\lambda).
\end{aligned}$$

Combining (4.29) with (4.31), we find that

$$\partial_t(\overline{\eta^2} - \overline{\eta}^2) + \operatorname{div}(u(\overline{\eta^2} - \overline{\eta}^2)) \leqslant C\overline{\eta}^4 \int_{\mathbb{R}} (\lambda - \overline{\eta})^2 \,\mathrm{d}\nu^2_{t,x}(\lambda).$$

However,

$$\overline{\eta}^4 \leqslant \overline{\eta^4} = \overline{\varrho_-}, \quad \int_{\mathbb{R}} (\lambda - \overline{\eta})^2 \,\mathrm{d}\nu^2_{t,x}(\lambda) = (\overline{\eta^2} - \overline{\eta}^2), \quad \overline{\varrho_+}\,\overline{\eta^2} = 0.$$

Therefore

$$\partial_t(\overline{\eta^2} - \overline{\eta}^2) + \operatorname{div}(u(\overline{\eta^2} - \overline{\eta}^2)) \leqslant C(\overline{\varrho_-} - \overline{\varrho_+})(\overline{\eta^2} - \overline{\eta}^2) = -C\bar{\varrho}(\overline{\eta^2} - \overline{\eta}^2),$$

where we have used the fact that $\bar{\varrho} = -(\overline{\varrho_-} - \overline{\varrho_+})$. In what follows, we denote $(\overline{\eta^2} - \overline{\eta}^2)$ by f, $f_\varepsilon = f * j_\varepsilon$. Then by Proposition 1.4 we obtain

$$(4.32) \qquad \partial_t f_\varepsilon + \operatorname{div}(u f_\varepsilon) \leqslant -C\bar{\varrho} f_\varepsilon + r_\varepsilon$$

with $r_\varepsilon \to 0$ in $L^s_{\mathrm{loc}}(\mathbb{R}^+ \times \mathbb{R}^2)$ for $s < \frac{4}{3}$. Let $\theta, \gamma > 0$ be small constants which will be determined later. Then multiplying (4.32) by $\theta(f_\varepsilon + \gamma)^{\theta-1}$ we find

$$\partial_t (f_\varepsilon + \gamma)^\theta + \operatorname{div}(u(f_\varepsilon + \gamma)^\theta) = (\theta(C-1) - 1)(\overline{\varrho_-} - \overline{\varrho_+})(f_\varepsilon + \gamma)^\theta + \theta(f_\varepsilon + \gamma)^{\theta-1} r_\varepsilon.$$

If we take $\varepsilon \to 0$ then $\gamma \to 0$ and picking the constant θ so small that $\theta(C-1)-1 \leqslant 0$ and $f^\theta(t,x) \in L^\infty(\mathbb{R}^+, L^4(\mathbb{R}^2))$, we arrive at

$$\partial_t f^\theta + \operatorname{div}(u f^\theta) \leqslant 0. \tag{4.33}$$

Due to (4.33), a proof similar to the last part of *Step a* implies that for almost all $\bar{t} \in \mathbb{R}^+$, we have

$$C \int_{\mathbb{R}^2} f^\theta(\bar{t}, x) \varphi\Big(\frac{x}{n}\Big) \,\mathrm{d}x \leqslant -\frac{C}{n} \int_0^{\bar{t}} \int_{\mathbb{R}^2} \bar{\varrho}(t,x) \nabla \Delta^{-1} \Big(\nabla \varphi\Big(\frac{x}{n}\Big) f^\theta\Big) \,\mathrm{d}x \,\mathrm{d}t. \tag{4.34}$$

On the other hand, note that

$$\begin{aligned} |\nabla \Delta^{-1} h| &= \left| \int_{\mathbb{R}^2} \frac{x-y}{|x-y|^2} \, h(y) \,\mathrm{d}y \right| \\ &\leqslant \left| \int_{|x-y| \leqslant r} \frac{x-y}{|x-y|^2} \, h(y) \,\mathrm{d}y \right| + \left| \int_{|x-y| \geqslant r} \frac{x-y}{|x-y|^2} \, h(y) \,\mathrm{d}y \right| \\ &\leqslant C\Big(r^{\frac{1}{2}} \|h\|_{L^4} + \frac{1}{r} \|h\|_{L^1} \Big). \end{aligned}$$

By taking $r = \big(\|h\|_{L^1} / \|h\|_{L^4}\big)^{\frac{2}{3}}$ in the above inequality, we obtain

$$\|\nabla \Delta^{-1} h\|_{L^\infty} \leqslant C \|h\|_{L^1}^{\frac{1}{3}} \|h\|_{L^4}^{\frac{2}{3}},$$

hence

$$\begin{aligned} \|\nabla \Delta^{-1} (\nabla \varphi\Big(\frac{x}{n}\Big) f^\theta)\|_{L^\infty} &\leqslant C \|\nabla \varphi\Big(\frac{x}{n}\Big)\|_{L^{\frac{4}{3}}}^{\frac{1}{3}} \|\nabla \varphi\|_{L^\infty}^{\frac{2}{3}} \|f^\theta\|_{L^4} \\ &\leqslant C n^{\frac{1}{2}} \|\nabla \varphi\|_{L^{\frac{4}{3}}}^{\frac{1}{3}} \|\nabla \varphi\|_{L^\infty}^{\frac{2}{3}} \|f^\theta\|_{L^4}, \end{aligned}$$

from which and (4.33), and using Fatou's Lemma, we arrive at

$$\int_{\mathbb{R}^2} f^\theta(t,x) \,\mathrm{d}x = 0$$

for almost all $t > 0$. This implies that

$$\nu^2_{t,x}(\lambda) = \delta_{\bar{\eta}(t,x)(\lambda)},$$

which together with (4.26) completes the proof of the lemma. □

With the above lemma, it is trivial to complete the proof of Theorem 4.1. One may check [26] for more details.

