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THE DENSITY OF INFINITELY DIFFERENTIABLE FUNCTIONS
IN SOBOLEV SPACES WITH MIXED BOUNDARY CONDITIONS*

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Abstract. We present a detailed proof of the density of the set $C^\infty(\overline{\Omega}) \cap V$ in the space of test functions $V \subset H^1(\Omega)$ that vanish on some part of the boundary $\partial\Omega$ of a bounded domain Ω .

Keywords: density theorems, finite element method

MSC 2000: 46E35

Let Ω be a nonempty bounded domain in \mathbb{R}^N ($N = 2$ or 3). The symbol $C^\infty(\overline{\Omega})$ denotes the set containing all restrictions to $\overline{\Omega}$ of infinitely smooth functions defined on \mathbb{R}^N (see [5, 1.2.1, 1.2.3 and 5.2.1]). Further, the symbol V denotes the set of test functions belonging to $H^1(\Omega)$ (for detailed definition of V see Theorem 1), where $H^1(\Omega) \equiv H^{1,2}(\Omega)$ is the Sobolev space in the notation defined in [5, 5.4.1].

In this paper we present a detailed proof of the density of $C^\infty(\overline{\Omega}) \cap V$ in V (see Theorem 1) the use of which is necessary when proving the convergence of the finite element method without any regularity assumptions on the exact solution u of a given variational problem, i.e., when proving the relation (which we present in $(*)$ for the case of a variational problem corresponding to a second order elliptic boundary value problem)

$$(*) \quad \lim_{h \rightarrow 0} \|\tilde{u} - u_h\|_{1, \Omega_h} = 0,$$

where \tilde{u} is the Calderon extension (see [7, p. 77]) of the exact solution and u_h is the approximate solution by the finite element method. Many authors consider the density of $C^\infty(\overline{\Omega}) \cap V$ in V to be evident and using it they do not give any reference

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(see, for example, [3, p. 135]). The assertion of Theorem 1 was given (in a little more general form) in [4]. However, the proof presented in [4] is so concise that almost no reader will have patience to read and understand it. For this reason we present in this paper a sufficiently detailed proof of this result which is of basic importance in the theory of convergence of the finite element method.

We also restrict ourselves to the class of domains $\tilde{\mathcal{C}}^{0,1} \subset \mathcal{C}^{0,1}$, where $\mathcal{C}^{0,1}$ denotes the set of domains with Lipschitz continuous boundary, in the following sense: If $\Omega \in \tilde{\mathcal{C}}^{0,1}$ then $\Omega \in \mathcal{C}^{0,1}$ and the boundary $\partial\Omega$ of Ω consists of a finite number of smooth parts which have a finite number of relative maxima and minima and inflexions and in the three-dimensional case also a finite number of saddle points. To consider such a class of domains is sufficient for applications.

Further, we will consider parts γ_i of $\partial\Omega$, on which homogeneous Dirichlet boundary condition will be prescribed, which satisfy the following condition. Let $\gamma_i \subset \partial\Omega$ be a relatively open set (i.e., open in the metric space $\partial\Omega$). We say that γ_i has a Lipschitz relative boundary $\partial\gamma_i$ (i.e., the boundary in the metric space $\partial\Omega$) and write $\gamma_i \in \text{LRB}$ if either $\dim\Omega = 2$, or if in the case $\dim\Omega = 3$ it has the following property:

Let X_0 be an arbitrary point of $\partial\gamma_i$ and let $N(X_0)$ be a neighbourhood of X_0 such that $N(X_0) \cap \partial\Omega$ is expressed as a graph $x_3 = a(x_1, x_2)$. Let further G_i be the image of $\gamma_i \cap N(X_0)$ in the projection to the plane x_1, x_2 with the boundary ∂G_i . Then G_i has the same property as Ω , i.e., ∂G_i is locally representable as a graph of a Lipschitz function in one variable (obviously this definition is independent of the description of $\partial\Omega$). (In this case we use the notation $\gamma_i \in \text{LRB}$ as mentioned above).

Theorem 1 (on the density of $C^\infty(\bar{\Omega}) \cap V$ in V). *Let $\Omega \in \tilde{\mathcal{C}}^{0,1}$ and let*

$$(1) \quad V = \{v \in H^1(\Omega) : v = 0 \text{ on } \Gamma_1, \text{ where } \Gamma_1 \subset \partial\Omega, \\ \text{meas}_{N-1} \Gamma_1 > 0 \text{ with } N = 2 \text{ or } N = 3\},$$

where Γ_1 consists of a finite number of relatively open parts in $\partial\Omega$, say $\gamma_1, \dots, \gamma_m$ such that $\gamma_i \in \text{LRB}$ ($i = 1, \dots, m$). Then the set $C^\infty(\bar{\Omega}) \cap V$ is dense in V .

The proof of Theorem 1 will be divided into four parts:

- 1) formulation of Lemma 2;
- 2) the idea of the proof of Lemma 2;
- 3) the proof of Theorem 1 by means of Lemma 2;
- 4) the detailed proof of Lemma 2.

Lemma 2. Let Θ be an N -dimensional parallelepiped

$$(2) \quad \Theta = \{X \in \mathbb{R}^N : X' \in \Delta, x_N \in (-\beta, 0)\},$$

where $\Delta = (-\alpha, \alpha)^{N-1}$ and α, β are positive numbers.

Let $G \subset \bar{G} \subset \Delta$ be a domain such that $G \in \tilde{C}^{0,1}$ and let $\bar{G} \subset \mathcal{U}$, where $\mathcal{U} \subset \Delta$ is an open set.

