Pavel Doktor; Alexander Ženíšek
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THE DENSITY OF INFINITELY DIFFERENTIABLE FUNCTIONS
IN SOBOLEV SPACES WITH MIXED BOUNDARY CONDITIONS*

Pavel Doktor, Praha, Alexander Ženíšek, Brno

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Abstract. We present a detailed proof of the density of the set $C^\infty(\overline{\Omega}) \cap V$ in the space of test functions $V \subset H^1(\Omega)$ that vanish on some part of the boundary $\partial\Omega$ of a bounded domain $\Omega$.

Keywords: density theorems, finite element method

MSC 2000: 46E35

Let $\Omega$ be a nonempty bounded domain in $\mathbb{R}^N$ ($N = 2$ or 3). The symbol $C^\infty(\overline{\Omega})$ denotes the set containing all restrictions to $\overline{\Omega}$ of infinitely smooth functions defined on $\mathbb{R}^N$ (see [5, 1.2.1, 1.2.3 and 5.2.1]). Further, the symbol $V$ denotes the set of test functions belonging to $H^1(\Omega)$ (for detailed definition of $V$ see Theorem 1), where $H^1(\Omega) \equiv H^{1,2}(\Omega)$ is the Sobolev space in the notation defined in [5, 5.4.1].

In this paper we present a detailed proof of the density of $C^\infty(\overline{\Omega}) \cap V$ in $V$ (see Theorem 1) the use of which is necessary when proving the convergence of the finite element method without any regularity assumptions on the exact solution $u$ of a given variational problem, i.e., when proving the relation (which we present in (*) for the case of a variational problem corresponding to a second order elliptic boundary value problem)

\[ \lim_{h \to 0} \|\tilde{u} - u_h\|_{1,\Omega_h} = 0, \]

where $\tilde{u}$ is the Calderon extension (see [7, p. 77]) of the exact solution and $u_h$ is the approximate solution by the finite element method. Many authors consider the density of $C^\infty(\overline{\Omega}) \cap V$ in $V$ to be evident and using it they do not give any reference.

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The assertion of Theorem 1 was given (in a little more general form) in [4]. However, the proof presented in [4] is so concise that almost no reader will have patience to read and understand it. For this reason we present in this paper a sufficiently detailed proof of this result which is of basic importance in the theory of convergence of the finite element method.

We also restrict ourselves to the class of domains $\tilde{C}^{0,1} \subset C^{0,1}$, where $C^{0,1}$ denotes the set of domains with Lipschitz continuous boundary, in the following sense: If $\Omega \in \tilde{C}^{0,1}$ then $\Omega \in C^{0,1}$ and the boundary $\partial \Omega$ of $\Omega$ consists of a finite number of smooth parts which have a finite number of relative maxima and minima and inflexions and in the three-dimensional case also a finite number of saddle points. To consider such a class of domains is sufficient for applications.

Further, we will consider parts $\gamma_i$ of $\partial \Omega$, on which homogeneous Dirichlet boundary condition will be prescribed, which satisfy the following condition. Let $\gamma_i \subset \partial \Omega$ be a relatively open set (i.e., open in the metric space $\partial \Omega$). We say that $\gamma_i$ has a Lipschitz relative boundary $\partial \gamma_i$ (i.e., the boundary in the metric space $\partial \Omega$) and write $\gamma_i \in \text{LRB}$ if either $\dim \Omega = 2$, or if in the case $\dim \Omega = 3$ it has the following property:

Let $X_0$ be an arbitrary point of $\partial \gamma_i$ and let $N(X_0)$ be a neighbourhood of $X_0$ such that $N(X_0) \cap \partial \Omega$ is expressed as a graph $x_3 = a(x_1, x_2)$. Let further $G_i$ be the image of $\gamma_i \cap N(X_0)$ in the projection to the plane $x_1, x_2$ with the boundary $\partial G_i$. Then $G_i$ has the same property as $\Omega$, i.e., $\partial G_i$ is locally representable as a graph of a Lipschitz function in one variable (obviously this definition is independent of the description of $\partial \Omega$). (In this case we use the notation $\gamma_i \in \text{LRB}$ as mentioned above).

**Theorem 1** (on the density of $C^\infty(\overline{\Omega}) \cap V$ in $V$). Let $\Omega \in \tilde{C}^{0,1}$ and let

\begin{equation}
V = \{v \in H^1(\Omega): v = 0 \text{ on } \Gamma_1, \text{ where } \Gamma_1 \subset \partial \Omega, \text{ meas}_{N-1} \Gamma_1 > 0 \text{ with } N = 2 \text{ or } N = 3\},
\end{equation}

where $\Gamma_1$ consists of a finite number of relatively open parts in $\partial \Omega$, say $\gamma_1, \ldots, \gamma_m$ such that $\gamma_i \in \text{LRB}$ ($i = 1, \ldots, m$). Then the set $C^\infty(\overline{\Omega}) \cap V$ is dense in $V$.

The proof of Theorem 1 will be divided into four parts:
1) formulation of Lemma 2;
2) the idea of the proof of Lemma 2;
3) the proof of Theorem 1 by means of Lemma 2;
4) the detailed proof of Lemma 2.
Lemma 2. Let $\Theta$ be an $N$-dimensional parallelepiped

$$\Theta = \{ X \in \mathbb{R}^N : X' \in \Delta, x_N \in (-\beta, 0) \},$$

where $\Delta = (-\alpha, \alpha)^{N-1}$ and $\alpha, \beta$ are positive numbers.

Let $G \subset \overline{G} \subset \Delta$ be a domain such that $G \in \mathcal{C}^{0,1}$ and let $\overline{G} \subset U$, where $U \subset \Delta$ is an open set.

Let us denote $\Theta_1 = (-\alpha, \alpha)^{N-1} \times (-\beta, \beta)$. Further, let us denote $\Gamma = G \times \{0\}$ and let $\mathcal{K} \subset \Theta_1$ be a compact set, $\Gamma \subset \mathcal{K}$. (See Fig. 1 in the case of $N = 2$.)

Then there exists a compact set $\mathcal{K}_1 \subset \Theta_1$, $\mathcal{K}_1 \supset \mathcal{K}$ (where $\mathcal{K}_1$ depends only on $\mathcal{K}$) with the following property ($\mathcal{K}_1$ will be defined at the end of the idea of the proof of Lemma 2 (see the text following relation (12))):

Let $u \in H^1(\Theta)$ be an arbitrary function which is equal to zero on $\Gamma$ (in the sense of traces) and $\text{supp} \; u \subset \mathcal{K}$.

Then there exists a sequence $\{ u_n \} \subset C^\infty(\Theta_1)$ such that $\text{supp} \; u_n \subset \mathcal{K}_1 \setminus \overline{\Gamma}$, where $\Gamma = G \times \{0\}$, and $u_n \to u$ in the space $H^1(\Theta)$.

Figure 1. A two-dimensional example with $G$ an interval.
The idea of the proof of Lemma 2. Due to the assumptions \( G \subset \Delta \), \( K \subset \Theta_1 \) we have (see Fig. 1 for \( N = 2 \))

\[
\text{dist}(K, \mathbb{R}^N \setminus \Theta_1) = \nu > 0.
\]

Let us denote successively

\[
\begin{align*}
U_\lambda(G) &= \{ X' \in \Delta : \text{dist}(X', G) < \lambda \}, \\
V_\lambda(G) &= \{ X' \in \Delta : \text{dist}(X', G) < \frac{3}{4} \lambda \}, \\
W_\lambda(G) &= \{ X' \in \Delta : \text{dist}(X', G) < \frac{1}{2} \lambda \}, \\
Z_\lambda(G) &= \{ X' \in \Delta : \text{dist}(X', G) < \frac{1}{4} \lambda \}
\end{align*}
\]

and, correspondingly (see Fig. 1 for \( N = 2 \)),

\[
\begin{align*}
U_\lambda(\Gamma) &= U_\lambda(G) \times \left( -\lambda, \frac{1}{4} \lambda \right), \\
V_\lambda(\Gamma) &= V_\lambda(G) \times \left( -\frac{3}{4} \lambda, \frac{1}{4} \lambda \right), \\
W_\lambda(\Gamma) &= W_\lambda(G) \times \left( -\frac{1}{2} \lambda, \frac{1}{4} \lambda \right), \\
Z_\lambda(\Gamma) &= Z_\lambda(G) \times \left( -\frac{1}{4} \lambda, \frac{1}{4} \lambda \right),
\end{align*}
\]

where \( \lambda \) is supposed sufficiently small, thus satisfying

\[
\lambda < \frac{1}{2} \nu.
\]

Let us put \( h = \frac{1}{4} \lambda \) and (see Fig. 2)

\[
(7) \quad u_\lambda(X', x_N) := u(X', x_N - h), \quad [X', x_N] \in \Theta
\]

with \( u_\lambda \in H^1(\Theta_2) \), \( \Theta_2 \equiv \Theta_2(h) = \Delta \times (-\beta, h) \),

\[
(8) \quad v_\lambda(X) = \begin{cases} 
0, & X \in W_\lambda(\Gamma), \\
u_\lambda(X), & X \in \Theta_2 \setminus W_\lambda(\Gamma),
\end{cases}
\]

\[
(9) \quad w_\lambda(X) = (\omega_h \ast v_\lambda)(X).
\]

In (9) we have used the brief notation for convolution

\[
(10) \quad (\omega_h \ast u)(X) = \int_{\mathbb{R}^N} \omega_h(X - Y)u(Y) \, dY = \int_{\mathbb{R}^N} \omega_h(Y)u(X - Y) \, dY,
\]

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where the mollifier $\omega_h(Z)$ is defined by the relations
\begin{equation}
\omega_h(Z) = \begin{cases} 
\kappa h^{-N} \exp\left(\frac{\|Z\|^2}{\|Z\|^2 - h^2}\right) & \text{for } \|Z\| < h, \\
0 & \text{for } \|Z\| \geq h;
\end{cases}
\end{equation}

the symbol $\| \cdot \|$ denotes the Euclidean norm and the constant $\kappa$ is defined by
\begin{equation}
\int_{\mathbb{R}^N} \omega_1(Z) \, dZ = \int_{\mathbb{R}^N} \omega_h(Z) \, dZ = 1,
\end{equation}
from which we obtain
\begin{equation*}
\kappa = \left(\int_{\mathbb{R}^N} \exp\left(\frac{\|Z\|^2}{\|Z\|^2 - 1}\right) \, dZ\right)^{-1}.
\end{equation*}
The first equality (12) follows from the fact that
\[ \|Z\|^2/(\|Z\|^2 - h^2) = \|Z/h\|^2/(\|Z/h\|^2 - 1) \]
and from the substitution $Z = hX = [hx_1, \ldots, hx_N]$ which implies $dZ = h^N \, dX$. Thus the convolution (10) is well-defined for $u \in L^2_\Omega$ and its restriction onto $\overline{\Omega}$ belongs to the space $C^\infty(\overline{\Omega})$.

