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ON A MULTIPLICATIVE TYPE SUM FORM FUNCTIONAL EQUATION AND ITS ROLE IN INFORMATION THEORY

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Abstract. In this paper, we obtain all possible general solutions of the sum form functional equations

$$\sum_{i=1}^{k}\sum_{j=1}^{l} f(p_iq_j) = \sum_{i=1}^{k} g(p_i)\sum_{j=1}^{l} h(q_j)$$

and

$$\sum_{i=1}^{k}\sum_{j=1}^{l} F(p_iq_j) = \sum_{i=1}^{k} G(p_i) + \sum_{j=1}^{l} H(q_j) + \lambda\sum_{i=1}^{k} G(p_i)\sum_{j=1}^{l} H(q_j)$$

valid for all complete probability distributions $(p_1,\ldots,p_k)$, $(q_1,\ldots,q_l)$, $k \geqslant 3$, $l \geqslant 3$ fixed integers; $\lambda \in \mathbb{R}$, $\lambda \neq 0$ and F, G, H, f, g, h are real valued mappings each having the domain $I = [0,1]$, the unit closed interval.

Keywords: sum form functional equation, additive function, multiplicative function

MSC 2000: 39B52, 39B82

1. Introduction

For $n = 1, 2, 3, \ldots$ let

$$\Gamma_n = \left\{(p_1,\ldots,p_n)\colon\ p_i \geqslant 0,\ i = 1,\ldots,n;\ \sum_{i=1}^{n} p_i = 1\right\}$$

denote the set of all n-component discrete probability distributions.

The Shannon entropy [17] of a probability distribution $(p_1,\dots,p_n)\in\Gamma_n$ is defined as

$$\text{(A)}\qquad H_n(p_1,\dots,p_n) = -\sum_{i=1}^{n} p_i \log_2 p_i$$

where $H_n\colon \Gamma_n\to\mathbb{R}$, $n=1,2,\dots$ and $\mathbb{R}$ denotes the set of real numbers. It is easy to see that (A) can be written in the sum form

$$H_n(p_1,\dots,p_n)=\sum_{i=1}^{n}\varphi(p_i)$$

where $\varphi\colon\ I\to\mathbb{R}$ is given by

$$\varphi(x) = -x\log_2 x,\quad 0\leqslant x\leqslant 1$$

subject to $0\log_2 0=0$ and $I=\{x\in\mathbb{R}\colon\ 0\leqslant x\leqslant 1\}$.

While studying some problems in statistical thermodynamics, Chaundy and Mcleod [4] came across the functional equation

$$\text{(B)}\qquad \sum_{i=1}^{k}\sum_{j=1}^{l} F(p_iq_j) = \sum_{i=1}^{k} F(p_i) + \sum_{j=1}^{l} F(q_j)$$

where $(p_1,\dots,p_k)\in\Gamma_k$, $(q_1,\dots,q_l)\in\Gamma_l$ and $F\colon\ I\to\mathbb{R}$. They proved that if $F\colon\ I\to\mathbb{R}$ is continuous and satisfies the functional equation (B) for all $(p_1,\dots,p_k)\in\Gamma_k$, $(q_1,\dots,q_l)\in\Gamma_l$, $k,l=1,2,\dots$ then F is of the form

$$\text{(C)}\qquad F(x)=\lambda x\log_2 x$$

for all $x\in I$, where λ is an arbitrary real constant.

The functional equation (B) is useful in the axiomatic characterization of the Shannon entropies $H_n\colon\ \Gamma_n\to\mathbb{R}$, $n=1,2,\dots$ given by (A).

The paper of Chaundy and Mcleod [4] gave birth to a new area of research work known as "Sum form functional equations in information theory". This area is of interest to the functional equationists as well as to those researchers in information theory who are primarily interested in discovering new measures of entropy.

As regards the functional equation (B), Aczél and Daróczy [1] considered it only for integers $k=l=2,3,\dots$ and proved that the continuous solutions of (B) are still of the form (C). Daróczy [5] also found the solutions of (B) by considering it only for $k=3$, $l=2$, $F(1)=0$ but assuming $F\colon\ I\to\mathbb{R}$ to be measurable in the sense of Lebesgue. Maksa [15] studied (B) for $k=3$, $l=2$ but he did not assume $F(1)=0$.

Instead of assuming $F\colon [0,1] \to \mathbb{R}$ to be measurable in the sense of Lebesgue, he assumed $F\colon [0,1] \to \mathbb{R}$ to be bounded on a set of positive measure.

Daróczy and Jarai [6] found the solutions of (B) by assuming it for $k = 2$, $l = 2$; and $F\colon I \to \mathbb{R}$ to be measurable in the sense of Lebesgue.

Finally, without imposing any condition on $F\colon [0,1] \to \mathbb{R}$, but assuming (B) only for a fixed pair (k,l), $k \geqslant 3$, $l \geqslant 3$ integers, Losonczi and Maksa [14] found the most general solutions of (B).

A generalization of the Shannon entropy (A) with which we shall be concerned in this paper is (with $H_n^\alpha\colon \Gamma_n \to \mathbb{R}$, $n = 1,2,3,\ldots$)

$$\text{(D)} \qquad H_n^\alpha(p_1,\ldots,p_n) = (1-2^{1-\alpha})^{-1}\left(1-\sum_{i=1}^n p_i^\alpha\right)$$

with $\alpha > 0$, $\alpha \neq 1$, $0^\alpha := 0$, $\alpha \in \mathbb{R}$. The entropies (D) are known as the nonadditive entropies of order α, $\alpha > 0$, $\alpha \neq 1$ and are due to Havrda and Charvát [9]. It can be easily seen that

$$\lim_{\alpha\to 1} H_n^\alpha(p_1,\ldots,p_n) = -\sum_{i=1}^n p_i \log_2 p_i = H_n(p_1,\ldots,p_n).$$

The axiomatic characterization of the entropies (D) leads to the study of the functional equation

$$\text{(1.1)} \qquad \sum_{i=1}^k\sum_{j=1}^l F(p_iq_j) = \sum_{i=1}^k F(p_i) + \sum_{j=1}^l F(q_j) + \lambda\sum_{i=1}^k F(p_i)\sum_{j=1}^l F(q_j)$$

where $(p_1,\ldots,p_k) \in \Gamma_k$, $(q_1,\ldots,q_l) \in \Gamma_l$. Clearly, (1.1) reduces to (B) when $\lambda = 0$.

By taking $\lambda = 2^{1-\alpha} - 1$, $\alpha \neq 1$, $\alpha \in \mathbb{R}$, $0^\alpha := 0$, the continuous solutions of (1.1) were found by Behara and Nath [3] for all positive integers $k = 2,3,\ldots$; $l = 2,3,\ldots$. Later on, the continuous solutions of (1.1) for $\lambda \neq 0$ and $k = 2,3,\ldots$; $l = 2,3,\ldots$ were obtained by Kannappan [11] and Mittal [16]. For fixed integers $k \geqslant 3$, $l \geqslant 2$ and assuming $F\colon I \to \mathbb{R}$ to be measurable in the sense of Lebesgue, Losonczi [12] obtained the measurable solutions of (1.1). For fixed integers $k \geqslant 3$, $l \geqslant 3$ and assuming $F\colon I \to \mathbb{R}$ to be measurable in the sense of Lebesgue, Kannappan [10] also obtained the Lebesgue measurable solutions of both the functional equations (B) and (1.1).