Let us denote $\Theta_1 = (-\alpha, \alpha)^{N-1} \times (-\beta, \beta)$. Further, let us denote $\Gamma = G \times \{0\}$ and let $\mathcal{K} \subset \Theta_1$ be a compact set, $\Gamma \subset \mathcal{K}$. (See Fig. 1 in the case of $N = 2$.)

Then there exists a compact set $\mathcal{K}_1 \subset \Theta_1$, $\mathcal{K}_1 \supset \mathcal{K}$ (where \mathcal{K}_1 depends only on \mathcal{K}) with the following property (\mathcal{K}_1 will be defined at the end of the idea of the proof of Lemma 2 (see the text following relation (12))):

Let $u \in H^1(\Theta)$ be an arbitrary function which is equal to zero on Γ (in the sense of traces) and $\text{supp } u \subset \mathcal{K}$.

Then there exists a sequence $\{u_n\} \subset C^\infty(\bar{\Theta}_1)$ such that $\text{supp } u_n \subset \mathcal{K}_1 \setminus \bar{\Gamma}$, where $\Gamma = G \times \{0\}$, and $u_n \rightarrow u$ in the space $H^1(\Theta)$.

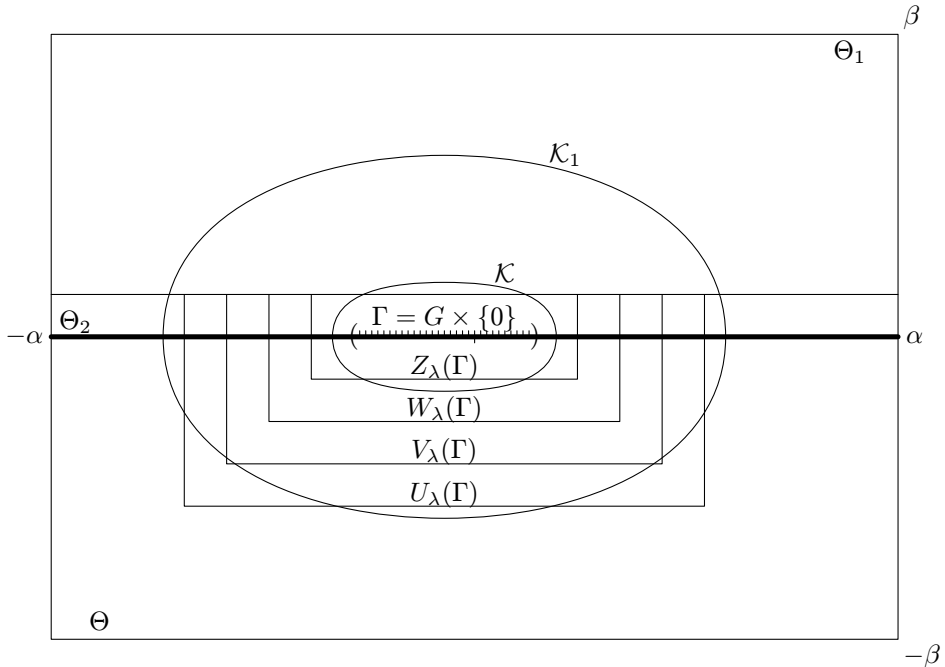


Figure 1. A two-dimensional example with G an interval.

The idea of the proof of Lemma 2. Due to the assumptions $\overline{G} \subset \Delta$, $\mathcal{K} \subset \Theta_1$ we have (see Fig. 1 for $N = 2$)

$$(3) \quad \text{dist}(\mathcal{K}, \mathbb{R}^N \setminus \Theta_1) = \nu > 0.$$

Let us denote successively

$$(4) \quad \begin{aligned} U_\lambda(G) &= \{X' \in \Delta: \text{dist}(X', G) < \lambda\}, \\ V_\lambda(G) &= \left\{X' \in \Delta: \text{dist}(X', G) < \frac{3}{4}\lambda\right\}, \\ W_\lambda(G) &= \left\{X' \in \Delta: \text{dist}(X', G) < \frac{1}{2}\lambda\right\}, \\ Z_\lambda(G) &= \left\{X' \in \Delta: \text{dist}(X', G) < \frac{1}{4}\lambda\right\} \end{aligned}$$

and, correspondingly (see Fig. 1 for $N = 2$),

$$(5) \quad \begin{aligned} U_\lambda(\Gamma) &= U_\lambda(G) \times \left(-\lambda, \frac{1}{4}\lambda\right), \\ V_\lambda(\Gamma) &= V_\lambda(G) \times \left(-\frac{3}{4}\lambda, \frac{1}{4}\lambda\right), \\ W_\lambda(\Gamma) &= W_\lambda(G) \times \left(-\frac{1}{2}\lambda, \frac{1}{4}\lambda\right), \\ Z_\lambda(\Gamma) &= Z_\lambda(G) \times \left(-\frac{1}{4}\lambda, \frac{1}{4}\lambda\right), \end{aligned}$$

where λ is supposed sufficiently small, thus satisfying

$$(6) \quad \lambda < \frac{1}{2}\nu.$$

Let us put $h = \frac{1}{4}\lambda$ and (see Fig. 2)

$$(7) \quad u_\lambda(X', x_N) := u(X', x_N - h), \quad [X', x_N] \in \Theta$$

with $u_\lambda \in H^1(\Theta_2)$, $\Theta_2 \equiv \Theta_2(h) = \Delta \times (-\beta, h)$,