We see immediately that $u_\lambda \in H^1(\Theta)$. It follows from (4)–(6) that $\text{supp} \, w_\lambda \subset K_1$, where $K_1 = \{X \in \mathbb{R}^N : \text{dist}(X,K) < \frac{1}{2} \nu\} \subset \Theta_1$ depends only on $K$ (because, according to (3), $\nu$ depends on $K$), and that $w_\lambda(X) = 0$ for $X \in Z_\lambda(\Gamma)$ (see Fig. 2). In what follows we show that $\|u - w_\lambda\|_{1,\Theta} \to 0$ for $\lambda \to 0$, which proves Lemma 2. □

\begin{figure}[h]
\centering
\begin{tikzpicture}
\draw[->] (-2,0) -- (2,0) node[right]{$v_\lambda \not= 0$};
\draw[->] (0,-2) -- (0,2) node[above]{$v_\lambda \equiv 0$};
\filldraw[black] (0,0) circle (2pt) node[below]{$W_\lambda(\Gamma)$};
\filldraw[black] (2,0) circle (2pt) node[below]{$V_\lambda(\Gamma)$};
\filldraw[black] (-2,0) circle (2pt) node[below]{$Z_\lambda(\Gamma)$};
\draw[->] (0,0) -- (0,-1) node[left]{$h$};
\end{tikzpicture}
\caption{Concerning relations (7)–(12).}
\end{figure}
Proof of Theorem 1 by means of Lemma 2. Let \( u \in V \) be an arbitrary but fixed function. We show that there exists such a sequence \( \{u_n\} \subset C^\infty(\overline{\Omega}) \) that

\[
\begin{align*}
  (13) & \quad u_n = 0 \text{ on an } N\text{-dimensional neighbourhood of } \Gamma_1, \\
  (14) & \quad u_n \to u \text{ in the space } H^1(\Omega).
\end{align*}
\]

For better clarity of exposition we restrict ourselves to the two-dimensional case, i.e., \( N = 2 \).

The domain \( \Omega \) has a Lipschitz continuous boundary and hence for any \( X \in \partial \Omega \) there exists a (local) Cartesian coordinate system \((x_1, x_2)\) and a Lipschitz function \( a(x_1) \) with the domain of definition \( \Delta = (-\alpha, \alpha) \subset \mathbb{R}^1 \) such that

\[
\begin{align*}
  \mathbb{U} & = \{[x_1, x_2]: x_1 \in \Delta, a(x_1) - \beta < x_2 < a(x_1)\} \subset \Omega, \\
  \mathbb{V} & = \{[x_1, x_2]: x_1 \in \Delta, a(x_1) - x_2 < a(x_1) + \beta\} \subset \mathbb{R}^2 \setminus \overline{\Omega},
\end{align*}
\]

where \( \alpha > 0, \beta > 0 \) are suitable constants. Let us denote

\[
\begin{align*}
  \mathbb{U}_r & = \bigcup \{[x_1, x_2]: x_1 \in \Delta, x_2 = a(x_1)\}, \\
  \mathbb{V}_r & = \bigcup \{[x_1, x_2]: x_1 \in \Delta, a(x_1) < x_2 < a(x_1) + \beta\} \subset \mathbb{R}^2 \setminus \overline{\Omega},
\end{align*}
\]

Owing to the compactness of \( \partial \Omega \) we can cover \( \partial \Omega \) by a finite number of such domains \( \mathbb{Z}_1, \ldots, \mathbb{Z}_m \). (The local Cartesian coordinate system \( x_1, x_2 \) and the function \( a(x_1) \) corresponding to \( \mathbb{Z}_r \) will be now denoted by \( x_1^r, x_2^r \) and \( a_r(x_1^r) \), respectively.) Further, we can find a domain \( \mathbb{Z}_0 \) such that \( \overline{\mathbb{Z}_0} \subset \Omega \) (\( \mathbb{Z}_0 \) is considered in the global Cartesian coordinate system \( x_1, x_2 \)) and

\[
\overline{\Omega} \subset \bigcup_{r=0}^{m} \mathbb{Z}_r.
\]

Owing to the compactness of \( \overline{\Omega} \) we can construct a partition of unity, i.e., a system of functions \( \varphi_r \in C_0^\infty(\mathbb{Z}_r) \) \((r = 0, 1, \ldots, m)\) which for \( X \in \overline{\Omega} \) (the points \( X \) are considered in the global system \( x_1, x_2 \)) satisfy

\[
0 \leq \varphi_r(X) \leq 1, \quad \sum_{r=0}^{m} \varphi_r(X) = 1.
\]

We can transform \( \mathbb{U}_r \) \((r = 1, \ldots, m)\) to the parallelepiped

\[
\Theta = (-\alpha, \alpha)^{N-1} \times (-\beta, 0) \quad \text{(in our case } N = 2)\]

\footnote{If \( N > 2 \) then we substitute \([x_1, x_2]\) by \([X', x_N] = [x_1, \ldots, x_{N-1}, x_N]\) and the function \( a(x_1) \) by \( a(X') \) with the domain of definition \( \Delta = (-\alpha, \alpha)^{N-1} \subset \mathbb{R}^{N-1} \).}
by means of the lipschitzian mapping

\[ T_r : \xi_1 = x_r, \xi_2 = x_r^r - a_r(x_r^r). \]

This transformation maps continuously \( H^1(\mathbb{U}_r) \) (\( 1 \leq r \leq m \)) onto \( H^1(\Theta) \) (see [7, Lemma 2.3.2 on p. 66]) and \( \text{supp} \varphi_r \) onto a compact set \( \mathcal{K}_r \subset \Theta_1 = T_r(\mathbb{Z}_r), \mathcal{K}_r = T_r(\text{supp} \varphi_r) \).

Let \( G' = T_r(\Gamma_1 \cap \text{supp} \varphi_r) \). We have \( G' \subset \overline{G'} \subset \Delta \). Thus \( \mathcal{K}_r \subset \Theta_1 \) is the compact set \( \mathcal{K} \) from Lemma 2. Let \( \mathcal{K}_{r,1} \subset \Theta_1 \) \( (\mathcal{K}_{r,1} \supset \mathcal{K}_r) \) be the compact set \( \mathcal{K}_1 \) from Lemma 2.

Hence, according to Lemma 2, we can approach \( T_r(\varphi_r u) \) by a sequence \( \{v_{n,r}\} \subset C^\infty(\overline{\Theta}_1), \text{supp} v_{n,r} \subset \mathcal{K}_{r,1} \setminus \overline{G'} \) \( (\Gamma' = G' \times \{0\}) \).

The main step of the proof consists in the following argument: The functions \( \tilde{u}_{n,r} = T_r^{-1}(v_{n,r}) \) belong to \( H^1(C) \) \( (C \subset \mathbb{R}^N \) is an \( N \)-dimensional cube which contains \( \overline{\Omega} \)),

\[ \tilde{u}_{n,r} = 0 \quad \text{in a neighbourhood of} \quad \Gamma_1 \cap \text{supp} \varphi_r \]

and \( \tilde{u}_{n,r} \rightarrow u\varphi_r \) in \( H^1(\Omega) \).

Applying the mollifier \( \omega_h \) to \( \tilde{u}_{n,r} \) we can replace \( \tilde{u}_{n,r} \) by \( u_{n,r} \subset C^\infty(\overline{\Omega}) \) having the same property (15) and converging to \( u\varphi_r \) in \( H^1(\Omega) \). Finally, we approach \( u\varphi_0 \) by the sequence \( \{u_{n,0}\} \subset C^\infty(\Omega) \) and write \( u_n = \sum_{r=0}^{m} u_{n,r} \), which proves the theorem. \( \square \)

The detailed proof of Lemma 2. Let us denote \( \sigma_\lambda = \Theta \setminus V_\lambda(\Gamma), P_\lambda = V_\lambda(\Gamma) \cap \Theta \) and recall that \( \Theta_2 = \Delta \times (-\beta, h) \). The proof proceeds as follows: Due to the fact that \( \overline{\Theta} = \overline{\sigma_\lambda} \cup \overline{\mathcal{P}_\lambda} \), we can write

\[ \|w_\lambda - u\|_{1,\Theta} \leq \|u - u_\lambda\|_{1,\Theta} + \|u_\lambda - w_\lambda\|_{1,\Theta} \]

\[ \leq \|u - u_\lambda\|_{1,\Theta} + \|u_\lambda - w_\lambda\|_{1,\sigma_\lambda} + \|u_\lambda - w_\lambda\|_{1,\mathcal{P}_\lambda} \]

and prove successively (in parts A–D) that all terms on the right-hand side of (16) tend to zero with \( \lambda \to 0 \). The main difficulty is to prove that \( \|w_\lambda\|_{1,\mathcal{P}_\lambda} \to 0 \) (if \( \lambda \to 0 \)), in particular to prove

\[ \left\| \frac{\partial w_\lambda}{\partial x_i} \right\|_{0,\mathcal{P}_\lambda} \to 0 \quad \text{if} \quad \lambda \to 0 \quad (i = 1, \ldots, N). \]

A. First we prove

\[ \lim_{\lambda \to 0} \|u - u_\lambda\|_{1,\Theta} = 0. \]

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Let us denote \( \overline{h} = (0, \ldots, 0, h) \).