As far as we know, Losonczi and Maksa [14] are the first to obtain the general solutions of (1.1) in the case when $\lambda \neq 0$ and fixing integers k and l, $k \geqslant 3$, $l \geqslant 3$.

If we define a mapping $f\colon I \to \mathbb{R}$ as

$$\text{(1.2)} \qquad f(x) = \lambda F(x) + x$$

for all $x \in I$, $\lambda \neq 0$, then the functional equation (1.1) reduces to the multiplicative type functional equation

$$\sum_{i=1}^{k}\sum_{j=1}^{l} f(p_i q_j) = \sum_{i=1}^{k} f(p_i) \sum_{j=1}^{l} f(q_j) \tag{1.3}$$

where $(p_1, \ldots, p_k) \in \Gamma_k$, $(q_1, \ldots, q_l) \in \Gamma_l$. Its general solutions, for fixed integers $k \geqslant 3$, $l \geqslant 3$, have been obtained by Losonczi and Maksa [14]. Then, by making use of the equation (1.2), they also obtained the corresponding solutions of (1.1). In this sense, both (1.1) and (1.3) are useful from the information-theoretic point of view.

In this paper, our object is to find all general solutions of the Pexiderized form of (1.3), that is,

$$\sum_{i=1}^{k}\sum_{j=1}^{l} f(p_i q_j) = \sum_{i=1}^{k} g(p_i) \sum_{j=1}^{l} h(q_j) \tag{1.4}$$

for all $(p_1, \ldots, p_k) \in \Gamma_k$, $(q_1, \ldots, q_l) \in \Gamma_l$; $k \geqslant 3$, $l \geqslant 3$ being fixed integers and $f\colon I \to \mathbb{R}$, $g\colon I \to \mathbb{R}$, $h\colon I \to \mathbb{R}$. The functional equation (1.4) is also useful from the information-theoretic point of view in the sense that it enables us to find the general solutions of the functional equation

$$\sum_{i=1}^{k}\sum_{j=1}^{l} F(p_i q_j) = \sum_{i=1}^{k} G(p_i) + \sum_{j=1}^{l} H(q_j) + \lambda \sum_{i=1}^{k} G(p_i) \sum_{j=1}^{l} H(q_j) \tag{1.5}$$

where $(p_1, \ldots, p_k) \in \Gamma_k$, $(q_1, \ldots, q_l) \in \Gamma_l$, $\lambda \neq 0$, $k \geqslant 3$, $l \geqslant 3$ being fixed integers and $F\colon I \to \mathbb{R}$, $G\colon I \to \mathbb{R}$, $H\colon I \to \mathbb{R}$. One can easily see that (1.5) is, indeed, a generalization of (1.1).

If $\lambda = 1$ then (1.5) reduces to

$$\sum_{i=1}^{k}\sum_{j=1}^{l} F(p_i q_j) = \sum_{i=1}^{k} G(p_i) + \sum_{j=1}^{l} H(q_j) + \sum_{i=1}^{k} G(p_i) \sum_{j=1}^{l} H(q_j). \tag{1.6}$$

As far as the authors know, the functional equation (1.6) seems to have been considered, for the first time, by Gulati [8] who found its Lebesgue measurable solutions in two cases (i) for $k = 1, 2, 3, \ldots$ and $l = 1, 2, 3, \ldots$ and (ii) for $k = l = 1, 2, 3, \ldots$. Surprisingly, the Lebesgue measurable solutions in the two cases are different.

The process of finding the general solutions of (1.4), for fixed integers $k \geqslant 3$, $l \geqslant 3$, needs determining the general solutions of the functional equation

$$\sum_{i=1}^{k}\sum_{j=1}^{l} T(p_i q_j) = \sum_{i=1}^{k} T(p_i) \sum_{j=1}^{l} T(q_j) + (l-k)T(0) \sum_{j=1}^{l} T(q_j) + l(k-1)T(0) \tag{1.7}$$

where $T\colon I \to \mathbb{R}$ and $k \geqslant 3$, $l \geqslant 3$ are fixed integers.

Equations (1.4) and (1.5), which are the respective extensions of (1.3) and (1.1), lead us to the meaningful entropies which cannot be obtained from the simpler equations (1.1) and (1.3) studied previously by Losonczi and Maksa [14]. This discussion is carried out in the last Section 5 of this paper; as such a discussion can be carried out only after obtaining the general solutions of (1.4) and (1.5) which are investigated in Sections 3 and 4 respectively.

2. The general solutions of functional equation (1.7)

Before investigating the general solutions of (1.7) for fixed integers k and l, $k \geqslant 3$, $l \geqslant 3$, we need some definitions and results already available in the literature (see Losonczi and Maksa [14]). Let

$$\Delta = \{(x, y)\colon\ 0 \leqslant x \leqslant 1,\ 0 \leqslant y \leqslant 1,\ 0 \leqslant x + y \leqslant 1\}.$$

In other words, Δ denotes the unit closed triangle in

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x, y)\colon\ x \in \mathbb{R},\ \ y \in \mathbb{R}\}.$$

A mapping $a\colon\ \mathbb{R} \to \mathbb{R}$ is said to be additive if it satisfies the equation

$$a(x + y) = a(x) + a(y) \tag{2.1}$$

for all $x \in \mathbb{R}$, $y \in \mathbb{R}$.

A mapping $a\colon\ I \to \mathbb{R}$, $I = [0, 1]$ is said to be additive on the triangle Δ if it satisfies (2.1) for all $(x, y) \in \Delta$.

A mapping $m\colon\ [0, 1] \to \mathbb{R}$ is said to be multiplicative if $m(0) = 0$, $m(1) = 1$ and $m(xy) = m(x)m(y)$ for all $x \in \,]0, 1[$, $y \in \,]0, 1[$.

Lemma 1. *Let $\psi\colon\ I \to \mathbb{R}$ be a mapping which satisfies the functional equation*

$$\sum_{i=1}^{n} \psi(p_i) = c \tag{2.2}$$

for all $(p_1, \ldots, p_n) \in \Gamma_n$; c a given constant and $n \geqslant 3$ a fixed integer. Then there exists an additive mapping $a\colon\ \mathbb{R} \to \mathbb{R}$ such that

$$\psi(p) = a(p) + \psi(0), \quad 0 \leqslant p \leqslant 1 \tag{2.3}$$

where

$$a(1) = c - n\psi(0). \tag{2.4}$$

Conversely, if (2.4) *holds, then the mapping* $\psi\colon\ I \to \mathbb{R}$ *defined by* (2.3) *satisfies the functional equation* (2.2).

This lemma appears on p. 74 in Losonczi and Maksa [14].