$$(8) \quad v_\lambda(X) = \begin{cases} 0, & X \in W_\lambda(\Gamma), \\ u_\lambda(X), & X \in \Theta_2 \setminus \overline{W_\lambda(\Gamma)}, \end{cases}$$

$$(9) \quad w_\lambda(X) = (\omega_h * v_\lambda)(X).$$

In (9) we have used the brief notation for convolution

$$(10) \quad (\omega_h * u)(X) = \int_{\mathbb{R}^N} \omega_h(X - Y)u(Y) dY = \int_{\mathbb{R}^N} \omega_h(Y)u(X - Y) dY,$$

where the mollifier $\omega_h(Z)$ is defined by the relations

$$(11) \quad \omega_h(Z) = \begin{cases} \varkappa h^{-N} \exp\left(\frac{\|Z\|^2}{\|Z\|^2 - h^2}\right) & \text{for } \|Z\| < h, \\ 0 & \text{for } \|Z\| \geq h; \end{cases}$$

the symbol $\|\cdot\|$ denotes the Euclidean norm and the constant \varkappa is defined by

$$(12) \quad \int_{\mathbb{R}^N} \omega_1(Z) dZ = \int_{\mathbb{R}^N} \omega_h(Z) dZ = 1,$$

from which we obtain

$$\varkappa = \left(\int_{\mathbb{R}^N} \exp\left(\frac{\|Z\|^2}{\|Z\|^2 - 1}\right) dZ \right)^{-1}.$$

The first equality (12) follows from the fact that

$$\|Z\|^2 / (\|Z\|^2 - h^2) = \|Z/h\|^2 / (\|Z/h\|^2 - 1)$$

and from the substitution $Z = hX = [hx_1, \dots, hx_N]$ which implies $dZ = h^N dX$. Thus the convolution (10) is well-defined for $u \in L_2\Omega$ and its restriction onto $\overline{\Omega}$ belongs to the space $C^\infty(\overline{\Omega})$.

We see immediately that $u_\lambda \in H^1(\Theta)$. It follows from (4)–(6) that $\text{supp } w_\lambda \subset \mathcal{K}_1$, where $\mathcal{K}_1 = \{X \in \mathbb{R}^N : \text{dist}(X, \mathcal{K}) < \frac{1}{2}\nu\} \subset \Theta_1$ depends only on \mathcal{K} (because, according to (3), ν depends on \mathcal{K}), and that $w_\lambda(X) = 0$ for $X \in Z_\lambda(\Gamma)$ (see Fig. 2). In what follows we show that $\|u - w_\lambda\|_{1,\Theta} \rightarrow 0$ for $\lambda \rightarrow 0$, which proves Lemma 2. \square

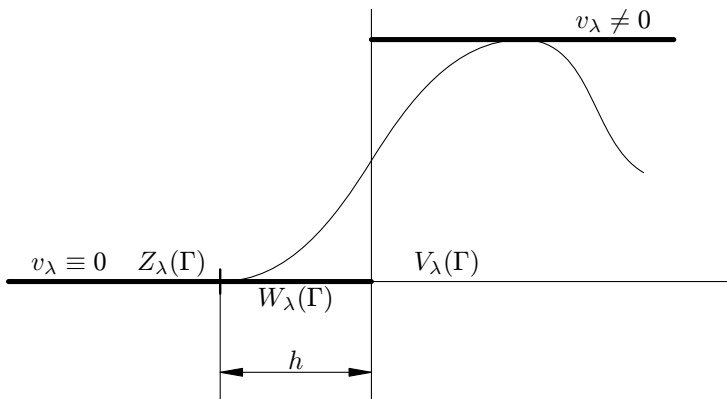


Figure 2. Concerning relations (7)–(12).

Proof of Theorem 1 by means of Lemma 2. Let $u \in V$ be an arbitrary but fixed function. We show that there exists such a sequence $\{u_n\} \subset C^\infty(\overline{\Omega})$ that

$$(13) \quad u_n = 0 \quad \text{on an } N\text{-dimensional neighbourhood of } \Gamma_1,$$

$$(14) \quad u_n \rightarrow u \quad \text{in the space } H^1(\Omega).$$

For better clarity of exposition we restrict ourselves to the two-dimensional case, i.e., $N = 2$.

The domain Ω has a Lipschitz continuous boundary and hence for any $X \in \partial\Omega$ there exists a (local) Cartesian coordinate system (x_1, x_2) and a Lipschitz function $a(x_1)$ with the domain of definition $\Delta = (-\alpha, \alpha) \subset \mathbb{R}^1$ such that¹

$$\begin{aligned} \mathbb{U} &= \{[x_1, x_2]: x_1 \in \Delta, a(x_1) - \beta < x_2 < a(x_1)\} \subset \Omega, \\ \mathbb{V} &= \{[x_1, x_2]: x_1 \in \Delta, a(x_1) < x_2 < a(x_1) + \beta\} \subset \mathbb{R}^2 \setminus \overline{\Omega}, \end{aligned}$$

where $\alpha > 0, \beta > 0$ are suitable constants. Let us denote

$$\mathbb{Z} = \mathbb{U} \cup \mathbb{V} \cup \{[x_1, x_2]: x_1 \in \Delta, x_2 = a(x_1)\}.$$