As
\[
\|u - u_\lambda\|_{0, \Theta} = \sqrt{\int_\Theta [u(X) - u(X - \overline{h})]^2 \, dX},
\]
we have, according to [5, Theorem 2.4.2] (the mean continuity theorem for \( L_2 \)-functions),
\[
\lim_{\lambda \to 0} \|u - u_\lambda\|_{0, \Theta} = 0.
\]

As \( D^\alpha u \in L_2(\Theta) \), \( D^\alpha u_\lambda \in L_2(\Theta) \) \((|\alpha| = 1)\) and as
\[
\|D^\alpha u - D^\alpha u_\lambda\|_{0, \Theta} = \sqrt{\int_\Theta [(D^\alpha u)(X) - (D^\alpha u)(X - \overline{h})]^2 \, dX},
\]
we have again, according to [5, Theorem 2.4.2],
\[
\lim_{\lambda \to 0} \|D^\alpha u - D^\alpha u_\lambda\|_{0, \Theta} = 0.
\]

Relations (18) and (19) together give relation (17).

B. Now we prove
\[
\lim_{\lambda \to 0} \|u_\lambda\|_{1, P_\lambda} = 0.
\]

We have mentioned at the end of the idea of the proof of Lemma 2 that
\[
u_\lambda \in H^1(\Theta).
\]

As (see Fig. 3)
\[
P_\lambda = V_\lambda(\Gamma) \cap \Theta,
\]
we have, due to (21),
\[
u_\lambda \in H^1(P_\lambda).
\]
Figure 3. Domains $P_\lambda$ and $Q_\lambda$.

By the definition of $V_\lambda(\Gamma)$ we have

$$\lim_{\lambda \to 0} (\text{meas}_N V_\lambda(\Gamma)) = 0;$$

hence, taking into account (22), we obtain from (24)

$$\lim_{\lambda \to 0} \text{meas}_N P_\lambda = 0.$$

Relation (23) yields

$$u_\lambda \in L_2(P_\lambda), \quad D^\alpha u_\lambda \in L_2(P_\lambda) \quad (|\alpha| = 1).$$

Relations (25) and (26) imply, according to the theorem on the absolute continuity of an integral, relation (20).

C. Our task in this subsection is to prove that

$$\lim_{\lambda \to 0} \|u_\lambda - w_\lambda\|_{1,\sigma_\lambda} \to 0,$$

where $u_\lambda$ and $w_\lambda$ are defined in (7)–(9). As

$$\sigma_\lambda = \Theta \setminus \overline{V_\lambda(\Gamma)}$$

we have

$$\Theta_2 \setminus \overline{W_\lambda(\Gamma)} \supset \sigma_\lambda.$$
As \( u_\lambda \in H^1(\Theta) \) and \( \text{supp } u \subset K \) we have (by (7)) \( u_\lambda \in H^1(\Theta_2) \). Hence \( v_\lambda \in H^1(\Theta_2 \setminus W_\lambda(\Gamma)) \) and (by (29)) \( v_\lambda \in H^1(\sigma_\lambda) \). By (3), (7), (8) and the assumption \( \text{supp } u \subset K \) we have \( \text{dist}(\text{supp } v_\lambda, \mathbb{R}^N \setminus \Theta_1) \geq h \). Thus we can repeat with another notation the proof of [10, Lemma 3.6] and prove the implication

\[
(30) \quad X \in \sigma_\lambda \cap \text{supp } v_\lambda \Rightarrow \frac{\partial}{\partial x_i}(\omega_h \ast v_\lambda) = \omega_h \ast \frac{\partial v_\lambda}{\partial x_i}.
\]

(Using the notation of [10, Chapter 3] we have \( \omega_h \ast u = R_h u \).) By [10, Theorem 3.7] implication (30) yields (together with the preceding text)

\[
\|u_\lambda - w_\lambda\|_{1,\sigma_\lambda} = \sqrt{\int_{\sigma_\lambda \cap \text{supp } v_\lambda} \sum_{|\alpha| \leq 1} [D^\alpha u_\lambda - D^\alpha(\omega_h \ast v_\lambda)]^2 \, dX} = \sqrt{\int_{\sigma_\lambda \cap \text{supp } v_\lambda} \sum_{|\alpha| \leq 1} [D^\alpha u_\lambda - \omega_h \ast D^\alpha u_\lambda]^2 \, dX} \rightarrow 0 \quad \text{if } \lambda \rightarrow 0.
\]

This proves relation (27).

**D.** Our task is now to estimate the function \( w_\lambda \) as an element of \( H^1(P_\lambda) \) (for the definition of \( P_\lambda \) see (22)). To this end, let us denote \( Q_\lambda = U_\lambda(\Gamma) \setminus \overline{W_\lambda(\Gamma)} \). Owing to the choice of \( h \) we obtain for \( X \in P_\lambda \) (cf. Fig. 3, properties of \( \omega_h \) and definitions of \( u_\lambda \) and \( v_\lambda \); we must realize that \( h \rightarrow 0 \)—in Fig. 1 the variable quantity \( h \) is relatively large—and that \( K \) and \( K_1 \) are fixed)

\[
(31) \quad w_\lambda(X) = \int_{\mathbb{R}^N} \omega_h(X - Y)v_\lambda(Y) \, dY
\]

\[
= \int_{\mathbb{R}^N \setminus W_\lambda(\Gamma)} \omega_h(X - Y)v_\lambda(Y) \, dY
\]

\[
= \int_{Q_\lambda} \omega_h(X - Y)u_\lambda(Y) \, dY \quad (X \in P_\lambda)
\]

(if \( X \in P_\lambda \) then \( \|X - Y\| < h \) only for the points \( Y \in Q_\lambda \) and similarly (again for \( X \in P_\lambda \))

\[
(32) \quad \psi_i(X) = \frac{\partial w_\lambda}{\partial x_i}(X) = \int_{Q_\lambda} \frac{\partial \omega_h}{\partial x_i}(X - Y)u_\lambda(Y) \, dY
\]

\[
= -\int_{Q_\lambda} \frac{\partial \omega_h}{\partial y_i}(X - Y)u_\lambda(Y) \, dY \quad (X \in P_\lambda).
\]

Now we apply [5, Theorem 2.5.3] (see also [10, Theorem 3.7]) which asserts that

\[
\lim_{\varepsilon \rightarrow 0^+} \|D^\alpha(R_\varepsilon u) - D^\alpha u\|_{L^2(\Omega^*)} = 0 \quad (\overline{\Omega^*} \subset \Omega).
\]
To this end let us denote by \( \tilde{u}_\lambda \) the extension of the function \( u_\lambda \) by zero onto the domain \( \Theta_1 \). In the notation of [5, Theorem 2.5.3] (or [10, Theorem 3.7]) we have

\[
\Omega = \Theta_1, \quad \Omega^* = P_\lambda.
\]

The extended function \( \tilde{u}_\lambda \) satisfies

\[
\tilde{u}_\lambda \in L_2(\Theta_1).
\]

An application of [5, Theorem 2.5.3] (or [10, Theorem 3.7]) with \( |\alpha| = 0 \) to the functions

\[
u = \tilde{u}_\lambda, \quad R_\varepsilon u = w_\lambda
\]

and domains (33) yields

\[
\lim_{\lambda \to 0} \| w_\lambda - u_\lambda \|_{0,P_\lambda} = 0.
\]

Let us combine (34) with the inequality

\[
\| w_\lambda \|_{0,P_\lambda} \leq \| u_\lambda \|_{0,P_\lambda} + \| w_\lambda - u_\lambda \|_{0,P_\lambda}
\]

and relation (20). Then we obtain

\[
\lim_{\lambda \to 0} \| w_\lambda \|_{0,P_\lambda} = 0.
\]

The same device cannot be used in the case of the functions \( \psi_i \) (given by (32)) because \( \tilde{u}_\lambda \notin H^1(\Theta_1) \) and hence

\[
\frac{\partial \tilde{u}_\lambda}{\partial x_i} \notin L_2(\Theta_1).
\]

Another approach must be used. Let us consider now the \( L_2(P_\lambda) \)-norm of \( \psi_i \). We shall distinguish two cases: \( i = N \) (considered in D1) and \( i < N \) (see D2.1 and D2.2).

**D1.** Let \( i = N \). As \( Q_\lambda \in \mathcal{C}^{0,1} \) we can use the Green-Gauss-Ostrogradskij formula and obtain from (32)—see Fig. 3 (\( X \in P_\lambda \)) (note that \( U_\lambda(G) \) and \( W_\lambda(G) \) are \((N-1)\)-dimensional sets)

\[
\psi_N(X) = \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_N}(Y) dY
+ \int_{U_\lambda(G)} \omega_h(X - [Y',-\lambda])u_\lambda(Y',-\lambda) dY'
- \int_{W_\lambda(G)} \omega_h(X - [Y',-1/2\lambda])u_\lambda(Y',-1/2\lambda) dY'
- \int_{U_\lambda(G) \setminus W_\lambda(G)} \omega_h(X - [Y',1/4\lambda])u_\lambda(Y',1/4\lambda) dY'.
\]
Let us note that we have used the Green-Gauss-Ostrogradskij theorem in the form
\[
\int_{Q_\lambda} \frac{\partial \omega_h(X - Y)}{\partial y_N} u_\lambda(Y) \, dY = \int_{\partial Q_\lambda} \omega_h(X - Y) u_\lambda(Y) n_N(Y) \, d\sigma \\
- \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda(Y)}{\partial y_N} \, dY
\]
where \( n_N = -1 \) on the base of \( Q_\lambda \), \( n_N = 1 \) on the parts of \( \partial Q_\lambda \) parallel with the base of \( Q_\lambda \) and \( n_N = 0 \) otherwise (see Fig. 3).