Lemma 2. *Every mapping* $a\colon\ I \to \mathbb{R}$, $I = [0,1]$, *additive on the unit triangle* Δ, *has a unique additive extension to the whole of* $\mathbb{R}$.

Note. This unique additive extension to the whole of $\mathbb{R}$ will also be denoted by the symbol a but now $a\colon\ \mathbb{R} \to \mathbb{R}$.

For Lemma 2, see Theorem (0.3.7) on p. 8 in Aczél and Daróczy [2] or Daróczy and Losonczi [7].

Theorem 1. *Let* $k \geqslant 3$, $l \geqslant 3$ *be fixed integers and* $T\colon\ [0,1] \to \mathbb{R}$ *be a mapping which satisfies the functional equation* (1.7) *for all* $(p_1, \ldots, p_k) \in \Gamma_k$ *and* $(q_1, \ldots, q_l) \in \Gamma_l$. *Then* T *is of the form*

$$T(p) = a(p) + T(0) \tag{2.5}$$

where $a\colon\ \mathbb{R} \to \mathbb{R}$ *is an additive function with*

$$\begin{cases} a(1) = -lT(0) \neq -1 + T(1) - T(0) \quad \textit{or} \\ a(1) = 1 - lT(0) = T(1) - T(0) \end{cases} \tag{2.6}$$

or

$$T(p) = M(p) - b(p) + T(0) \tag{2.7}$$

where $b\colon\ \mathbb{R} \to \mathbb{R}$ *is an additive function with*

$$b(1) = lT(0) \tag{2.8}$$

and $M\colon\ [0,1] \to \mathbb{R}$ *is a nonconstant nonadditive multiplicative function with*

$$M(0) = 0, \tag{2.9}$$

$$M(1) = 1 \tag{2.10}$$

and

$$M(pq) = M(p)M(q) \tag{2.11}$$

for all $p \in \,]0,1[$, $q \in \,]0,1[$.

P r o o f. Let us put $q_1 = 1$, $q_2 = \ldots = q_l = 0$ in (1.7). We obtain

$$[1 - T(1) - (l-1)T(0)]\Big[\sum_{i=1}^{k} T(p_i) - (k-l)T(0)\Big] = 0. \tag{2.12}$$

Case 1. $1 - T(1) - (l-1)T(0) \neq 0$. Then (2.12) reduces to

$$\sum_{i=1}^{k} T(p_i) = (k-l)T(0).$$

Hence, by Lemma 1, T is of the form (2.5) in which $a\colon \mathbb{R} \to \mathbb{R}$ is an additive mapping with $a(1) = -lT(0) \neq -1 + T(1) - T(0)$ as mentioned in (2.6).

Case 2. $1 - T(1) - (l-1)T(0) = 0$.

The functional equation (1.7) may be written in the form

$$\sum_{i=1}^{k}\Big[\sum_{j=1}^{l} T(p_i q_j) - T(p_i)\sum_{j=1}^{l} T(q_j) - (l-k)T(0)p_i \sum_{j=1}^{l} T(q_j)\Big] = l(k-1)T(0).$$

Hence, by Lemma 1,

$$\begin{aligned}\sum_{j=1}^{l} T(pq_j) - T(p)\sum_{j=1}^{l} T(q_j) - (l-k)T(0)p\sum_{j=1}^{l} T(q_j)\\ = A_1(p, q_1, \ldots, q_l) - \frac{1}{k}A_1(1, q_1, \ldots, q_l) + \frac{l}{k}(k-1)T(0)\end{aligned} \tag{2.13}$$

where $A_1\colon \mathbb{R} \times \Gamma_l \to \mathbb{R}$ is additive in the first variable. The substitution $p = 0$ in (2.13) gives

$$A_1(1, q_1, \ldots, q_l) = T(0)\Big[k\sum_{j=1}^{l} T(q_j) - l\Big]. \tag{2.14}$$

This holds for all $(q_1, \ldots, q_l) \in \Gamma_l$.

Let $x \in [0,1]$, $(r_1, \ldots, r_l) \in \Gamma_l$. Put successively $p = xr_t$, $t = 1, \ldots, l$ in (2.13); add the resulting l equations and use the additivity of A_1. We get

$$\begin{aligned}\sum_{t=1}^{l}\sum_{j=1}^{l} T(xr_t q_j) - \sum_{t=1}^{l} T(xr_t)\sum_{j=1}^{l} T(q_j) - (l-k)T(0)x\sum_{j=1}^{l} T(q_j)\\ = A_1(x, q_1, \ldots, q_l) - \frac{l}{k}A_1(1, q_1, \ldots, q_l) + \frac{l^2}{k}(k-1)T(0).\end{aligned} \tag{2.15}$$

Now put $p = x$, $q_1 = r_1, \dots, q_l = r_l$ in (2.13). We obtain

$$\begin{aligned} \sum_{t=1}^{l} T(xr_t) = T(x)\sum_{t=1}^{l} T(r_t) + (l-k)T(0)x\sum_{t=1}^{l} T(r_t) \\ + A_1(x, r_1, \dots, r_l) - \frac{1}{k}A_1(1, r_1, \dots, r_l) \\ + \frac{l}{k}(k-1)T(0). \end{aligned} \tag{2.16}$$

From (2.15) and (2.16), it follows that

$$\begin{aligned} &\sum_{t=1}^{l}\sum_{j=1}^{l} T(xr_tq_j) - T(x)\sum_{t=1}^{l} T(r_t)\sum_{j=1}^{l} T(q_j) \\ &\qquad - (l-k)T(0)x\sum_{t=1}^{l} T(r_t)\sum_{j=1}^{l} T(q_j) - \frac{l^2}{k}(k-1)T(0) \\ &= A_1(x, r_1, \dots, r_l)\sum_{j=1}^{l} T(q_j) - \frac{1}{k}A_1(1, r_1, \dots, r_l)\sum_{j=1}^{l} T(q_j) \\ &\qquad + \frac{l}{k}(k-1)T(0)\sum_{j=1}^{l} T(q_j) + (l-k)T(0)x\sum_{j=1}^{l} T(q_j) \\ &\qquad + A_1(x, q_1, \dots, q_l) - \frac{l}{k}A_1(1, q_1, \dots, q_l). \end{aligned} \tag{2.17}$$

The left-hand side of (2.17) does not undergo any change if we interchange q_j and r_j, $j = 1, \dots, l$. So, the right-hand side of (2.17) must also remain unchanged after interchanging q_j and r_j, $j = 1, \dots, l$. Consequently, we obtain

$$\begin{aligned} &A_1(x, q_1, \dots, q_l)\left[\sum_{t=1}^{l} T(r_t) - 1\right] - \frac{1}{k}A_1(1, q_1, \dots, q_l)\left[\sum_{t=1}^{l} T(r_t) - l\right] \\ &\qquad + \frac{l}{k}(k-1)T(0)\sum_{t=1}^{l} T(r_t) + (l-k)T(0)x\sum_{t=1}^{l} T(r_t) \\ &= A_1(x, r_1, \dots, r_l)\left[\sum_{j=1}^{l} T(q_j) - 1\right] \\ &\qquad - \frac{1}{k}A_1(1, r_1, \dots, r_l)\left[\sum_{j=1}^{l} T(q_j) - l\right] \\ &\qquad + \frac{l}{k}(k-1)T(0)\sum_{j=1}^{l} T(q_j) + (l-k)T(0)x\sum_{j=1}^{l} T(q_j). \end{aligned} \tag{2.18}$$

Now we divide our discussion into two cases depending upon whether $\sum_{t=1}^{l} T(r_t) - 1$ vanishes identically on Γ_l or does not vanish identically on Γ_l.