Remarks and further references

The presentation of Theorem 1.1 and Proposition 1.3 is taken from [20], Theorem 1.2 is from [11], the general compensated compactness theorem can be found in [30] and Proposition 1.4 is taken from [6]. For more general compensated compactness result induced by H-measure or micro-local defect measure, one may see [21] and [31] for more details.

Theorem 2.1 is taken from [35]. For the uniqueness of the dissipative weak solutions to (2.1), and the related results on the conservative weak solutions to (2.1), one may check [35] for more details. For the general asymptotic equations, one may see [1] for the recent progress.

Theorem 2.2 is taken from [32], where it is proved via the vanishing viscosity method.

Theorem 3.1 is taken from [38], and one may check [39] for the related existence result for general H^1 initial data.

The last result, Theorem 4.1, is taken from [26], where we have also proved an existence result for related model equations arising from sup-conductivity. The definition of renormalized solutions was first introduced by R. J. DiPerna and P.-L. Lions in the proof of the global existence of solutions to transport and kinetic equations [6], [7] to overcome possible concentrations in the approximate solution sequence. This notion was applied later by the authors in [12] to the isotropic compressible Navier-Stokes equations. Remark 4.2 shows the necessity of the renormalized solutions for (4.1).

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References

[1] *A. Bressan, Ping Zhang, and Yuxi Zheng*: On the asymptotic variational wave equations. Archive for Rational Mechanics and Analysis. Online 2006.

[2] *R. Camassa, D. Holm*: An integrable shallow water equation with peaked solitons. Phys. Rev. Lett. *71* (1993), 1661–1664. Zbl 0972.35521

[3] *J. S. Chapman, J. Rubinstein, and M. Schatzman*: A mean-field model of superconducting vortices. Eur. J. Appl. Math. *7* (1996), 97–111. Zbl 0849.35135

[4] *A. Constantin, J. Escher*: Wave breaking for nonlinear nonlocal shallow water equations. Acta Math. *181* (1998), 229–243. Zbl 0923.76025

[5] *R. J. DiPerna*: Convergence of the viscosity method for isentropic gas dynamics. Comm. Math. Phys. *91* (1983), 1–30. Zbl 0533.76071

[6] *R. J. DiPerna, P.-L. Lions*: Ordinary differential equations, transport theory and Sobolev spaces. Invent. Math. *98* (1989), 511–547. Zbl 0696.34049