First we shall prove that the first integral on the right-hand side of (36) tends to zero in \( L_2(P_\lambda) \); this means that we shall prove
\[
\lim_{\lambda \to 0} \int_{P_\lambda} \left( \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_N}(Y) \, dY \right)^2 \, dX = 0.
\]
We have
\[
\int_{P_\lambda} \left( \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_N}(Y) \, dY \right)^2 \, dX \\
= \kappa \int_{P_\lambda} \left\{ \int_{Q_\lambda} h^{-N} \exp\left( \frac{\|X - Y\|^2}{\|X - Y\|^2 - h^2} \right) \frac{\partial u_\lambda}{\partial y_N}(Y) \, dY \right\}^2 \, dX.
\]
For the sake of brevity, let us denote
\[
E_h(X, Y) := \exp\left( \frac{\|X - Y\|^2}{\|X - Y\|^2 - h^2} \right), \quad F(Y) := \frac{\partial u_\lambda}{\partial y_N}(Y).
\]
Using this notation we further denote
\[
F_h(X) = \frac{1}{h^N} \int_{Q_\lambda} E_h(X, Y) F(Y) \, dY.
\]
By (39), (40) we can write
\[
|F_h(X)| \leq \frac{1}{h^N} \int_{Q_\lambda} \sqrt{E_h(X, Y)} \sqrt{E_h(X, Y)} |F(Y)| \, dY.
\]
Let us square inequality (41) and to the resulting right-hand side let us apply the Schwarz inequality; we thus obtain
\[
|F_h(X)|^2 \leq \frac{1}{h^N} \int_{Q_\lambda} E_h(X, Y) \, dY \cdot \frac{1}{h^N} \int_{Q_\lambda} E_h(X, Y) |F(Y)|^2 \, dY.
\]
We have (see [8, p. 218, relation (168)], where the case $N = 2$ is considered)
\[
\int_{Q_\lambda} E_h(X,Y) \, dY = Ch^N.
\]
Hence
\[
(43) \quad \frac{1}{h^N} \int_{Q_\lambda} E_h(X,Y) \, dY = C.
\]
Let us use (43) and let us integrate inequality (42) over $P_\lambda$ with respect to $X$. We obtain
\[
(44) \quad \int_{P_\lambda} |F_h(X)|^2 \, dX \leq C \int_{Q_\lambda} \left\{ \int_{Q_\lambda} E_h(X,Y)|F(Y)|^2 \, dY \right\} \, dX.
\]
Now we use the Fubini theorem on the right-hand side of (44) and then relation (43) (where we write now $P_\lambda$ instead of $Q_\lambda$). We conclude (if we use also notation (39)\textsuperscript{2}):
\[
(45) \quad \int_{P_\lambda} |F_h(X)|^2 \, dX \leq C \int_{Q_\lambda} \left\{ \frac{1}{h^N} \int_{P_\lambda} E_h(X,Y) \, dX \right\} \, dY
\]
\[
= C^2 \int_{Q_\lambda} |F(Y)|^2 \, dY = C^2 \int_{Q_\lambda} \left( \frac{\partial u_\lambda}{\partial y_N}(Y) \right)^2 \, dY.
\]
Using the relation
\[
\lim_{\lambda \to 0} (\text{meas}_N Q_\lambda) = 0
\]
and the theorem on the absolute continuity of the Lebesgue integral we see that the right-hand side of (45) tends to zero if $\lambda \to 0$. This proves (37).

Now we prove that the second and the fourth integrals on the right-hand side of (36) are equal to zero for $X \in P_\lambda$. We recall that
\[
h = \frac{1}{4}\lambda.
\]
As to the second integral, for $X \in P_\lambda = V_\lambda(\Gamma) \cap \Theta$ we have
\[
(46) \quad \omega_h(X - [Y',-\lambda]) = \omega_h([X',x_N] - [Y',-\lambda])
\]
\[
= \omega_h(X' - Y', x_N + \lambda)
\]
\[
= \omega_h(X' - Y', x_N + 4h) = \omega_h(Z),
\]
where we set
\[
Z = [X' - Y', x_N + 4h].
\]
We further have

(47) \[ \|Z\| = \sqrt{(X' - Y')^2 + (x_N + 4h)^2}, \]

where we set for the sake of brevity

\[ (X' - Y')^2 = (x_1 - y_1)^2 + \ldots + (x_{N-1} - y_{N-1})^2. \]

The most inconvenient case is

\[ (X' - Y')^2 = 0. \]

Since in the case \( X \in P_\lambda = V_\lambda(\Gamma) \cap \Theta \) we have

(48) \[ x_N \in (-3h, 0) \]

we obtain from (47)

(49) \[ \|Z\| \geq h, \]

which implies by (46) (and by the fact that \( \omega_h(Z) = 0 \) for \( \|Z\| \geq h \)) that

\[ \omega_h(X - [Y', -\lambda]) = 0. \]

This proves that the second integral on the right-hand side of (36) is equal to zero for \( X \in P_\lambda \).

As to the fourth integral on the right-hand side of (36), we have

\[ \omega_h\left(X - \left[Y', \frac{1}{4}\lambda\right]\right) = \omega_h(X' - Y', x_N - h). \]

As (48) holds we obtain again (49) with \( Z = [X' - Y', x_N - h] \). This proves that the fourth integral on the right-hand side of (36) is equal to zero for \( X \in P_\lambda \).

Let us consider the third integral on the right-hand side of (36) (in the second equality we use (7)):

\[ \psi_{N,3}(X) = \int_{W_\lambda(G)} \omega_h\left(X - \left[Y', \frac{-1}{2}\lambda\right]\right) u_\lambda\left(Y', -\frac{1}{2}\lambda\right) \, dY' \]
\[ = \int_{W_\lambda(G)} \omega_h\left(X' - Y', x_N + \frac{1}{2}\lambda\right) u\left(Y', -\frac{3}{4}\lambda\right) \, dY'. \]

As

\[ \omega_h(X' - Y', x_N + \frac{1}{2}\lambda) = 0 \quad \forall \|X' - Y'\| \geq h \]

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we can write
\[ \psi_{N,3}(X) = \int_{\|X' - Y'\| < h} \omega_h \left( X' - Y', x_N + \frac{1}{2} \lambda \right) u \left( Y', -\frac{3}{4} \lambda \right) dY'. \]

From this identity we obtain by the Schwarz inequality (and by extending the domain of integration in the case of the first integral)

(50) \[ |\psi_{N,3}(X)|^2 = \left( \int_{\|X' - Y'\| < h} \omega_h \left( X' - Y', x_N + \frac{1}{2} \lambda \right) u \left( Y', -\frac{3}{4} \lambda \right) dY' \right)^2 \]

\[ \leq \int_{\|X' - Y'\| < h} \omega_h^2 \left( X' - Y', x_N + \frac{1}{2} \lambda \right) dY' \times \int_{\|X' - Y'\| < h} u^2 \left( Y', -\frac{3}{4} \lambda \right) dY' \]

\[ \leq \int_{R^{N-1}} \omega_h^2 \left( X' - Y', x_N + \frac{1}{2} \lambda \right) dY' \times \int_{\|Y' - X'\| < h} u^2 \left( Y', -\frac{3}{4} \lambda \right) dY' \]

\[ = \int_{R^{N-1}} \omega_h^2 \left( Z', x_N + \frac{1}{2} \lambda \right) dZ' \int_{\|Y' - X'\| < h} u^2 \left( Y', -\frac{3}{4} \lambda \right) dY'. \]

(We have used also the fact that \( \omega_h(X' - Y', x_N + \frac{1}{2} \lambda) = \omega_h(Y' - X', x_N + \frac{1}{2} \lambda). \))

Integrating (50) over \( P_{\lambda} \) with respect to \( X \) (and extending the domain of integration in the case of the first integral) we obtain

(51) \[ \|\psi_{N,3}\|_{0,P_{\lambda}}^2 = \int_{-3\lambda/4}^{0} \left( \int_{R^{N-1}} \omega_h^2 \left( Z', x_N + \frac{1}{2} \lambda \right) dZ' \right) dx_N \]

\[ \times \int_{V_\lambda(G)} \left( \int_{\|Y' - X'\| < h} u^2 \left( Y', -\frac{3}{4} \lambda \right) dY' \right) dX' \]

\[ \leq \left( \int_{R^{N}} \omega_h^2(Z) dZ \right) \int_{V_\lambda(G)} \left( \int_{\|Y' - X'\| < h} u^2 \left( Y', -\frac{3}{4} \lambda \right) dY' \right) dX'. \]

Now we prove that

(52) \[ \int_{R^{N}} \omega_h^2(Z) dZ = Ch^{-N}. \]

Let \( K_h \) be the disc (or the sphere) with its center at the origin and its radius equal to \( h \). Then

\[ \text{meas}_N K_h = \begin{cases} 
\pi h^2 & \text{for } N = 2, \\
\frac{4}{3} \pi h^3 & \text{for } N = 3 
\end{cases} \]
and the mean value theorem yields
\[
\int_{R^N} \omega^2_h(Z) \, dZ = \int_{K_h} \omega^2_h(Z) \, dZ = \chi h^{-2N} \cdot C_0 \cdot \text{meas}_N K_h
\]
\[
= \chi h^{-2N} \cdot C_0 \cdot \tilde{C} h^N = Ch^{-N},
\]
where we set
\[
C_0 = \exp \left( \frac{\|Z_0\|^2}{\|Z_0\|^2 - h^2} \right) \leq 1, \quad C = \chi C_0 \tilde{C}.
\]
This proves relation (52).