Case 2.1. $\sum_{t=1}^{l} T(r_t) - 1$ vanishes identically on Γ_l. Then

$$\sum_{t=1}^{l} T(r_t) = 1$$

for all $(r_1, \ldots, r_l) \in \Gamma_l$. By using Lemma 1, it follows that T is of the form (2.5) in which $a(1) = 1 - lT(0) = T(1) - T(0)$ as mentioned in (2.6).

Case 2.2. $\sum_{t=1}^{l} T(r_t) - 1$ does not vanish identically on Γ_l.

In this case, there exists a probability distribution $(r_1^*, \ldots, r_l^*) \in \Gamma_l$ such that

$$\sum_{t=1}^{l} T(r_t^*) - 1 \neq 0. \tag{2.19}$$

Putting $r_1 = r_1^*, \ldots, r_l = r_l^*$ in (2.18), making use of (2.19) and (2.14) and performing tedious calculations, it follows that

$$A_1(x, q_1, \ldots, q_l) = A_2(x)\left[\sum_{j=1}^{l} T(q_j) - 1\right] - (l-k)T(0)x \tag{2.20}$$

where $A_2 \colon \mathbb{R} \to \mathbb{R}$ is such that

$$A_2(x) = \left[\sum_{t=1}^{l} T(r_t^*) - 1\right]^{-1} [A_1(x, r_1^*, \ldots, r_l^*) + (l-k)T(0)x]. \tag{2.21}$$

From (2.21) it is easy to conclude that $A_2 \colon \mathbb{R} \to \mathbb{R}$ is additive as the mapping $x \mapsto A_1(x, r_1^*, \ldots, r_l^*)$ is additive. Also, putting $x = 1$ in (2.21) and making use of (2.14) by taking $q_j = r_j^*$, $j = 1, \ldots, l$ it follows that

$$A_2(1) = kT(0). \tag{2.22}$$

Moreover, from (2.13), (2.14), (2.20) and (2.22) one can derive

$$\begin{aligned} &\sum_{j=1}^{l} [T(pq_j) + A_2(pq_j) + (l-k)T(0)pq_j - T(0)] \\ &\qquad - [T(p) + A_2(p) + (l-k)T(0)p - T(0)] \\ &\qquad\quad \times \sum_{j=1}^{l} [T(q_j) + A_2(q_j) + (l-k)T(0)q_j - T(0)] = 0. \end{aligned} \tag{2.23}$$

Define a mapping $M\colon\ I \to \mathbb{R},\ I = [0,1]$, as

$$M(p) = T(p) + A_2(p) + (l-k)T(0)p - T(0) \tag{2.24}$$

for all $p \in I$. Then (2.23) reduces to the equation

$$\sum_{j=1}^{l}[M(pq_j) - M(p)M(q_j)] = 0. \tag{2.25}$$

Hence, by Lemma 1,

$$M(pq) - M(p)M(q) = E_1(p,q) - \frac{1}{l}E_1(p,1) \tag{2.26}$$

where $E_1\colon\ [0,1]\times\mathbb{R} \to \mathbb{R}$ is additive in the second variable with

$$E_1(p,0) = 0. \tag{2.27}$$

Since $A_2(0) = 0$ and $A_2(1) = kT(0)$, (2.9) and (2.10) follow from (2.24). Also, putting $q = 0$ in (2.26) and making use of (2.9) and (2.27), it follows that

$$E_1(p,1) = 0 \tag{2.28}$$

for all p, $0 \leqslant p \leqslant 1$. Now, (2.26) reduces to

$$M(pq) - M(p)M(q) = E_1(p,q) \tag{2.29}$$

for all $p \in [0,1]$ and $q \in [0,1]$. The left-hand side of (2.29) is symmetric in p and q. Hence $E_1(p,q) = E_1(q,p)$ for all $p \in [0,1]$, $q \in [0,1]$. Consequently, E_1 is also additive in the first variable. Also, we may suppose that $E_1(\cdot,q)$ has been extended additively to the whole of $\mathbb{R}$ and this extension is unique by Lemma 2.

From (2.29), as on p. 77 in Losonczi and Maksa [14], it follows that

$$\begin{aligned} M(pqr) - M(p)M(q)M(r) &= E_1(pq,r) + M(r)E_1(p,q) \\ &= E_1(qr,p) + M(p)E_1(q,r) \end{aligned} \tag{2.30}$$

for all p, q, r in $[0,1]$. Now, we prove that $E_1(p,q) = 0$ for all p, q, $0 \leqslant p \leqslant 1$, $0 \leqslant q \leqslant 1$. To the contrary, suppose there exist p^* and q^*, $0 \leqslant p^* \leqslant 1$, $0 \leqslant q^* \leqslant 1$, such that $E_1(p^*,q^*) \neq 0$. Then, from (2.30),

$$M(r) = [E_1(p^*,q^*)]^{-1}[E_1(q^*r,p^*) + M(p^*)E_1(q^*,r) - E_1(p^*q^*,r)],$$

from which it is easy to conclude that M is additive. Now, making use of (2.10), (2.19), (2.22), (2.24) and the additivity of A_2 and M, we have

$$1 \neq \sum_{t=1}^{l} T(r_t^*) = M(1) - A_2(1) - (l-k)T(0) + lT(0) = 1,$$

a contradiction. Hence, $E_1(p,q) = 0$ for all p and q, $0 \leqslant p \leqslant 1$, $0 \leqslant q \leqslant 1$. Thus, (2.29) reduces to $M(pq) = M(p)M(q)$ for all p and q, $0 \leqslant p \leqslant 1$, $0 \leqslant q \leqslant 1$. Hence, (2.11) also holds. So, M is a nonconstant nonadditive multiplicative function. By virtue of (2.24), T is of the form (2.7) in which M is a nonconstant nonadditive multiplicative function; $b\colon \mathbb{R} \to \mathbb{R}$ is additive, it is given by $b(p) = A_2(p) + (l-k)T(0)p$ for all $p \in [0,1]$ and (2.8) holds. $\square$