Owing to the compactness of $\partial\Omega$ we can cover $\partial\Omega$ by a finite number of such domains $\mathbb{Z}_1, \dots, \mathbb{Z}_m$. (The local Cartesian coordinate system x_1, x_2 and the function $a(x_1)$ corresponding to \mathbb{Z}_r will be now denoted by x_1^r, x_2^r and $a_r(x_1^r)$, respectively.) Further, we can find a domain \mathbb{Z}_0 such that $\overline{\mathbb{Z}_0} \subset \Omega$ (\mathbb{Z}_0 is considered in the global Cartesian coordinate system x_1, x_2) and

$$\overline{\Omega} \subset \bigcup_{r=0}^m \mathbb{Z}_r.$$

Owing to the compactness of $\overline{\Omega}$ we can construct a partition of unity, i.e., a system of functions $\varphi_r \in C_0^\infty(\mathbb{Z}_r)$ ($r = 0, 1, \dots, m$) which for $X \in \overline{\Omega}$ (the points X are considered in the global system x_1, x_2) satisfy

$$0 \leq \varphi_r(X) \leq 1, \quad \sum_{r=0}^m \varphi_r(X) = 1.$$

We can transform \mathbb{U}_r ($r = 1, \dots, m$) to the parallelepiped

$$\Theta = (-\alpha, \alpha)^{N-1} \times (-\beta, 0) \quad (\text{in our case } N = 2)$$

¹ If $N > 2$ then we substitute $[x_1, x_2]$ by $[X', x_N] = [x_1, \dots, x_{N-1}, x_N]$ and the function $a(x_1)$ by $a(X')$ with the domain of definition $\Delta = (-\alpha, \alpha)^{N-1} \subset \mathbb{R}^{N-1}$.

by means of the lipschitzian mapping

$$\mathcal{T}_r: \xi_1 = x_1^r, \xi_2 = x_2^r - a_r(x_1^r).$$

This transformation maps continuously $H^1(\mathbb{U}_r)$ ($1 \leq r \leq m$) onto $H^1(\Theta)$ (see [7, Lemma 2.3.2 on p. 66]) and $\text{supp } \varphi_r$ onto a compact set $\mathcal{K}_r \subset \Theta_1 = \mathcal{T}_r(\mathbb{Z}_r)$, $\mathcal{K}_r = \mathcal{T}_r(\text{supp } \varphi_r)$.

Let $G^r = \mathcal{T}_r(\Gamma_1 \cap \text{supp } \varphi_r)$. We have $G^r \subset \overline{G^r} \subset \Delta$. Thus $\mathcal{K}_r \subset \Theta_1$ is the compact set \mathcal{K} from Lemma 2. Let $\mathcal{K}_{r,1} \subset \Theta_1$ ($\mathcal{K}_{r,1} \supset \mathcal{K}_r$) be the compact set \mathcal{K}_1 from Lemma 2.

Hence, according to Lemma 2, we can approach $\mathcal{T}_r(\varphi_r u)$ by a sequence $\{v_{n,r}\} \subset C^\infty(\overline{\Theta}_1)$, $\text{supp } v_{n,r} \subset \mathcal{K}_{r,1} \setminus \overline{\Gamma^r}$ ($\Gamma^r = G^r \times \{0\}$).

The main step of the proof consists in the following argument: The functions $\tilde{u}_{n,r} = \mathcal{T}_r^{-1}(v_{n,r})$ belong to $H^1(C)$ ($C \subset \mathbb{R}^N$ is an N -dimensional cube which contains $\overline{\Omega}$),

$$(15) \quad \tilde{u}_{n,r} = 0 \quad \text{in a neighbourhood of } \overline{\Gamma}_1 \cap \text{supp } \varphi_r$$

and $\tilde{u}_{n,r} \rightarrow u\varphi_r$ in $H^1(\Omega)$.

Applying the mollifier ω_h to $\tilde{u}_{n,r}$ we can replace $\tilde{u}_{n,r}$ by $u_{n,r} \subset C^\infty(\overline{\Omega})$ having the same property (15) and converging to $u\varphi_r$ in $H^1(\Omega)$. Finally, we approach $u\varphi_0$ by the sequence $\{u_{n,0}\} \subset C_0^\infty(\Omega)$ and write $u_n = \sum_{r=0}^m u_{n,r}$, which proves the theorem. \square

The detailed proof of Lemma 2. Let us denote $\sigma_\lambda = \Theta \setminus \overline{V_\lambda(\Gamma)}$, $P_\lambda = V_\lambda(\Gamma) \cap \Theta$ and recall that $\Theta_2 = \Delta \times (-\beta, h)$. The proof proceeds as follows: Due to the fact that $\overline{\Theta} = \overline{\sigma}_\lambda \cup \overline{P}_\lambda$, we can write

$$(16) \quad \begin{aligned} \|w_\lambda - u\|_{1,\Theta} &\leq \|u - u_\lambda\|_{1,\Theta} + \|u_\lambda - w_\lambda\|_{1,\Theta} \\ &\leq \|u - u_\lambda\|_{1,\Theta} + \|u_\lambda - w_\lambda\|_{1,\sigma_\lambda} + \|u_\lambda - w_\lambda\|_{1,P_\lambda} \\ &\leq \|u - u_\lambda\|_{1,\Theta} + \|u_\lambda - w_\lambda\|_{1,\sigma_\lambda} + \|u_\lambda\|_{1,P_\lambda} + \|w_\lambda\|_{1,P_\lambda} \end{aligned}$$

and prove successively (in parts A–D) that all terms on the right-hand side of (16) tend to zero with $\lambda \rightarrow 0$. The main difficulty is to prove that $\|w_\lambda\|_{1,P_\lambda} \rightarrow 0$ (if $\lambda \rightarrow 0$), in particular to prove

$$\left\| \frac{\partial w_\lambda}{\partial x_i} \right\|_{0,P_\lambda} \rightarrow 0 \quad \text{if } \lambda \rightarrow 0 \quad (i = 1, \dots, N).$$