From now on we shall assume that the functions considered are sufficiently smooth and we will extend our result by the density argument (i.e., by means of [5, Theorem 5.5.9]). For easier understanding we shall distinguish two cases: \( N = 2 \) and \( N = 3 \).

a) \( N = 2 \): In this case (considering for simplicity that \( G = (a, b) \)) we have by (4)

\[
(53) \quad U_\lambda(G) = (a - \lambda, b + \lambda),
\]
\[
V_\lambda(G) = \left( a - \frac{3}{4} \lambda, b + \frac{3}{4} \lambda \right).
\]

We recall that
\[
h = \frac{1}{4} \lambda.
\]

Let us consider the second integral on the right-hand side of (51). In the case of \( N = 2 \) we have

\[
(54) \quad \int_{V_\lambda(G)} \left( \int_{\|Y' - X'\| < h} u^2(Y', -\frac{3}{4} \lambda) \, dY' \right) \, dX' = \int_{a-3\lambda/4}^{b+3\lambda/4} \left( \int_{x_1-h}^{x_1+h} u^2(y_1, -\frac{3}{4} \lambda) \, dy_1 \right) \, dx_1.
\]

Using the mean value theorem we obtain (owing to the sufficient smoothness of functions considered)

\[
\int_{a-3\lambda/4}^{b+3\lambda/4} \left( \int_{x_1-h}^{x_1+h} u^2(y_1, -\frac{3}{4} \lambda) \, dy_1 \right) \, dx_1 = \int_{a-3\lambda/4}^{b+3\lambda/4} u^2(\tilde{y}_1, -\frac{3}{4} \lambda) \, dx_1 \cdot \int_{x_1-h}^{x_1+h} \, dy_1 = 2h \int_{a-3\lambda/4}^{b+3\lambda/4} u^2(\tilde{y}_1, -\frac{3}{4} \lambda) \, dx_1,
\]

where

\[
(55) \quad \tilde{y}_1 = x_1 + \eta, \quad \eta \in (-h, h).
\]

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The transformation $x_1 + \eta = t$ with (55) yields
\[ 2h \int_{a-3\lambda/4}^{b+3\lambda/4} u^2\left(\bar{y}_1, -\frac{3}{4}\lambda\right) \, dx_1 = 2h \int_{a-3\lambda/4+\eta}^{b+3\lambda/4+\eta} u^2\left(t, -\frac{3}{4}\lambda\right) \, dt \leq 2h \int_{a-\lambda}^{b+\lambda} u^2\left(t, -\frac{3}{4}\lambda\right) \, dt. \]

Using (53) and combining the result just obtained with (54) we obtain (in the case $N = 2$)
\[ (56) \int_{V_{\lambda}(G)} \left( \int_{\|Y' - X'\| < h} u^2\left(Y', -\frac{3}{4}\lambda\right) \, dY' \right) \, dX' \leq 2h \int_{U_{\lambda}(G)} u^2\left(Y', -\frac{3}{4}\lambda\right) \, dY' \]
and thus we have (in the case of $N = 2$), due to (51) and (52),
\[ (57) \|\psi_{N,3}\|_{0,P,\lambda}^2 \leq C_2 h^{-1} \int_{U_{\lambda}(G)} u^2\left(Y', -\frac{3}{4}\lambda\right) \, dY'. \]

b) $N = 3$: Our task is now to prove relation (57) in the case $N = 3$ (i.e., to prove relation (62)). Now the domains $U_{\lambda}(G)$ and $V_{\lambda}(G)$ are given by (4)_1 and (4)_2, respectively. Using the mean value theorem we obtain
\[ (58) \int_{V_{\lambda}(G)} \left( \int_{\|Y' - X'\| < h} u^2\left(Y', -\frac{3}{4}\lambda\right) \, dY' \right) \, dX' = \int_{V_{\lambda}(G)} u^2\left(\bar{Y}', -\frac{3}{4}\lambda\right) \, dX' \cdot \int_{\|Y' - X'\| < h} dY' \]
\[ = \pi h^2 \int_{V_{\lambda}(G)} u^2\left(\bar{Y}', -\frac{3}{4}\lambda\right) \, dX', \]
where
\[ (59) \bar{Y}' = X' + [\eta_1, \eta_2], \quad [\eta_1, \eta_2] \in \{\|Y' - X'\| < h\}. \]

This means that
\[ (60) \eta_1 \in (-h \cos \alpha, h \cos \alpha), \quad \eta_2 \in (-h \sin \alpha, h \sin \alpha), \quad \alpha \in (0, \pi). \]
The transformation $x_1 + \eta_1 = s_1, \quad x_2 + \eta_2 = s_2$ with (59) and (60) yields
\[ \pi h^2 \int_{V_{\lambda}(G)} u^2\left(\bar{Y}', -\frac{3}{4}\lambda\right) \, dX' \leq \pi h^2 \int_{U_{\lambda}(G)} u^2\left(S', -\frac{3}{4}\lambda\right) \, dS'. \]
Using \( (4)_1, (4)_2 \) and combining the result just obtained with (58) we obtain (in the case of \( N = 3 \))

\[
\int_{V_{\lambda}(G)} \left( \int_{\|Y'-X'\| < h} u^2(Y', -\frac{3}{4}\lambda) \, dX' \right) \, dY' \leq \pi h^2 \int_{U_{\lambda}(G)} u^2(Y', -\frac{3}{4}\lambda) \, dY'
\]

and thus we have (if \( N = 3 \)), due to (51) and (52),

\[
\|\psi_{N,3}\|_{0,P_\lambda}^2 \leq C_2 h^{-1} \int_{U_{\lambda}(G)} u^2(Y', -\frac{3}{4}\lambda) \, dY'.
\]

Let us consider the integral on the right-hand side of (62) or on the right-hand side of (57). (Note that relation (62) has the same form as relation (57) which is written in the case of \( N = 2 \).) We obtain for \( Y' \in U_{\lambda}(G) \) (let us point out that we assume \( u \) to be smooth enough and extend then the result by the density argument (see [5, Theorem 5.5.9])):

\[
u(Y', -\frac{3}{4}\lambda) = u(Y', 0) + \int_0^{-\frac{3}{4}\lambda} \frac{\partial u}{\partial y_N}(Y', \xi) \, d\xi;
\]

now we square this relation, use the inequality \((a + b)^2 \leq 2(a^2 + b^2)\) and then apply the Schwarz inequality; finally, we extend the interval of integration \((-\frac{3}{4}\lambda, 0)\) to \((-\lambda, 0)\) arriving at

\[
\begin{align*}
u^2(Y', -\frac{3}{4}\lambda) & \leq 2 \left[ u^2(Y', 0) + \left( \int_0^{-\frac{3}{4}\lambda} \frac{\partial u}{\partial y_N}(Y', \xi) \, d\xi \right)^2 \right] \\
& = 2 \left[ u^2(Y', 0) + \left( -\int_{-\frac{3}{4}\lambda}^0 \frac{\partial u}{\partial y_N}(Y', \xi) \, d\xi \right)^2 \right] \\
& \leq 2 \left[ u^2(Y', 0) + \frac{3}{4} \lambda \int_{-\frac{3}{4}\lambda}^0 \left( \frac{\partial u}{\partial y_N}(Y', \xi) \right)^2 \, d\xi \right] \\
& \leq 2 \left[ u^2(Y', 0) + \frac{3}{4} \lambda \int_{-\lambda}^0 \left( \frac{\partial u}{\partial y_N}(Y', \xi) \right)^2 \, d\xi \right].
\end{align*}
\]

Hence \((h = \frac{1}{4}\lambda)\)

\[
h^{-1} \int_{U_{\lambda}(G)} u^2(Y', -\frac{3}{4}\lambda) \, dY' \leq C_3 \left[ h^{-1} \int_{U_{\lambda}(G)} u^2(Y', 0) \, dY' \right. \\
\left. + \int_{U_{\lambda}(G)} \left( \frac{\partial u}{\partial y_N}(Y', \xi) \right)^2 \, dY' \, d\xi \right].
\]

The theorem on the absolute continuity of an integral implies

\[
\int_{U_{\lambda}(\Gamma)} \left( \frac{\partial u}{\partial y_N}(Y', \xi) \right)^2 \, dY' \, d\xi \to 0 \quad \text{with} \quad \lambda \to 0
\]

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and further we have (owing to the fact that \( u \mid_\Gamma = 0, \Gamma = G \times \{0\} \))

\[
(65) \quad \int_{U_\lambda(G)} u^2(Y', 0) \, dY' = \int_{U_\lambda(G) \setminus G} u^2(Y', 0) \, dY'.
\]

When analyzing the integral on the right-hand side of (65) we shall distinguish between the cases \( N = 2 \) and \( N \geq 3 \) (in applications it is sufficient to consider the case \( N = 3 \)).

1) The case \( N = 2 \). Again, let for simplicity \( G = (a, b) \) (see Fig. 1). Then

\[
\int_{U_\lambda(G) \setminus G} u^2(Y', 0) \, dY' = \int_{a-\lambda}^{a} u^2(y_1, 0) \, dy_1 + \int_{b}^{b+\lambda} u^2(y_1, 0) \, dy_1 = I_1 + I_2.
\]

The integral \( I_1 \) will be written in the form

\[
I_1 = -\int_{a}^{a-\lambda} u^2(y_1, 0) \, dy_1
\]

and the substitution \( y_1 = a - t \), where \( t \in (0, \lambda) \), will be used. We obtain, according to the theorem on substitution in a simple integral,

\[
I_1 = \int_{0}^{\lambda} u^2(a - t, 0) \, dt.
\]

Similarly, using in the integral \( I_2 \) the substitution \( y_1 = b + t \) where \( t \in (0, \lambda) \), we obtain

\[
I_2 = \int_{0}^{\lambda} u^2(b + t, 0) \, dt.
\]

Hence, according to the theorem on differentiation of an integral as a function of the upper limit and by the l’Hospital rule (and taking into account that the first integral on the right-hand side of (63) is multiplied by \( h^{-1} \)), we obtain

\[
(66) \quad \lim_{\lambda \to 0^+} \frac{1}{\lambda} \int_{U_\lambda(G) \setminus G} u^2(Y', 0) \, dY' = \\
= \lim_{\lambda \to 0^+} \frac{1}{\lambda} \left( \int_{0}^{\lambda} u^2(a - t, 0) \, dt + \int_{0}^{\lambda} u^2(b + t, 0) \, dt \right) \\
= \lim_{\lambda \to 0^+} (u^2(a - \lambda, 0) + u^2(b + \lambda, 0)) \\
= u^2(a, 0) + u^2(b, 0) = 0,
\]

because \( u(a, 0) = 0, u(b, 0) = 0 \) (\( u \) is, according to our assumption, a smooth function). All these results yield \( \|\psi_N\|_{0, P_\lambda} \to 0 \) with \( \lambda \to 0 \) (when \( N = 2 \)).
2) In this part of the proof we use, beside the Trace Theorem in the form of [5, Theorem 6.8.13] (which concerns the mapping \( \mathcal{R} : H^{1,p} \to H^{1-\frac{1}{p},p}(\partial \Omega) \)), the following theorem (we cite it from [1], where it is introduced by the following words: “We shall not attempt any proof” and references to the works of Besov [2], Uspenskij [9] and Lizorkin [6] are given. Adams further writes: “The theorem is stated for \( \mathbb{R}^N \) but can obviously be extended to domains with sufficient regularity” (as, for example, domains \( \Omega \in \mathcal{C}^{0,1} \)).