3. The general solutions of functional equation (1.4)

Now we prove

Theorem 2. *Let $k \geqslant 3$, $l \geqslant 3$ be fixed integers and let $f\colon I \to \mathbb{R}$, $g\colon I \to \mathbb{R}$, $h\colon I \to \mathbb{R}$, $I = [0,1]$, be mappings which satisfy the functional equation* (1.4) *for all $(p_1,\dots,p_k) \in \Gamma_k$ and $(q_1,\dots,q_l) \in \Gamma_l$. Then any general solution of* (1.4) *is of the form*

$$\begin{cases} f(p) = b_1(p) - \dfrac{1}{kl} b_1(1), \\ g(p) = b_2(p) - \dfrac{1}{k} b_2(1), \\ h \ \text{any arbitrary function} \end{cases} \tag{3.1}$$

or

$$\begin{cases} f(p) = b_1(p) - \dfrac{1}{kl} b_1(1), \\ g \ \text{any arbitrary function}, \\ h(p) = b_3(p) - \dfrac{1}{l} b_3(1) \end{cases} \tag{3.2}$$

or

$$\begin{cases} f(p) = [g(1) + (k-1)g(0)][h(1) + (l-1)h(0)]a(p) + A(p) + f(0), \\ g(p) = [g(1) + (k-1)g(0)]a(p) + A^*(p) + g(0), \\ h(p) = [h(1) + (l-1)h(0)]a(p) + h(0) \end{cases} \tag{3.3}$$

with

$$
\text{(3.3a)} \qquad \begin{cases} a(1) = 1 - \dfrac{lh(0)}{[h(1) + (l-1)h(0)]}, \\ A(1) = l\{[g(1) + (k-1)g(0)]h(0) - kf(0)\}, \\ A^*(1) = l\Big[\dfrac{g(1) + (k-1)g(0)}{h(1) + (l-1)h(0)}\Big]h(0) - kg(0); \end{cases}
$$

or

$$
\text{(3.4)} \qquad \begin{cases} f(p) = [g(1) + (k-1)g(0)][h(1) + (l-1)h(0)][M(p) - b(p)] \\ \qquad\qquad + A(p) + f(0), \\ g(p) = [g(1) + (k-1)g(0)][M(p) - b(p)] + A^*(p) + g(0), \\ h(p) = [h(1) + (l-1)h(0)][M(p) - b(p)] + h(0) \end{cases}
$$

with

$$
\text{(3.4a)} \qquad \begin{cases} b(1) = \dfrac{lh(0)}{h(1) + (l-1)h(0)}, \\ A(1) = l\{[g(1) + (k-1)g(0)]h(0) - kf(0)\}, \\ A^*(1) = l\Big[\dfrac{g(1) + (k-1)g(0)}{h(1) + (l-1)h(0)}\Big]h(0) - kg(0), \end{cases}
$$

where $b_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$; $a\colon \mathbb{R} \to \mathbb{R}$, $b\colon \mathbb{R} \to \mathbb{R}$, $A\colon \mathbb{R} \to \mathbb{R}$, $A^*\colon \mathbb{R} \to \mathbb{R}$ *are additive functions;* $M\colon [0,1] \to \mathbb{R}$ *is a nonconstant nonadditive multiplicative function; and* $[g(1) + (k-1)g(0)][h(1) + (l-1)h(0)] \neq 0$ *in* (3.3), (3.3a) *and* (3.4), (3.4a).

To prove this theorem, we need to prove some lemmas:

Lemma 3. *If a mapping* $f\colon I \to \mathbb{R}$ *satisfies the functional equation*

$$
\text{(3.5)} \qquad \sum_{i=1}^{k} \sum_{j=1}^{l} f(p_i q_j) = 0
$$

for all $(p_1, \ldots, p_k) \in \Gamma_k$, $(q_1, \ldots, q_l) \in \Gamma_l$, $k \geqslant 3$, $l \geqslant 3$ *fixed integers, then*

$$
\text{(3.6)} \qquad f(p) = b_1(p) - \frac{1}{kl} b_1(1)
$$

where $b_1\colon \mathbb{R} \to \mathbb{R}$ *is an additive function.*

P r o o f. Choose $q_1 = 1$, $q_2 = \ldots = q_l = 0$. Then equation (3.5) reduces to

$$\sum_{i=1}^{k} f(p_i) = -k(l-1)f(0).$$

Hence, by Lemma 1,

$$f(p) = b_1(p) - \frac{1}{k}b_1(1) - (l-1)f(0) \tag{3.7}$$

for all p, $0 \leqslant p \leqslant 1$, $b_1\colon \mathbb{R} \to \mathbb{R}$ being any additive function. Putting $p = 0$ in (3.7), we obtain $f(0) = -b_1(1)/kl$. Putting this value of $f(0)$ in (3.7), (3.6) readily follows.

□

Lemma 4. *Under the assumption stated in the statement of Theorem 2, the following conclusions hold:*

$$f(p) = [g(1) + (k-1)g(0)]h(p) + A(p) - [g(1) + (k-1)g(0)]h(0) + f(0), \tag{3.8}$$

$$\begin{aligned}[g(1) + (k-1)g(0)] &\sum_{i=1}^{k}\sum_{j=1}^{l} h(p_i q_j) \\ &= \sum_{i=1}^{k} g(p_i) \sum_{j=1}^{l} h(q_j) + l(k-1)[g(1) + (k-1)g(0)]h(0),\end{aligned} \tag{3.9}$$

$$\begin{aligned}[h(1) + (l-1)h(0)] &\sum_{i=1}^{k}\sum_{j=1}^{l} g(p_i q_j) \\ &= \sum_{i=1}^{k} g(p_i) \sum_{j=1}^{l} h(q_j) + k(l-1)[h(1) + (l-1)h(0)]g(0),\end{aligned} \tag{3.10}$$

$$\begin{aligned}[h(1) + (l-1)h(0)] &\sum_{i=1}^{k} g(p_i) \\ &= [g(1) + (k-1)g(0)] \sum_{i=1}^{k} h(p_i) + (l-k)[g(1) + (k-1)g(0)]h(0),\end{aligned} \tag{3.11}$$

$$\begin{aligned}[g(1) + (k-1)g(0)][h(1) + (l-1)h(0)] &\sum_{i=1}^{k}\sum_{j=1}^{l} h(p_i q_j) \\ &= [g(1) + (k-1)g(0)] \sum_{i=1}^{k} h(p_i) \sum_{j=1}^{l} h(q_j) \\ &\quad + (l-k)[g(1) + (k-1)g(0)]h(0) \sum_{j=1}^{l} h(q_j) \\ &\quad + l(k-1)[g(1) + (k-1)g(0)][h(1) + (l-1)h(0)]h(0),\end{aligned} \tag{3.12}$$

where $h(1) + (l-1)h(0) \neq 0$ *and* $A\colon \mathbb{R} \to \mathbb{R}$ *is an additive function.*

Proof. Putting $p_1 = 1$, $p_2 = \ldots = p_k = 0$ in (1.4), we obtain

$$\sum_{j=1}^{l} \{f(q_j) - [g(1) + (k-1)g(0)]h(q_j)\} = -l(k-1)f(0).$$

Hence, by Lemma 1 (changing q to p),

$$f(p) = [g(1) + (k-1)g(0)]h(p) + A(p) - \frac{1}{l}A(1) - (k-1)f(0) \tag{3.13}$$

for all p, $0 \leqslant p \leqslant 1$, where $A\colon\ \mathbb{R} \to \mathbb{R}$ is an additive function with

$$A(1) = l\{[g(1) + (k-1)g(0)]h(0) - kf(0)\}. \tag{3.14}$$

From equations (3.13) and (3.14), equation (3.8) follows.