A. First we prove

$$(17) \quad \lim_{\lambda \rightarrow 0} \|u - u_\lambda\|_{1,\Theta} = 0.$$

Let us denote

$$\bar{h} = (0, \dots, 0, h).$$

As

$$\|u - u_\lambda\|_{0,\Theta} = \sqrt{\int_{\Theta} [u(X) - u(X - \bar{h})]^2 dX},$$

we have, according to [5, Theorem 2.4.2] (the mean continuity theorem for L_2 -functions),

$$(18) \quad \lim_{\lambda \rightarrow 0} \|u - u_\lambda\|_{0,\Theta} = 0.$$

As $D^\alpha u \in L_2(\Theta)$, $D^\alpha u_\lambda \in L_2(\Theta)$ ($|\alpha| = 1$) and as

$$\|D^\alpha u - D^\alpha u_\lambda\|_{0,\Theta} = \sqrt{\int_{\Theta} [(D^\alpha u)(X) - (D^\alpha u)(X - \bar{h})]^2 dX},$$

we have again, according to [5, Theorem 2.4.2],

$$(19) \quad \lim_{\lambda \rightarrow 0} \|D^\alpha u - D^\alpha u_\lambda\|_{0,\Theta} = 0.$$

Relations (18) and (19) together give relation (17).

B. Now we prove

$$(20) \quad \lim_{\lambda \rightarrow 0} \|u_\lambda\|_{1,P_\lambda} = 0.$$

We have mentioned at the end of the *idea of the proof of Lemma 2* that

$$(21) \quad u_\lambda \in H^1(\Theta).$$

As (see Fig. 3)

$$(22) \quad P_\lambda = V_\lambda(\Gamma) \cap \Theta,$$

we have, due to (21),

$$(23) \quad u_\lambda \in H^1(P_\lambda).$$

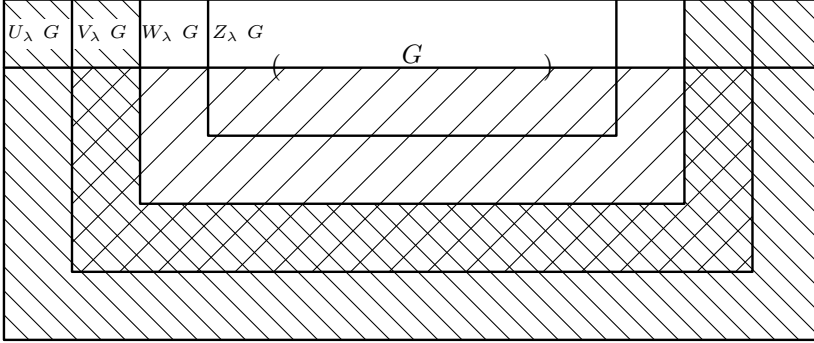


Figure 3. Domains P_λ and Q_λ .

By the definition of $V_\lambda(\Gamma)$ we have

$$(24) \quad \lim_{\lambda \rightarrow 0} (\text{meas}_N V_\lambda(\Gamma)) = 0;$$

hence, taking into account (22), we obtain from (24)

$$(25) \quad \lim_{\lambda \rightarrow 0} \text{meas}_N P_\lambda = 0.$$

Relation (23) yields

$$(26) \quad u_\lambda \in L_2(P_\lambda), \quad D^\alpha u_\lambda \in L_2(P_\lambda) \quad (|\alpha| = 1).$$

Relations (25) and (26) imply, according to the theorem on the absolute continuity of an integral, relation (20).

C. Our task in this subsection is to prove that

$$(27) \quad \lim_{\lambda \rightarrow 0} \|u_\lambda - w_\lambda\|_{1, \sigma_\lambda} \rightarrow 0,$$

where u_λ and w_λ are defined in (7)–(9). As

$$(28) \quad \sigma_\lambda = \Theta \setminus \overline{V_\lambda(\Gamma)}$$

we have

$$(29) \quad \Theta_2 \setminus \overline{W_\lambda(\Gamma)} \supset \sigma_\lambda.$$

As $u_\lambda \in H^1(\Theta)$ and $\text{supp } u \subset \mathcal{K}$ we have (by (7)) $u_\lambda \in H^1(\Theta_2)$. Hence $v_\lambda \in H^1(\Theta_2 \setminus \overline{W_\lambda(\Gamma)})$ and (by (29)) $v_\lambda \in H^1(\sigma_\lambda)$. By (3), (7), (8) and the assumption $\text{supp } u \subset \mathcal{K}$ we have $\text{dist}(\text{supp } v_\lambda, \mathbb{R}^N \setminus \Theta_1) \geq h$. Thus we can repeat with another notation the proof of [10, Lemma 3.6] and prove the implication

$$(30) \quad X \in \sigma_\lambda \cap \text{supp } v_\lambda \Rightarrow \frac{\partial}{\partial x_i}(\omega_h * v_\lambda) = \omega_h * \frac{\partial v_\lambda}{\partial x_i}.$$

(Using the notation of [10, Chapter 3] we have $\omega_h * u = R_h u$.) By [10, Theorem 3.7] implication (30) yields (together with the preceding text)

$$\begin{aligned} \|u_\lambda - w_\lambda\|_{1, \sigma_\lambda} &= \sqrt{\int_{\sigma_\lambda \cap \text{supp } v_\lambda} \sum_{|\alpha| \leq 1} [D^\alpha u_\lambda - D^\alpha(\omega_h * v_\lambda)]^2 dX} \\ &= \sqrt{\int_{\sigma_\lambda \cap \text{supp } v_\lambda} \sum_{|\alpha| \leq 1} [D^\alpha u_\lambda - \omega_h * D^\alpha u_\lambda]^2 dX} \rightarrow 0 \quad \text{if } \lambda \rightarrow 0. \end{aligned}$$

This proves relation (27).