**Theorem A.** Let \( s > 0, 1 < p \leq q < \infty, \) and \( 1 \leq k \leq n. \) Let \( \chi = s - n/p + k/q. \) If

1. \( \chi \geq 0 \) and \( p < q, \) or
2. \( \chi > 0 \) and \( \chi \) is not an integer, or
3. \( \chi \geq 0 \) and \( 1 < p \leq 2, \)

then (direct imbedding theorem)

\[
W^{s,p}(\mathbb{R}^n) \rightarrow W^{\chi,q}(\mathbb{R}^k).
\]

Imbedding (67) does not necessarily hold for \( p = q > 2 \) and \( \chi \) a nonnegative integer.

In this part of the proof we restrict ourselves to the case \( N = 3. \) Our considerations start now again from relation (65). Using the Trace Theorem in the form of [5, Theorem 6.8.13] we see that

\[
u(Y', 0) \in H^{\frac{1}{2}}(\Delta).
\]

Now we use Theorem A: We have (using the notation of Theorem A)

\[
s = \frac{1}{2}, \quad p = q = 2, \quad n = N - 1, \quad k = n - 1 = N - 2.
\]

Hence

\[
\chi = s - \frac{n}{p} + \frac{k}{q} = \frac{1}{2} - \frac{N - 1}{2} + \frac{N - 2}{2} = 0.
\]

Thus, according to the assertion of Theorem A,

\[
H^{\frac{1}{2}}(\mathbb{R}^{N-1}) = W^{\frac{1}{2},2}(\mathbb{R}^{N-1}) \rightarrow H^0(\mathbb{R}^{N-2}) = L_2(\mathbb{R}^{N-2}),
\]

and consequently

\[
H^{\frac{1}{2}}(\Delta) \rightarrow L_2(\partial \Delta)
\]

and also (again according to Theorem A, because \( G \subset \Delta \))

\[
H^{\frac{1}{2}}(\Delta) \rightarrow L_2(\partial G).
\]
Thus (as $u(Y',0) = u\big|_G$)

$$u(Y',0)|_{\partial G} \in L_2(\partial G).$$

Then the convergence

$$\frac{1}{\lambda} \int_{U_\lambda(G) \setminus G} u^2(Y',0) \, dY' \to 0 \quad \text{(with } \lambda \to 0)$$

follows from the properties of traces: Of course, \textit{locally} we have

$$\int_{\mathcal{R}} u^2(Y',0) \, dY' \leq \int_{\mathcal{R}'} \left( \int_{a(z_1) - \gamma}^{a(z_1) + \gamma} u^2(z_1, z_2, 0) \, dz_2 \right) \, dz_1,$$

where $\mathcal{R}$ is the intersection of $U_\lambda(G) \setminus G$ with some suitable neighbourhood

$$\mathcal{R}_1 = \{[z_1, z_2]: z_1 \in \mathcal{R}', z_2 \in (a(z_1) - \gamma, a(z_1) + \gamma)\}$$

of any fixed point of $\partial G$ (see Fig. 4) and $a(z_1)$ is the function which represents $\partial G$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4.png}
\caption{Concerning the case $N = 3$ (see 1b) of part D1.)}
\end{figure}
with respect to a local Cartesian coordinate system of axes \((z_1, z_2)\) with a Lipschitz constant \(L\) and \(K \leq \sqrt{L^2 + 1}\). The function

\begin{equation}
\Phi(\eta) = \int_{\mathcal{R}'} u^2(z_1, a(z_1) - \eta, 0) \, dz_1
\end{equation}

is a continuous function of \(\eta \in (-\gamma, 0)\): indeed, the function \(u(z_1, z_2, 0)\) is assumed sufficiently smooth on \(\mathcal{R}' \times (-\gamma, \gamma)\), hence it is continuous on \(\overline{\mathcal{R}'} \times (-\gamma, \gamma)\); by Cantor’s theorem it is there uniformly continuous. As \(a(z_1)\) is Lipschitz continuous on \(\mathcal{R}'\), we have \(|a(z_1) - a(z'_1)| < L|z_1 - z'_1|\). All these facts imply that (according to the definition of uniform continuity of a function of two variables)

\[|u(z_1, a(z_1) - \eta, 0) - u(z'_1, a(z'_1) - \eta', 0)| < \varepsilon \quad \text{for} \quad |z_1 - z'_1| < \frac{\delta}{2L}, \quad |\eta - \eta'| < \frac{\delta}{2}.
\]

It suffices now to use the theorem on continuity of an integral with respect to a parameter.

Thus, using the Fubini theorem and making the change of variables in the form \(\eta = a(z_1) - z_2\), we obtain by means of the l’Hôpital rule and the theorem on differentiation of the integral as a function of the upper limit

\[
\lim_{\lambda \to 0^+} \frac{1}{\lambda} \int_{\mathcal{R}} u^2(Y', 0) \, dY' \leq K \lim_{\lambda \to 0^+} \frac{1}{K\lambda} \int_{-K\lambda}^{0} \Phi(\eta) \, d\eta = -K\Phi(0) = 0,
\]

because \(u|_\Gamma = 0\) and the function \(\Phi(\eta)\) is given by relation (69); we must keep in mind that \(u^2(z_1, a(z_1), 0)|_{z_1 \in \mathcal{R}'} = u^2|_{\partial G} = u^2|_{G} = u^2|_{\Gamma} = 0\).

**D2.1.** \(N = 2\). First we shall discuss the case \(N = 2\). Let, for simplicity, the set \(G\) be an open interval on the axis \(x_1\):

\[G = (a, b).
\]

Then

\[
U_\lambda(G) = (a - \lambda, b + \lambda), \quad V_\lambda(G) = (a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda),
\]

\[
W_\lambda(G) = (a - \frac{1}{2}\lambda, b + \frac{1}{2}\lambda), \quad Z_\lambda(G) = (a - \frac{1}{4}\lambda, b + \frac{1}{4}\lambda),
\]

and

\[
U_\lambda(\Gamma) = (a - \lambda, b + \lambda) \times (-\frac{1}{4}\lambda),
\]

\[
V_\lambda(\Gamma) = (a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda) \times (-\frac{3}{4}\lambda, \frac{1}{4}\lambda),
\]

\[
W_\lambda(\Gamma) = (a - \frac{1}{2}\lambda, b + \frac{1}{2}\lambda) \times (-\frac{1}{2}\lambda, \frac{1}{4}\lambda),
\]

\[
Z_\lambda(\Gamma) = (a - \frac{1}{4}\lambda, b + \frac{1}{4}\lambda) \times (-\frac{1}{4}\lambda, \frac{1}{4}\lambda).
\]
By (32), Fig. 3 and the Green-Gauss-Ostrogradskij theorem we have

\[ \psi_1(X) = \frac{\partial w_\lambda}{\partial x_1}(X) = \int_{Q_\lambda} \frac{\partial \omega}{\partial x_1}(X - Y)u_\lambda(Y) \, dY = \]

\[ = - \int_{Q_\lambda} \frac{\partial \omega}{\partial y_1}(X - Y)u_\lambda(Y) \, dY = \]

\[ = \int_{Q_\lambda} \omega(X - Y) \frac{\partial u_\lambda}{\partial y_1}(Y) \, dY \]

\[ - \int_{-\lambda}^{\lambda/4} \omega_h(X - [b + \lambda, y_2])u_\lambda(b + \lambda, y_2) \, dy_2 \]

\[ + \int_{-\lambda/2}^{\lambda/4} \omega_h(X - [b + \frac{1}{2}\lambda, y_2])u_\lambda(b + \frac{1}{2}\lambda, y_2) \, dy_2 \]

\[ - \int_{-\lambda/2}^{\lambda/4} \omega_h(X - [a - \frac{1}{2}\lambda, y_2])u_\lambda(a - \frac{1}{2}\lambda, y_2) \, dy_2 \]

\[ + \int_{-\lambda}^{\lambda/4} \omega_h(X - [a - \lambda, y_2])u_\lambda(a - \lambda, y_2) \, dy_2. \]

Similarly to Section D1 the first integral on the right-hand side of (70) tends to zero in \( L_2(P_\lambda); \) this means

\[ \lim_{\lambda \to 0} \int_{P_\lambda} \left( \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_1}(Y) \, dY \right)^2 \, dX = 0, \]

where (see Fig. 3)

\[ P_\lambda = V_\lambda(\Gamma) \cap \Theta, \quad Q_\lambda = U_\lambda(\Gamma) \setminus \overline{W_\lambda(\Gamma)}. \]

Now we prove that the second and fifth integrals on the right-hand side of (70) are equal to zero. We have

\[ \omega_h(X - [b + \lambda, y_2]) = \omega_h([x_1, x_2] - [b + \lambda, y_2]) \]

\[ = \omega_h(x_1 - b - \lambda, x_2 - y_2) \]

\[ = \omega_h(x_1 - b - 4h, x_2 - y_2) = \omega_h(Z), \]

where we set

\[ Z = [x_1 - b - 4h, x_2 - y_2]. \]

Thus

\[ \|Z\| = \sqrt{(x_1 - b - 4h)^2 + (x_2 - y_2)^2}. \]
The most inconvenient situation occurs when

\[(x_2 - y_2)^2 = 0.\]

As \(X \in P_\lambda = V_\lambda(\Gamma) \cap \Theta\) we have

\[(72) \quad x_1 \in \left( a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda \right) = (a - 3h, b + 3h).\]

In such a situation the inequality

\[(73) \quad \|Z\| \geq h\]

holds because for every \(x_1\) satisfying (72) we have

\[(x_1 - b - 4h)^2 > [(b + 3h) - b - 4h]^2 = h^2.\]

Hence the second integral on the right-hand side of (70) is equal to zero.