[7] *R. J. DiPerna, P.-L. Lions*: On the Cauchy problem for Boltzmann equations: Global existence and weak stability. Ann. Math. *130* (1989), 321–366. Zbl 0698.45010
[8] *R. J. DiPerna, A. J. Majda*: Oscillations and concentration in weak solutions of the incompressible fluid equations. Commun. Math. Phys. *108* (1987), 667–689. Zbl 0626.35059
[9] *Qiang Du, Ping Zhang*: Existence of weak solutions to some vortex density models. SIAM J. Math. Anal. *34* (2003), 1279–1299. Zbl 1044.35085
[10] *E. Weinan*: Dynamics of vortex liquids in Ginsburg-Landau theories with application to superconductivity. Phys. Rev. B *50* (1994), 1126–1135.
[11] *L. C. Evans*: Weak Convergence Methods for Nonlinear Partial Differential Equations. CBMS No. 74. AMS, Providence, 1990. Zbl 0698.35004
[12] *E. Feireisl, A. Novotný, and H. Petzeltová*: On the existence of globally defined weak solutions to the Navier-Stokes equations. J. Math. Fluid Mech. *3* (2001), 358–392. Zbl 0997.35043
[13] *B. Fuchssteiner, A. S. Fokas*: Symplectic structures, their Bäcklund transformations and hereditary symmetries. Phys. D *4* (1981/1982), 47–66.
[14] *R. T. Glassey, J. K. Hunter, and Yuxi Zheng*: Singularities and oscillations in a nonlinear variational wave equation. In: Singularities and Oscillations. IMA, Vol. 91 (J. Rauch, M. Taylor, eds.). Springer-Verlag, New York, 1997, pp. 37–60. Zbl 0958.35084
[15] *R. T. Glassey, J. K. Hunter, and Yuxi Zheng*: Singularities of a variational wave equation. J. Differ. Equations *129* (1996), 49–78. Zbl 0879.35107
[16] *A. Grundland E. Infeld*: A family of nonlinear Klein-Gordon equations and their solutions. J. Math. Phys. *33* (1992), 2498–2503. Zbl 0764.35088
[17] *J. K. Hunter, R. A. Saxton*: Dynamics of director fields. SIAM J. Appl. Math. *51* (1991), 1498–1521. Zbl 0761.35063
[18] *J. K. Hunter, Yuxi Zheng*: On a nonlinear hyperbolic variational equation I and II. Arch. Ration. Mech. Anal. *129* (1995), 305–353, 355–383. Zbl 0834.35085
[19] *R. L. Jerrard, H. M. Soner*: Dynamics of Ginzburg-Landau vortices. Arch. Ration. Mech. Anal. *142* (1998), 99–125. Zbl 0923.35167
[20] *J. L. Joly, G. Metivier, and J. Rauch*: Focusing at a point and absorption of nonlinear oscillations. Trans. Am. Math. Soc. *347* (1995), 3921–3969. Zbl 0857.35087
[21] *P. Gérard*: Microlocal defect measures. Commun. Partial Differ. Equations *16* (1991), 1761–1794. Zbl 0770.35001
[22] *Fanghua Lin*: Some dynamical properties of Ginzburg-Landau vortices. Commun. Pure Appl. Math. *49* (1996), 323–359. Zbl 0853.35058
[23] *Fanghua Lin, Ping Zhang*: On the hydrodynamic limit of Ginzburg-Landau vortices. Discrete Contin. Dyn. Syst. *6* (2000), 121–142. Zbl 1034.35128
[24] *P.-L. Lions*: Mathematical Topics in Fluid Mechanics, Vol. 2, Compressible Models. Oxford Lecture Series in Mathematics and Its Applications. Clarendon Press, Oxford, 1998. Zbl 0908.76004
[25] *P.-L. Lions, N. Masmoudi*: Global solutions for some Oldroyd models of non-Newtonian flows. Chin. Ann. Math., Ser. B *21* (2000), 131–146. Zbl 0957.35109
[26] *N. Masmoudi, Ping Zhang*: Weak solutions to the vortex density equations arising from sup-conductivity. Ann. Inst. Henri Poincaré, Anal. Non Linéaire *22* (2005), 441–458.
[27] *H. Mckean*: Breakdown of shallow water equation. Asian J. Math. *2* (1998), 867–874. Zbl 0959.35140
[28] *F. Murat*: Compacité par compensation. Ann. Sc. Norm. Super. Pisa, Cl. Sci, IV *5* (1978), 489–507. Zbl 0399.46002

[29] *R. A. Saxton*: Dynamic instability of the liquid crystal director. In: Contemp. Math. Vol. 100: Current Progress in Hyperbolic Systems (W. B. Lindquist, ed.). AMS, Providence, 1989, pp. 325–330. Zbl 325–330
[30] *L. Tartar*: Compensated compactness and applications to partial differential equations. Nonlinear Anal. Mech. Heriot-Watt Symposium. Research Notes in Math., Vol. 39, (R. J. Knops, ed.). Pitman Press, 1979. Zbl 0437.35004
[31] *L. Tartar*: H-measures, a new approach for studying homogenisation, oscillations and concentration effects in partial differential equations. Proc. R. Soc. Edinb., Sect. A *115* (1990), 193–230. Zbl 0774.35008
[32] *Zhouping Xin, Ping Zhang*: On the weak solutions to a shallow water equation. Comm. Pure. Appl. Math. *LIII* (2000), 1411–1433.
[33] *L. C. Young*: Lectures on the Calculus of Variations and Optimal Control Theory. Saunders, Philadelphia-London-Toronto, 1969. Zbl 0177.37801
[34] *Ping Zhang, Yuxi Zheng*: On oscillations of an asymptotic equation of a nonlinear variational wave equation. Asymptotic Anal. *18* (1998), 307–327. Zbl 0935.35031
[35] *Ping Zhang, Yuxi Zheng*: Existence and uniqueness of solutions to an asymptotic equation arising from a variational wave equation with general data. Arch. Ration. Mech. Anal. *155* (2000), 49–83. Zbl 0982.35062
[36] *Ping Zhang, Yuxi Zheng*: Rarefactive solutions to a nonlinear variational wave equation. Commun. Partial Differ. Equations *26* (2001), 381–419. Zbl 0989.35112
[37] *Ping Zhang, Yuxi Zheng*: Singular and rarefactive solutions to a nonlinear variational wave equation. Chin Ann. Math., Ser. B *22* (2001), 159–170. Zbl 0980.35137
[38] *Ping Zhang, Yuxi Zheng*: Weak solutions to nonlinear variational wave equation. Arch. Ration. Mech. Anal. *166* (2003), 303–319. Zbl 1029.35173
[39] *Ping Zhang, Yuxi Zheng*: Weak solutions to a nonlinear variational wave equation with general data. Ann. Inst. Henri Poincaré, Anal. Non Linéaire *22* (2005), 207–226. Zbl pre 02165099
[40] *H. Zorski, E. Infeld*: New soliton equations for dipole chains. Phys. Rev. Lett. *68* (1992), 1180–1183.

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