D. Our task is now to estimate the function w_λ as an element of $H^1(P_\lambda)$ (for the definition of P_λ see (22)). To this end, let us denote $Q_\lambda = U_\lambda(\Gamma) \setminus \overline{W_\lambda(\Gamma)}$. Owing to the choice of h we obtain for $X \in P_\lambda$ (cf. Fig. 3, properties of ω_h and definitions of u_λ and v_λ ; we must realize that $h \rightarrow 0$ —in Fig. 1 the variable quantity h is relatively large—and that \mathcal{K} and \mathcal{K}_1 are fixed)

$$(31) \quad \begin{aligned} w_\lambda(X) &= \int_{\mathbb{R}^N} \omega_h(X - Y)v_\lambda(Y) dY \\ &= \int_{\mathbb{R}^N \setminus W_\lambda(\Gamma)} \omega_h(X - Y)v_\lambda(Y) dY \\ &= \int_{Q_\lambda} \omega_h(X - Y)u_\lambda(Y) dY \quad (X \in P_\lambda) \end{aligned}$$

(if $X \in P_\lambda$ then $\|X - Y\| < h$ only for the points $Y \in Q_\lambda$) and similarly (again for $X \in P_\lambda$)

$$(32) \quad \begin{aligned} \psi_i(X) &= \frac{\partial w_\lambda}{\partial x_i}(X) = \int_{Q_\lambda} \frac{\partial \omega_h}{\partial x_i}(X - Y)u_\lambda(Y) dY \\ &= - \int_{Q_\lambda} \frac{\partial \omega_h}{\partial y_i}(X - Y)u_\lambda(Y) dY \quad (X \in P_\lambda). \end{aligned}$$

Now we apply [5, Theorem 2.5.3] (see also [10, Theorem 3.7]) which asserts that

$$\lim_{\varepsilon \rightarrow 0^+} \|D^\alpha(R_\varepsilon u) - D^\alpha u\|_{L_2(\Omega^*)} = 0 \quad (\overline{\Omega^*} \subset \Omega).$$

To this end let us denote by \tilde{u}_λ the extension of the function u_λ by zero onto the domain Θ_1 . In the notation of [5, Theorem 2.5.3] (or [10, Theorem 3.7]) we have

$$(33) \quad \Omega = \Theta_1, \quad \Omega^* = P_\lambda.$$

The extended function \tilde{u}_λ satisfies

$$\tilde{u}_\lambda \in L_2(\Theta_1).$$

An application of [5, Theorem 2.5.3] (or [10, Theorem 3.7]) with $|\alpha| = 0$ to the functions

$$u = \tilde{u}_\lambda, \quad R_\varepsilon u = w_\lambda$$

and domains (33) yields

$$(34) \quad \lim_{\lambda \rightarrow 0} \|w_\lambda - u_\lambda\|_{0, P_\lambda} = 0.$$

Let us combine (34) with the inequality

$$\|w_\lambda\|_{0, P_\lambda} \leq \|u_\lambda\|_{0, P_\lambda} + \|w_\lambda - u_\lambda\|_{0, P_\lambda}$$

and relation (20). Then we obtain

$$(35) \quad \lim_{\lambda \rightarrow 0} \|w_\lambda\|_{0, P_\lambda} = 0.$$

The same device cannot be used in the case of the functions ψ_i (given by (32)) because $\tilde{u}_\lambda \notin H^1(\Theta_1)$ and hence

$$\frac{\partial \tilde{u}_\lambda}{\partial x_i} \notin L_2(\Theta_1).$$

Another approach must be used. Let us consider now the $L_2(P_\lambda)$ -norm of ψ_i . We shall distinguish two cases: $i = N$ (considered in D1) and $i < N$ (see D2.1 and D2.2).

D1. Let $i = N$. As $Q_\lambda \in \tilde{C}^{0,1}$ we can use the Green-Gauss-Ostrogradskij formula and obtain from (32)—see Fig. 3 ($X \in P_\lambda$) (note that $U_\lambda(G)$ and $W_\lambda(G)$ are $(N-1)$ -dimensional sets)

$$(36) \quad \begin{aligned} \psi_N(X) = & \int_{Q_\lambda} \omega_h(X-Y) \frac{\partial u_\lambda}{\partial y_N}(Y) \, dY \\ & + \int_{U_\lambda(G)} \omega_h(X - [Y', -\lambda]) u_\lambda(Y', -\lambda) \, dY' \\ & - \int_{W_\lambda(G)} \omega_h\left(X - \left[Y', -\frac{1}{2}\lambda\right]\right) u_\lambda\left(Y', -\frac{1}{2}\lambda\right) \, dY' \\ & - \int_{U_\lambda(G) \setminus W_\lambda(G)} \omega_h\left(X - \left[Y', \frac{1}{4}\lambda\right]\right) u_\lambda\left(Y', \frac{1}{4}\lambda\right) \, dY'. \end{aligned}$$

Let us note that we have used the Green-Gauss-Ostrogradskij theorem in the form

$$\int_{Q_\lambda} \frac{\partial \omega_h}{\partial y_N} (X - Y) u_\lambda(Y) \, dY = \int_{\partial Q_\lambda} \omega_h(X - Y) u_\lambda(Y) n_N(Y) \, d\sigma - \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_N}(Y) \, dY$$

where $n_N = -1$ on the base of Q_λ , $n_N = 1$ on the parts of ∂Q_λ parallel with the base of Q_λ and $n_N = 0$ otherwise (see Fig. 3).