From equations (3.8) and (3.14), it is easy to see that

$$\begin{aligned} \sum_{i=1}^{k}\sum_{j=1}^{l} f(p_i q_j) = [g(1) + (k-1)g(0)] &\sum_{i=1}^{k}\sum_{j=1}^{l} h(p_i q_j) \\ &- l(k-1)[g(1) + (k-1)g(0)]h(0). \end{aligned} \tag{3.15}$$

From (1.4) and (3.15) we get (3.9). The proof of (3.10) is similar and hence omitted.

Now, put $q_1 = 1$, $q_2 = \ldots = q_l = 0$ in (3.9). We obtain equation (3.11). Multiplying equation (3.9) by $h(1) + (l-1)h(0) \neq 0$, we obtain

$$\begin{aligned} [g(1) + (k-1)g(0)][h(1) + (l-1)h(0)] &\sum_{i=1}^{k}\sum_{j=1}^{l} h(p_i q_j) \\ = [h(1) + (l-1)h(0)] &\sum_{i=1}^{k} g(p_i) \sum_{j=1}^{l} h(q_j) \\ &+ l(k-1)[g(1) + (k-1)g(0)][h(1) + (l-1)h(0)]h(0). \end{aligned} \tag{3.16}$$

From (3.11) and (3.16), we obtain equation (3.12). □

Proof of Theorem 2. We divide our discussion into three cases:

Case 1. $\sum\limits_{i=1}^{k} g(p_i)$ vanishes identically on Γ_k, that is,

$$\sum_{i=1}^{k} g(p_i) = 0 \tag{3.17}$$

for all $(p_1, \dots, p_k) \in \Gamma_k$. Then (1.4) reduces to (3.5) and h can be an arbitrary function. So, f is of the form (3.6) for all p, $0 \leqslant p \leqslant 1$. Applying Lemma 1 to (3.17), we obtain

$$g(p) = b_2(p) - \frac{1}{k} b_2(1) \tag{3.18}$$

for all p, $0 \leqslant p \leqslant 1$, $b_2\colon \mathbb{R} \to \mathbb{R}$ being an additive function. Equations (3.6), (3.18) together with an arbitrary function h constitute the solution (3.1) of (1.4).

Case 2. $\sum\limits_{j=1}^{l} h(q_j)$ vanishes identically on Γ_l, that is,

$$\sum_{j=1}^{l} h(q_j) = 0 \tag{3.19}$$

for all $(q_1, \dots, q_l) \in \Gamma_l$. Then (1.4) reduces to (3.5) and g can be an arbitrary function. So, f is of the form (3.6) for all p, $0 \leqslant p \leqslant 1$. Applying Lemma 1 to (3.19) we obtain

$$h(p) = b_3(p) - \frac{1}{l} b_3(1) \tag{3.20}$$

for all p, $0 \leqslant p \leqslant 1$, $b_3\colon \mathbb{R} \to \mathbb{R}$ being an additive function. Equations (3.6), (3.20) together with an arbitrary function g constitute the solution (3.2) of (1.4).

Case 3. Neither $\sum\limits_{i=1}^{k} g(p_i)$ vanishes identically on Γ_k nor $\sum\limits_{j=1}^{l} h(q_j)$ vanishes identically on Γ_l. Then there exist a $(p_1^*, \dots, p_k^*) \in \Gamma_k$ and a $(q_1^*, \dots, q_l^*) \in \Gamma_l$ such that $\sum\limits_{i=1}^{k} g(p_i^*) \neq 0$ and $\sum\limits_{j=1}^{l} h(q_j^*) \neq 0$, respectively, and consequently

$$\sum_{i=1}^{k} g(p_i^*) \sum_{j=1}^{l} h(q_j^*) \neq 0. \tag{3.21}$$

We prove that $g(1)+(k-1)g(0) \neq 0$. To the contrary, suppose $g(1)+(k-1)g(0) = 0$. Then (3.9) reduces to

$$\sum_{i=1}^{k} g(p_i) \sum_{j=1}^{l} h(q_j) = 0$$

valid for all $(p_1, \dots, p_k) \in \Gamma_k$ and $(q_1, \dots, q_l) \in \Gamma_l$. In particular, $\sum\limits_{i=1}^{k} g(p_i^*) \sum\limits_{j=1}^{l} h(q_j^*) = 0$, which contradicts (3.21). Hence $g(1) + (k-1)g(0) \neq 0$.

Similarly, making use of (3.10), we can prove that $h(1)+(l-1)h(0)\neq 0$. Now using the fact that $h(1)+(l-1)h(0)\neq 0$, equation (3.11) can be written as

$$\sum_{i=1}^{k}\Big\{g(p_i)-\Big[\frac{g(1)+(k-1)g(0)}{h(1)+(l-1)h(0)}\Big]h(p_i)\Big\}=(l-k)\Big[\frac{g(1)+(k-1)g(0)}{h(1)+(l-1)h(0)}\Big]h(0).$$

Hence, by Lemma 1,

$$\begin{aligned}g(p)=&\Big[\frac{g(1)+(k-1)g(0)}{h(1)+(l-1)h(0)}\Big]h(p)+A^*(p)-\frac{1}{k}A^*(1)\\&+\frac{(l-k)}{k}\Big[\frac{g(1)+(k-1)g(0)}{h(1)+(l-1)h(0)}\Big]h(0)\end{aligned}\tag{3.22}$$

for all p, $0\leqslant p\leqslant 1$ where $A^*\colon\ \mathbb{R}\to\mathbb{R}$ is an additive function with

$$A^*(1)=l\Big[\frac{g(1)+(k-1)g(0)}{h(1)+(l-1)h(0)}\Big]h(0)-kg(0).\tag{3.23}$$

From (3.22) and (3.23) we obtain

$$\begin{aligned}g(p)=&\Big[\frac{g(1)+(k-1)g(0)}{h(1)+(l-1)h(0)}\Big]h(p)+A^*(p)\\&-\Big[\frac{g(1)+(k-1)g(0)}{h(1)+(l-1)h(0)}\Big]h(0)+g(0).\end{aligned}\tag{3.24}$$

Since $[g(1)+(k-1)g(0)][h(1)+(l-1)h(0)]\neq 0$, equation (3.12) gives

$$\begin{aligned}&\frac{\sum_{i=1}^{k}\sum_{j=1}^{l}h(p_iq_j)}{[h(1)+(l-1)h(0)]}\\&=\frac{\sum_{i=1}^{k}h(p_i)}{[h(1)+(l-1)h(0)]}\,\frac{\sum_{j=1}^{l}h(q_j)}{[h(1)+(l-1)h(0)]}\\&\quad+(l-k)\frac{h(0)}{[h(1)+(l-1)h(0)]}\,\frac{\sum_{j=1}^{l}h(q_j)}{[h(1)+(l-1)h(0)]}\\&\quad+l(k-1)\frac{h(0)}{[h(1)+(l-1)h(0)]}.\end{aligned}\tag{3.25}$$