A similar situation arises in the case of the fifth integral on the right-hand side of (70) because

\[\omega_h(X - [a - \lambda, y_2]) = \omega_h(x_1 - a + 4h, x_2 - y_2) = \omega_h(Z)\]

and again we have

\[(x_1 - a + 4h)^2 > [(a - 3h) - a + 4h]^2 = h^2\]

for every \(x_1\) satisfying (72), hence relation (73) holds again.

As to the third and fourth integrals on the right-hand side of (70) they can be different from zero. We see it (in the case of the third integral) from the relation

\[\omega_h(X - \left[ b + \frac{1}{2}\lambda, y_2 \right]) = \omega_h(x_1 - b - 2h, x_2 - y_2) = \omega_h(Z)\]

where, due to (72), we can have \(x_1 = b + 2h\), and thus it is possible that

\[\|Z\| < h.\]

Thus, let us consider

\[(74) \quad \psi_{1,3}(X) = \int_{-\lambda/2}^{\lambda/4} \omega_h(X - \left[ b + \frac{1}{2}\lambda, y_2 \right]) u_\lambda \left( b + \frac{1}{2}\lambda, y_2 \right) dy_2,\]

\[(75) \quad \psi_{1,4}(X) = \int_{-\lambda/2}^{\lambda/4} \omega_h(X - \left[ a - \frac{1}{2}\lambda, y_2 \right]) u_\lambda \left( a - \frac{1}{2}\lambda, y_2 \right) dy_2.\]
Taking into account the preceding text and relation (70) we can write

\[(76) \quad |\psi_1(X)|^2 = (\psi_{1,3}(X) + \psi_{1,4}(X))^2 \leq 2(|\psi_{1,3}(X)|^2 + |\psi_{1,4}(X)|^2);\]

thus it suffices to consider the case of relation (74); the considerations in the case of (75) follow the same lines.

Using the Schwarz inequality and relation (74) we obtain

\[(77) \quad |\psi_{1,3}(X)|^2 \leq \int_{-\lambda/2}^{\lambda/4} \omega_h^2 \left( X - \left[ b + \frac{1}{2} \lambda, y_2 \right] \right) \, dy_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left( b + \frac{1}{2} \lambda, y_2 \right) \, dy_2.\]

Inequality (77) will be rewritten to the form

\[(78) \quad |\psi_{1,3}(X)|^2 \leq \int_{-\lambda/2}^{\lambda/4} \omega_h^2 \left( x_1 - b - \frac{1}{2} \lambda, x_2 - y_2 \right) \, dy_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left( b + \frac{1}{2} \lambda, y_2 \right) \, dy_2 \]

\[\leq \int_{-\infty}^{\infty} \omega_h^2 \left( x_1 - b - \frac{1}{2} \lambda, y_2 - x_2 \right) \, dy_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left( b + \frac{1}{2} \lambda, y_2 \right) \, dy_2.\]

In the first integral on the right-hand side of (78) we transform the variables in the form

\[y_2 - x_2 = z_2 \Rightarrow dy_2 = dz_2\]

and in the second integral we change the notation of the integration variable. We obtain

\[(79) \quad |\psi_{1,3}(X)|^2 \leq \int_{-\infty}^{\infty} \omega_h^2 \left( x_1 - b - \frac{1}{2} \lambda, z_2 \right) \, dz_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left( b + \frac{1}{2} \lambda, s \right) \, ds.\]

Let us integrate inequality (79) over \(P_\lambda:\)

\[(80) \quad \int_{P_\lambda} |\psi_{1,3}(X)|^2 \, dX \]

\[\leq \int_{P_\lambda} \left\{ \int_{-\infty}^{\infty} \omega_h^2 \left( x_1 - b - \frac{1}{2} \lambda, z_2 \right) \, dz_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left( b + \frac{1}{2} \lambda, s \right) \, ds \right\} \, dX \]

\[= \int_{a-3\lambda/4}^{b+3\lambda/4} \left( \int_{-\infty}^{\infty} \omega_h^2 \left( x_1 - \frac{1}{2} \lambda - b, z_2 \right) \, dz_2 \right) \, dx_1 \]

\[\times \int_{-3\lambda/4}^{0} \, dx_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda \left( b + \frac{1}{2} \lambda, s \right) \, ds \]

\[\leq \int_{-\infty}^{\infty} \left( \int_{-\infty}^{\infty} \omega_h^2 \left( x_1 - b - \frac{1}{2} \lambda, z_2 \right) \, dz_2 \right) \, dx_1 \cdot 3h \int_{-\lambda/2}^{\lambda/4} u_\lambda \left( b + \frac{1}{2} \lambda, s \right) \, ds.\]
In the first integral on the right-hand side of (80) we transform the variables in the form
\[ x_1 - b - \frac{1}{2} \lambda = z_1 \Rightarrow dx_1 = dz_1; \]
hence we obtain
\[
\| \psi_{1,3} \|_{0, P_\lambda}^2 \leq \int_{\mathbb{R}^2} \omega_h^2(Z) \, dZ \cdot 3h \int_{-\lambda/2}^{\lambda/4} u_\lambda \left( b + \frac{1}{2} \lambda, s \right) \, ds.
\]

Relations (81) and (52) (with \( N = 2 \)) yield
\[
\| \psi_{1,3} \|_{0, P_\lambda}^2 \leq C_2 h^{-1} \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left( b + \frac{1}{2} \lambda, s \right) \, ds.
\]

Inequality (82) is a one-dimensional analogue of inequalities (57) and (62). In this case we proceed as follows (we use definition (7) of the function \( u_\lambda \) and the definition of \( h \):
\[
h^{-1} \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left( b + \frac{1}{2} \lambda, s \right) \, ds = h^{-1} \int_{-2h}^{h} u_\lambda \left( b + \frac{1}{2} \lambda, s \right) \, ds
\]
\[
= \frac{1}{h} \int_{-2h}^{0} u^2 \left( b + \frac{1}{2} \lambda, s - h \right) \, ds + \frac{1}{h} \int_{0}^{h} u^2 \left( b + \frac{1}{2} \lambda, s - h \right) \, ds.
\]

Let \( h \to 0 \) and let us evaluate each limit on the right-hand side of (83) separately. In both cases we have the limit of type \( \frac{0}{0} \); thus we can use the l’Hospital rule and the theorem on differentiation of an integral as a function of the upper limit. Again, as in part D1, we assume the function \( u \) to be sufficiently smooth and then extend the result by a density argument (see [5, Theorem 5.5.9]). First we consider the limit of the second term on the right-hand side of (83):
\[
\lim_{h \to 0} \frac{1}{h} \int_{0}^{h} u^2 \left( b + \frac{1}{2} \lambda, s - h \right) \, ds = \lim_{h \to 0} u^2(b + 2h, 0) = 0,
\]
because \([b, 0] \in \overline{\Gamma} = \overline{G} \times \{0\} \) and \( u \big|_{\Gamma} = 0 \). (The function \( u \) is smooth.)

As to the first term on the right-hand side of (83), we proceed as follows:
\[
-2 \lim_{h \to 0} \frac{1}{2h} \int_{0}^{-2h} u^2 \left( b + \frac{1}{2} \lambda, s - h \right) \, ds = -2 \lim_{h \to 0} u^2(b + 2h, -3h)
\]
\[
= -2u^2(b, 0) = 0.
\]

Relations (81)–(85) yield
\[
\lim_{\lambda \to 0} \| \psi_{1,3} \|_{0, P_\lambda}^2 = 0.
\]
We can prove in the same way that

\[
\lim_{\lambda \to 0} \|\psi_{1,4}\|_{0, P_\lambda}^2 = 0.
\]

Relations (76) and (86), (87) imply

\[
\lim_{\lambda \to 0} \|\psi_1\|_{0, P_\lambda}^2 = 0.
\]

The proof of Lemma 2 in the case \( N = 2 \) is complete.

**D2.2.** \( N = 3 \). Now we shall consider the situation \( i < N \) in the case \( N = 3 \), this means the cases \( i = 1 \) and \( i = 2 \).

For greater simplicity we assume that \( G \subset \overline{G} \subset \Delta \) is a simply connected domain. Further we assume that the boundary \( \partial G \) consists of a finite number of smooth arcs; this assumption is sufficient for applications. In the case \( N = 3 \) the sets \( U_\lambda(G), \ldots, Z_\lambda(G) \) and \( U_\lambda(\Gamma), \ldots, Z_\lambda(\Gamma) \) are defined by relations (4) and (5), respectively. Domains (5) are three-dimensional cylinders with bases parallel to the coordinate plane \((x_1, x_2)\) and with the lateral area of a cylinder formed by straightlines parallel to the axis \( x_3 \). The projections of both bases of the cylinder \( M_\lambda(\Gamma) \) \((M = U, \ldots, Z)\) are identical with the two-dimensional domain \( M_\lambda(G) \) which lies in the coordinate plane \((x_1, x_2)\).