First we shall prove that the first integral on the right-hand side of (36) tends to zero in $L_2(P_\lambda)$; this means that we shall prove

$$(37) \quad \lim_{\lambda \rightarrow 0} \int_{P_\lambda} \left(\int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_N}(Y) \, dY \right)^2 \, dX = 0.$$

We have

$$(38) \quad \int_{P_\lambda} \left(\int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_N}(Y) \, dY \right)^2 \, dX = \varkappa \int_{P_\lambda} \left\{ \int_{Q_\lambda} h^{-N} \exp\left(\frac{\|X - Y\|^2}{\|X - Y\|^2 - h^2}\right) \frac{\partial u_\lambda}{\partial y_N}(Y) \, dY \right\}^2 \, dX.$$

For the sake of brevity, let us denote

$$(39) \quad E_h(X, Y) := \exp\left(\frac{\|X - Y\|^2}{\|X - Y\|^2 - h^2}\right), \quad F(Y) := \frac{\partial u_\lambda}{\partial y_N}(Y).$$

Using this notation we further denote

$$(40) \quad F_h(X) = \frac{1}{h^N} \int_{Q_\lambda} E_h(X, Y) F(Y) \, dY.$$

By (39), (40) we can write

$$(41) \quad |F_h(X)| \leq \frac{1}{h^N} \int_{Q_\lambda} \sqrt{E_h(X, Y)} \sqrt{E_h(X, Y)} |F(Y)| \, dY.$$

Let us square inequality (41) and to the resulting right-hand side let us apply the Schwarz inequality; we thus obtain

$$(42) \quad |F_h(X)|^2 \leq \frac{1}{h^N} \int_{Q_\lambda} E_h(X, Y) \, dY \cdot \frac{1}{h^N} \int_{Q_\lambda} E_h(X, Y) |F(Y)|^2 \, dY.$$

We have (see [8, p. 218, relation (168)], where the case $N = 2$ is considered)

$$\int_{Q_\lambda} E_h(X, Y) \, dY = Ch^N.$$

Hence

$$(43) \quad \frac{1}{h^N} \int_{Q_\lambda} E_h(X, Y) \, dY = C.$$

Let us use (43) and let us integrate inequality (42) over P_λ with respect to X . We obtain

$$(44) \quad \int_{P_\lambda} |F_h(X)|^2 \, dX \leq \frac{C}{h^N} \int_{P_\lambda} \left\{ \int_{Q_\lambda} E_h(X, Y) |F(Y)|^2 \, dY \right\} \, dX.$$

Now we use the Fubini theorem on the right-hand side of (44) and then relation (43) (where we write now P_λ instead of Q_λ). We conclude (if we use also notation (39)₂):

$$(45) \quad \begin{aligned} \int_{P_\lambda} |F_h(X)|^2 \, dX &\leq C \int_{Q_\lambda} \left\{ |F(Y)|^2 \frac{1}{h^N} \int_{P_\lambda} E_h(X, Y) \, dX \right\} \, dY \\ &= C^2 \int_{Q_\lambda} |F(Y)|^2 \, dY = C^2 \int_{Q_\lambda} \left(\frac{\partial u_\lambda}{\partial y_N}(Y) \right)^2 \, dY. \end{aligned}$$

Using the relation

$$\lim_{\lambda \rightarrow 0} (\text{meas}_N Q_\lambda) = 0$$

and the theorem on the absolute continuity of the Lebesgue integral we see that the right-hand side of (45) tends to zero if $\lambda \rightarrow 0$. This proves (37).

Now we prove that the second and the fourth integrals on the right-hand side of (36) are equal to zero for $X \in P_\lambda$. We recall that

$$h = \frac{1}{4}\lambda.$$

As to the second integral, for $X \in P_\lambda = V_\lambda(\Gamma) \cap \Theta$ we have

$$(46) \quad \begin{aligned} \omega_h(X - [Y', -\lambda]) &= \omega_h([X', x_N] - [Y', -\lambda]) \\ &= \omega_h(X' - Y', x_N + \lambda) \\ &= \omega_h(X' - Y', x_N + 4h) = \omega_h(Z), \end{aligned}$$

where we set

$$Z = [X' - Y', x_N + 4h].$$

We further have

$$(47) \quad \|Z\| = \sqrt{(X' - Y')^2 + (x_N + 4h)^2},$$

where we set for the sake of brevity

$$(X' - Y')^2 = (x_1 - y_1)^2 + \dots + (x_{N-1} - y_{N-1})^2.$$

The most inconvenient case is

$$(X' - Y')^2 = 0.$$

Since in the case $X \in P_\lambda = V_\lambda(\Gamma) \cap \Theta$ we have

$$(48) \quad x_N \in (-3h, 0)$$

we obtain from (47)

$$(49) \quad \|Z\| \geq h,$$

which implies by (46) (and by the fact that $\omega_h(Z) = 0$ for $\|Z\| \geq h$) that

$$\omega_h(X - [Y', -\lambda]) = 0.$$

This proves that the second integral on the right-hand side of (36) is equal to zero for $X \in P_\lambda$.