Let us define a mapping $T\colon\ [0,1]\to\mathbb{R}$ as

$$T(x)=[h(1)+(l-1)h(0)]^{-1}h(x)\tag{3.26}$$

for all $x \in [0,1]$. Then, with the aid of (3.26), (3.25) reduces to the functional equation (1.7). Also, from (3.26) it is easy to see that $T(1) + (l-1)T(0) = 1$. Consequently, Theorem 1 implies that T is of the form (2.5), along with (2.6) or (2.7). Equations (2.5), (2.7), (3.8), (3.24) and (3.26) yield the solutions (3.3) along with (3.3a), and (3.4) along with (3.4a) of the functional equation (1.4). The details are omitted for the sake of brevity. □

4. The general solutions of functional equation (1.5) when $\lambda \neq 0$

In this section, we prove

Theorem 3. *Let $k \geqslant 3$, $l \geqslant 3$ be fixed integers and let $F\colon I \to \mathbb{R}$, $G\colon I \to \mathbb{R}$, $H\colon I \to \mathbb{R}$, $I = [0,1]$, be mappings which satisfy the functional equation* (1.5) *for all $(p_1, \dots, p_k) \in \Gamma_k$ and $(q_1, \dots, q_l) \in \Gamma_l$. Then any general solution of* (1.5) *is of the form*

$$\text{(4.1)} \qquad \begin{cases} F(p) = \dfrac{1}{\lambda}\Big[b_1(p) - \dfrac{1}{kl}b_1(1) - p\Big], \\[2mm] G(p) = \dfrac{1}{\lambda}\Big[b_2(p) - \dfrac{1}{k}b_2(1) - p\Big], \\ H \ \text{ any arbitrary function} \end{cases}$$

or

$$\text{(4.2)} \qquad \begin{cases} F(p) = \dfrac{1}{\lambda}\Big[b_1(p) - \dfrac{1}{kl}b_1(1) - p\Big], \\ G \ \text{ any arbitrary function,} \\ H(p) = \dfrac{1}{\lambda}\Big[b_3(p) - \dfrac{1}{l}b_3(1) - p\Big] \end{cases}$$

or

$$\text{(4.3)} \qquad \begin{cases} F(p) = \dfrac{1}{\lambda}\{[\lambda(G(1) + (k-1)G(0)) + 1][\lambda(H(1) + (l-1)H(0)) + 1]a(p) \\ \qquad\qquad + A(p) + \lambda F(0) - p\}, \\[2mm] G(p) = \dfrac{1}{\lambda}\{[\lambda(G(1) + (k-1)G(0)) + 1]a(p) + A^*(p) + \lambda G(0) - p\}, \\[2mm] H(p) = \dfrac{1}{\lambda}\{[\lambda(H(1) + (l-1)H(0)) + 1]a(p) + \lambda H(0) - p\} \end{cases}$$

with

$$
\text{(4.3a)}\qquad \begin{cases} a(1) = 1 - \dfrac{\lambda l H(0)}{[\lambda(H(1) + (l-1)H(0)) + 1]}, \\ A(1) = \lambda l\{[\lambda(G(1) + (k-1)G(0)) + 1]H(0) - kF(0)\}, \\ A^*(1) = \lambda l\Big[\dfrac{\lambda(G(1) + (k-1)G(0)) + 1}{\lambda(H(1) + (l-1)H(0)) + 1}\Big]H(0) - \lambda k G(0) \end{cases}
$$

or

$$
\text{(4.4)}\qquad \begin{cases} F(p) = \dfrac{1}{\lambda}\{[\lambda(G(1) + (k-1)G(0)) + 1][\lambda(H(1) + (l-1)H(0)) + 1] \\ \qquad\quad \times [M(p) - b(p)] + A(p) + \lambda F(0) - p\}, \\ G(p) = \dfrac{1}{\lambda}\{[\lambda(G(1) + (k-1)G(0)) + 1][M(p) - b(p)] \\ \qquad\quad + A^*(p) + \lambda G(0) - p\}, \\ H(p) = \dfrac{1}{\lambda}\{[\lambda(H(1) + (l-1)H(0)) + 1][M(p) - b(p)] + \lambda H(0) - p\} \end{cases}
$$

with

$$
\text{(4.4a)}\qquad \begin{cases} b(1) = \dfrac{\lambda l H(0)}{[\lambda(H(1) + (l-1)H(0)) + 1]}, \\ A(1) = \lambda l\{[\lambda(G(1) + (k-1)G(0)) + 1]H(0) - kF(0)\}, \\ A^*(1) = \lambda l\Big[\dfrac{\lambda(G(1) + (k-1)G(0)) + 1}{\lambda(H(1) + (l-1)H(0)) + 1}\Big]H(0) - \lambda k G(0), \end{cases}
$$

where $b_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$; $a\colon \mathbb{R} \to \mathbb{R}$, $b\colon \mathbb{R} \to \mathbb{R}$, $A\colon \mathbb{R} \to \mathbb{R}$, $A^*: \mathbb{R} \to \mathbb{R}$ *are additive functions;* $M\colon [0,1] \to \mathbb{R}$ *is a nonconstant nonadditive multiplicative function, and*

$$
[\lambda(G(1) + (k-1)G(0)) + 1][\lambda(H(1) + (l-1)H(0)) + 1] \neq 0.
$$

Proof. Let us write (1.5) in the form

$$
\text{(4.5)}\qquad \sum_{i=1}^{k}\sum_{j=1}^{l}[\lambda F(p_i q_j) + p_i q_j] = \sum_{i=1}^{k}[\lambda G(p_i) + p_i]\sum_{j=1}^{l}[\lambda H(q_j) + q_j].
$$

Define mappings $f\colon I \to \mathbb{R}$, $g\colon I \to \mathbb{R}$, $h\colon I \to \mathbb{R}$ as

$$
\text{(4.6)}\qquad \begin{cases} f(x) = \lambda F(x) + x, \\ g(x) = \lambda G(x) + x, \\ h(x) = \lambda H(x) + x \end{cases}
$$

for all $x \in I$. Then (4.5) reduces to the functional equation (1.4) whose solutions are given by (3.1), (3.2), (3.3) along with (3.3a) and (3.4) along with (3.4a), in which $b_i\colon \mathbb{R} \to \mathbb{R}$, $i = 1, 2, 3$; $a\colon \mathbb{R} \to \mathbb{R}$, $b\colon \mathbb{R} \to \mathbb{R}$, $A\colon \mathbb{R} \to \mathbb{R}$, $A^*\colon \mathbb{R} \to \mathbb{R}$ are additive functions and $M\colon [0, 1] \to \mathbb{R}$ is a nonconstant nonadditive multiplicative function. Now, making use of (4.6) and (3.1), (3.2), (3.3) along with (3.3a), (3.4) along with (3.4a), the required solutions (4.1), (4.2), (4.3) along with (4.3a) and (4.4) along with (4.4a) follow. The details are omitted. □