Let \( M^1_\lambda(\Gamma) \) and \( M^2_\lambda(\Gamma) \) be the lower and upper bases of the cylinder \( M_\lambda(\Gamma) \), respectively, and let \( M^3_\lambda(\Gamma) = \partial M_\lambda(\Gamma) \setminus (M^1_\lambda(\Gamma) \cup M^2_\lambda(\Gamma)) \). Let \( X \in \partial M_\lambda(\Gamma) \) be an arbitrary point except for the points at which \( \partial M_\lambda(\Gamma) \) is not smooth. Let \( n(X) \) be the unit outer normal to \( \partial M_\lambda(\Gamma) \) at the point \( X \). Then we have

\[
X \in M^1_\lambda(\Gamma) \Rightarrow n(X) = (0, 0, -1),
\]

\[
X \in M^2_\lambda(\Gamma) \Rightarrow n(X) = (0, 0, 1),
\]

\[
X \in M^3_\lambda(\Gamma) \Rightarrow n(X) = (n_1(X), n_2(X), 0), \quad n_1^2(X) + n_2^2(X) = 1.
\]

Now we shall compute the functions \( \psi_i(X) \) given by relation (32) for \( i = 1 \) and \( i = 2 \). By the Green-Gauss-Ostrogradskij theorem and relations (89)–(91) we have

\[
\psi_i(X) = \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_i}(Y) \, dY
\]

\[
- \int_{U^3_\lambda(\Gamma)} \omega_h(X - Y) u_\lambda(Y) n_i(Y) \, d\sigma_Y
\]

\[
+ \int_{W^3_\lambda(\Gamma)} \omega_h(X - Y) u_\lambda(Y) n_i(Y) \, d\sigma_Y.
\]

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The last two integrals appearing on the right-hand side of (92) are surface integrals of the first kind. Similarly to Section D1 the first integral on the right-hand side of (92) tends to zero in \( L_2(P_\lambda) \); this means

\[
\lim_{\lambda \to 0} \int_{P_\lambda} \left( \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_1}(Y) \, dY \right)^2 \, dX = 0,
\]

where (see Fig. 3)

\[
P_\lambda = V_\lambda(\Gamma) \cap \Theta, \quad Q_\lambda = U_\lambda(\Gamma) \setminus \overline{W_\lambda(\Gamma)}.
\]

Now we prove that the second integral on the right-hand side of (92) is equal to zero. We have

\[
\|X - Y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.
\]

The most inconvenient situation occurs when \( x_3 - y_3 = 0 \) (in this case the points \( X, Y \) lie in a plane which is parallel to the coordinate plane \((x_1, x_2)\)) and when the point \( X \) is very close to \( V_\lambda^3(\Gamma) \) or lies on \( V_\lambda^3(\Gamma) \). As \( Y \in U_\lambda^3(\Gamma) \) and \( X \in P_\lambda \) we see that in every case we have \( \|X - Y\| \geq h \). Thus \( \omega_h(X - Y) = 0 \) and the second integral on the right-hand side of (92) is equal to zero. Hence

\[
\psi_i(X) = \int_{W_\lambda^3(\Gamma)} \omega_h(X - Y)u_\lambda(Y)n_i(Y) \, d\sigma_Y.
\]

Using the Schwarz inequality and the fact that \( |n_i(Y)| \leq 1 \) we obtain

\[
|\psi_i(X)|^2 \leq \int_{W_\lambda^3(\Gamma)} \omega_h^2(X - Y) \, d\sigma_Y \int_{W_\lambda^3(\Gamma)} u_\lambda^2(Y) \, d\sigma_Y
\]

\[
= \int_{W_\lambda^3(\Gamma)} \omega_h^2(X - Y) \, d\sigma_Y \int_{W_\lambda^3(\Gamma)} u_\lambda^2(Z) \, d\sigma_Z.
\]

Integration of inequality (94) over \( P_\lambda \) yields

\[
\|\psi_i\|_{0,P_\lambda}^2 \leq \int_{P_\lambda} \left\{ \int_{W_\lambda^3(\Gamma)} \omega_h^2(X - Y) \, d\sigma_Y \right\} \, dX \cdot \int_{W_\lambda^3(\Gamma)} u_\lambda^2(Z) \, d\sigma_Z.
\]

First let us consider the second surface integral appearing on the right-hand side of (95). As \( W_\lambda^3(\Gamma) = \partial W_\lambda(\Gamma) \times (-\frac{1}{4} \lambda, \frac{1}{4} \lambda) \) we can write using definition (7) of the function \( u_\lambda \) and the properties of a line integral of the first kind:

\[
\int_{W_\lambda^3(\Gamma)} u_\lambda^2(Z) \, d\sigma_Z = \int_{\partial W_\lambda(\Gamma) \times (-\frac{1}{4} \lambda, \frac{1}{4} \lambda)} u^2(z_1, z_2, z_3 - h) \, d\sigma_Z
\]

\[
= \sum_{k=1}^m \int_{-\lambda/2}^{\lambda/4} \left\{ \int_{R_k} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) \sqrt{1 + \|a_{k,\lambda}(z_1)\|^2} \, dz_1 \right\} \, dz_3,
\]

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where $a_{k,\lambda}$ is a local representation of the curve $\partial W_\lambda(G)$. (The situation is similar to that which is sketched in Fig. 4.) We have $|a_{k,\lambda}(z_1)|^2 < C_k$, where the constant $C_k$ depends only on the Lipschitz constant of the function $\alpha_k(z_1)$ which represents locally the curve $\partial G$ (which does not depend on $\lambda$; however, $\partial W_\lambda(G)$ depends, similarly as $W_\lambda(G)$, on $\lambda$—see (4)); attention: the symbols $a_{k,\lambda}$ and $\alpha_k$ represent two different parallel arcs). The symbol $R'_k$ ($k = 1, \ldots, m$) denotes a segment on the $z_1$-axis of the $k$th local coordinate system $(z_1, z_2)$. (It would be more precise to use the notation $(z_1^{(k)}, z_2^{(k)})$ instead of the notation $(z_1, z_2)$; however, it would have been very cumbersome.) Thus

$$
\int_{\Gamma} u^2(Z) \, d\sigma_Z \leq C_0 \sum_{k=1}^{m} \int_{-\lambda/2}^{\lambda/2} \left\{ \int_{R'_k} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) \, dz_1 \right\} \, dz_3,
$$

where $C_0 = \max(C_1, \ldots, C_m)$. Let us consider the expression

$$
\frac{1}{h} \int_{-\lambda/2}^{\lambda/4} \left\{ \int_{R'_k} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) \, dz_1 \right\} \, dz_3 = \frac{1}{h} \int_{R'_k} \left\{ \int_{-2h}^{0} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) \, dz_3 + \frac{1}{h} \int_{0}^{h} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) \, dz_3 \right\} \, dz_1,
$$

where we have used the Fubini theorem. Let $h \to 0$ in (98). First let us consider the second expression appearing on the right-hand side of (98). To this end, the following fact should be noted: if $h \to 0$ then $\partial W_\lambda(G)$ tends to $\partial G$. The corresponding $k$th part of $\partial G$ is described in the local coordinate system $(z_1, z_2)$ by the function $\alpha_k(z_1)$. Hence (as the limit is of the type $0/0$, we use the l'Hospital rule and the theorem on differentiation of an integral as a function of the upper limit)

$$
\lim_{h \to 0} \frac{1}{h} \int_{0}^{h} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) \, dz_3 = \lim_{h \to 0} u^2(z_1, a_{k,\lambda}(z_1), 0) = u^2(z_1, \alpha_k(z_1), 0).
$$

As $[z_1, \alpha_k(z_1), 0] \in \partial G$ and $u|_{\Gamma} = 0$, where $\Gamma = G \times \{0\}$, we have by the assumed continuity of the function $u$

$$
u^2(z_1, a_{k,\lambda}(z_1), 0) = 0,$$

which yields

$$
\lim_{h \to 0} \frac{1}{h} \int_{0}^{h} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) \, dz_3 = 0.
$$
As to the first expression appearing on the right-hand side of (98) we have by the same argument

\[
\lim_{h \to 0} \frac{1}{h} \int_{-2h}^{0} u^2(z_1, a_k, \lambda(z_1), z_3 - h) \, dz_3 = -2 \lim_{h \to 0} u^2(z_1, a_k, \lambda(z_1), -3h) = -2u^2(z_1, \alpha_k(z_1), 0) = 0.
\]

We see from relations (95)–(100) that to complete the proof of Lemma 2 in the case \( N = 3 \) means to prove

\[
\int_{P_\lambda} \left\{ \int_{W^3_\lambda(\Gamma)} \omega_h^2(X - Y) \, d\sigma_Y \right\} \, dX = Ch^{-1}.
\]

The Fubini theorem and the mean value theorem yield

\[
\int_{P_\lambda} \left\{ \int_{W^3_\lambda(\Gamma)} \omega_h^2(X - Y) \, d\sigma_Y \right\} \, dX = \int_{W^3_\lambda(\Gamma)} \left\{ \int_{P_\lambda} \omega_h^2(X - Y) \, dX \right\} \, d\sigma_Y = \int_{W^3_\lambda(\Gamma)} \omega_h^2(X_0 - Y) \, d\sigma_Y \int_{\|X - Y\| < h} \frac{\|X - Y\|}{Y \in W^3_\lambda(\Gamma)} \, dX.
\]

We have

\[
\int_{\|X - Y\| < h} \frac{\|X - Y\|}{Y \in W^3_\lambda(\Gamma)} \, dX = \frac{4}{3} \pi h^3.
\]

As to the first integral on the right-hand side of (102), the case \( X_0 \in W^3_\lambda(\Gamma) \) is most inconvenient. As the inequality \( \|X_0 - Y\| < h \) must hold for \( \omega_h^2(X_0 - Y) > 0 \) we find

\[
\int_{W^3_\lambda(\Gamma)} \omega_h^2(X_0 - Y) \, d\sigma_Y = \int_{\sigma(X_0)} \omega_h^2(X_0 - Y) \, d\sigma_Y,
\]

where

\[
\sigma(X_0) \subset W^3_\lambda(\Gamma), \quad \operatorname{meas_2} \sigma(X_0) \leq Ch^2.
\]

Hence, taking into account (11) with \( N = 3 \), we conclude that

\[
\int_{W^3_\lambda(\Gamma)} \omega_h^2(X_0 - Y) \, d\sigma_Y = \omega_h^2(X_0 - Y_0) \int_{\sigma(X_0)} \omega_h^2(X_0 - Y) \, d\sigma_Y \leq Ch^{-4}.
\]

Relations (102)–(104) imply the desired result (101). Lemma 2 is completely proved. ∎
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Authors’ addresses: P. Doktor, Českolipská 12, 190 00 Prague 9, Czech Republic; A. Ženišek, Department of Mathematics FME, Technická 2, 616 69 Brno, Czech Republic, e-mail: zenisek@fme.vutbr.cz.