5. Comments

Losonczi [13] considered the functional equation

$$\sum_{i=1}^{k}\sum_{j=1}^{l} F_{ij}(p_i q_j) = \sum_{i=1}^{k} G_i(p_i) + \sum_{j=1}^{l} H_j(q_j) + \lambda \sum_{i=1}^{k} G_i(p_i) \sum_{j=1}^{l} H_j(q_j) \tag{5.1}$$

with $(p_1, \ldots, p_k) \in \Gamma_k$, $(q_1, \ldots, q_l) \in \Gamma_l$, $\lambda \neq 0$, $F_{ij}\colon I \to \mathbb{R}$, $G_i\colon I \to \mathbb{R}$, $H_j\colon I \to \mathbb{R}$ as unknown functions. He found measurable (in the sense of Lebesgue) solutions of (5.1) by taking $k \geqslant 3$, $l \geqslant 3$ as fixed integers; $i = 1, \ldots, k$, $j = 1, \ldots, l$; in Theorem 6 on p. 69 in Losonczi [13]. Even if we take $k = 3$ and $l = 3$, it is obvious that the functional equation (5.1) contains fifteen unknown functions, which is a significantly large number. Hence, it seems improbable that the measurable solutions of (5.1) will be of direct importance from the information-theoretic point of view. However, some special cases of (5.1) are certainly useful from the information-theoretic point of view. For instance, if we take $F_{ij} = F$, $G_i = G$, $H_j = H$, $i = 1$ to k, $j = 1$ to l, then (5.1) reduces to (1.5). This is the reason for considering (1.5).

Losonczi and Maksa [14] have shown that if a function $F\colon [0, 1] \to \mathbb{R}$ satisfies equation (1.1) for $\lambda \neq 0$, $k \geqslant 3$, $l \geqslant 3$ fixed integers, then it is of the form

$$F(p) = \frac{a(p) + \alpha_1 - p}{\lambda}, \quad p \in [0, 1] \tag{5.2}$$

or

$$F(p) = \frac{M(p) - A(p) - p}{\lambda}, \quad p \in [0, 1] \tag{5.3}$$

where a, A are additive functions, α_1 is an arbitrary constant, $A(1) = 0$; M is a multiplicative function and $a(1)$ satisfies

$$a(1) + kl\alpha_1 = (a(1) + k\alpha_1)(a(1) + l\alpha_1). \tag{5.4}$$

From (5.2) it follows that

$$\sum_{i=1}^{k} F(p_i) = \frac{a(1) + k\alpha_1 - 1}{\lambda},$$

which is independent of $p_1, \ldots, p_k$. So, the solution (5.2) is not of any use from the information-theoretic point of view. From (5.3) we have

$$\sum_{i=1}^{k} F(p_i) = L_k^{\lambda}(p_1, \ldots, p_k)$$

where

$$L_k^{\lambda}(p_1, \ldots, p_k) = \frac{1}{\lambda}\Bigl(\sum_{i=1}^{k} M(p_i) - 1\Bigr), \tag{5.5}$$

which is certainly useful from the information-theoretic point of view. The nonadditive measure of entropy, given by Havrda and Charvát [9], is a particular case of (5.5) when $\lambda = 2^{1-\alpha} - 1$, $\alpha \neq 1$ and $M\colon\ [0,1] \to \mathbb{R}$ is of the form $M(p) = p^{\alpha}$, $0 \leqslant p \leqslant 1$, $\alpha \neq 1$, $\alpha > 0$, $0^{\alpha} := 0$, $1^{\alpha} := 1$.

The solution (4.1) is not of any relevance in information theory as the mapping H in it is an arbitrary function. The same is true of (4.2) as the mapping G in it is also an arbitrary function. As regards (4.3), each of the summands $\sum\limits_{i=1}^{k} F(p_i)$, $\sum\limits_{i=1}^{k} G(p_i)$, $\sum\limits_{i=1}^{k} H(p_i)$ is independent of $p_1, \ldots, p_k$ and hence (4.3) is not of much significance from the point of view of information theory, either.

Solution (4.4) is certainly useful from the information-theoretic point of view. Let us put

$$\begin{aligned}
\beta_1 &= \lambda(G(1) + (k-1)G(0)) + 1,\\
\beta_2 &= \lambda(H(1) + (l-1)H(0)) + 1,\\
\beta_3 &= \lambda k(l-1)F(0)
\end{aligned}$$

and

$$\beta_4 = \lambda(k-l)H(0).$$

Then from the last line in the statement of Theorem 3, it follows that $\beta_1 \neq 0$, $\beta_2 \neq 0$. Now, from (4.4) it can be easily seen that

$$\text{(5.6)} \qquad \begin{cases} \sum_{i=1}^{k} F(p_i) = \beta_1\beta_2 L_k^{\lambda}(p_1,\dots,p_k) + \frac{1}{\lambda}(\beta_1\beta_2 - \beta_3 - 1), \\ \sum_{i=1}^{k} G(p_i) = \beta_1 L_k^{\lambda}(p_1,\dots,p_k) + \frac{1}{\lambda}(\beta_1 - 1), \\ \sum_{i=1}^{k} H(p_i) = \beta_2 L_k^{\lambda}(p_1,\dots,p_k) + \frac{1}{\lambda}(\beta_2 + \beta_4 - 1), \end{cases}$$

where $\beta_1 \neq 0$, $\beta_2 \neq 0$, β_3 and β_4 are arbitrary real constants.

Each of the summands $\sum_{i=1}^{k} F(p_i)$, $\sum_{i=1}^{k} G(p_i)$, $\sum_{i=1}^{k} H(p_i)$ reduces to $L_k^{\lambda}(p_1,\dots,p_k)$, given by (5.5) and characterized by Losonczi and Maksa [14] when $\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = 0$ (which means $F(0) = 0$) and $\beta_4 = 0$ (which means $H(0) = 0$). So, the functional equations (1.4) and (1.5) lead certainly to the meaningful entropies given by (5.6), and the measure $L_k^{\lambda}(p_1,\dots,p_k)$ given by (5.5) follows as a special case of them when $\beta_1 = 1$, $\beta_2 = 1$, $\beta_3 = 0$ (that is, $F(0) = 0$) and $\beta_4 = 0$ (that is, $H(0) = 0$).

If we take $M(p) = p^2$, $0 \leqslant p \leqslant 1$, then each of the summands $\sum_{i=1}^{k} F(p_i)$, $\sum_{i=1}^{k} G(p_i)$, $\sum_{i=1}^{k} H(p_i)$ in (5.6) reduces to an expression of the form

$$\text{(5.7)} \qquad \mu\left\{\frac{1}{\lambda}\left(\sum_{i=1}^{k} p_i^2 - 1\right)\right\} + \frac{\nu}{\lambda} \qquad \text{(both } \mu \neq 0 \text{ and } \nu \text{ real)},$$

which is the quadratic entropy (upto additive and nonzero multiplicative constants) due to Vajda [18].

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