As to the fourth integral on the right-hand side of (36), we have

$$\omega_h\left(X - \left[Y', \frac{1}{4}\lambda\right]\right) = \omega_h(X' - Y', x_N - h).$$

As (48) holds we obtain again (49) with $Z = [X' - Y', x_N - h]$. This proves that the fourth integral on the right-hand side of (36) is equal to zero for $X \in P_\lambda$.

Let us consider the third integral on the right-hand side of (36) (in the second equality we use (7)):

$$\begin{aligned} \psi_{N,3}(X) &= \int_{W_\lambda(G)} \omega_h\left(X - \left[Y', -\frac{1}{2}\lambda\right]\right) u_\lambda\left(Y', -\frac{1}{2}\lambda\right) dY' \\ &= \int_{W_\lambda(G)} \omega_h\left(X' - Y', x_N + \frac{1}{2}\lambda\right) u\left(Y', -\frac{3}{4}\lambda\right) dY'. \end{aligned}$$

As

$$\omega_h\left(X' - Y', x_N + \frac{1}{2}\lambda\right) = 0 \quad \forall \|X' - Y'\| \geq h$$

we can write

$$\psi_{N,3}(X) = \int_{\|X'-Y'\|<h} \omega_h\left(X'-Y', x_N + \frac{1}{2}\lambda\right) u\left(Y', -\frac{3}{4}\lambda\right) dY'.$$

From this identity we obtain by the Schwarz inequality (and by extending the domain of integration in the case of the first integral)

$$\begin{aligned} (50) \quad |\psi_{N,3}(X)|^2 &= \left(\int_{\|X'-Y'\|<h} \omega_h\left(X'-Y', x_N + \frac{1}{2}\lambda\right) u\left(Y', -\frac{3}{4}\lambda\right) dY' \right)^2 \\ &\leq \int_{\|X'-Y'\|<h} \omega_h^2\left(X'-Y', x_N + \frac{1}{2}\lambda\right) dY' \\ &\quad \times \int_{\|X'-Y'\|<h} u^2\left(Y', -\frac{3}{4}\lambda\right) dY' \\ &\leq \int_{\mathbb{R}^{N-1}} \omega_h^2\left(X'-Y', x_N + \frac{1}{2}\lambda\right) dY' \\ &\quad \times \int_{\|Y'-X'\|<h} u^2\left(Y', -\frac{3}{4}\lambda\right) dY' \\ &= \int_{\mathbb{R}^{N-1}} \omega_h^2\left(Z', x_N + \frac{1}{2}\lambda\right) dZ' \int_{\|Y'-X'\|<h} u^2\left(Y', -\frac{3}{4}\lambda\right) dY'. \end{aligned}$$

(We have used also the fact that $\omega_h(X'-Y', x_N + \frac{1}{2}\lambda) = \omega_h(Y'-X', x_N + \frac{1}{2}\lambda)$.) Integrating (50) over P_λ with respect to X (and extending the domain of integration in the case of the first integral) we obtain

$$\begin{aligned} (51) \quad \|\psi_{N,3}\|_{0,P_\lambda}^2 &= \int_{-3\lambda/4}^0 \left(\int_{\mathbb{R}^{N-1}} \omega_h^2\left(Z', x_N + \frac{1}{2}\lambda\right) dZ' \right) dx_N \\ &\quad \times \int_{V_\lambda(G)} \left(\int_{\|Y'-X'\|<h} u^2\left(Y', -\frac{3}{4}\lambda\right) dY' \right) dX' \\ &\leq \left(\int_{\mathbb{R}^N} \omega_h^2(Z) dZ \right) \int_{V_\lambda(G)} \left(\int_{\|Y'-X'\|<h} u^2\left(Y', -\frac{3}{4}\lambda\right) dY' \right) dX'. \end{aligned}$$

Now we prove that

$$(52) \quad \int_{\mathbb{R}^N} \omega_h^2(Z) dZ = Ch^{-N}.$$

Let K_h be the disc (or the sphere) with its center at the origin and its radius equal to h . Then

$$\text{meas}_N K_h = \begin{cases} \pi h^2 & \text{for } N = 2, \\ \frac{4}{3}\pi h^3 & \text{for } N = 3 \end{cases}$$

and the mean value theorem yields

$$\begin{aligned} \int_{\mathbb{R}^N} \omega_h^2(Z) dZ &= \int_{K_h} \omega_h^2(Z) dZ = \varkappa h^{-2N} \cdot C_0 \cdot \text{meas}_N K_h \\ &= \varkappa h^{-2N} \cdot C_0 \cdot \tilde{C} h^N = C h^{-N}, \end{aligned}$$

where we set

$$C_0 = \exp\left(\frac{\|Z_0\|^2}{\|Z_0\|^2 - h^2}\right) \leq 1, \quad C = \varkappa C_0 \tilde{C}.$$

This proves relation (52).

From now on we shall assume that the functions considered are sufficiently *smooth* and we will extend our result by the density argument (i.e., by means of [5, Theorem 5.5.9]). For easier understanding we shall distinguish two cases: $N = 2$ and $N = 3$.

a) $N = 2$: In this case (considering for simplicity that $G = (a, b)$) we have by (4)

$$(53) \quad \begin{aligned} U_\lambda(G) &= (a - \lambda, b + \lambda), \\ V_\lambda(G) &= \left(a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda\right). \end{aligned}$$

We recall that

$$h = \frac{1}{4}\lambda.$$

Let us consider the second integral on the right-hand side of (51). In the case of $N = 2$ we have

$$(54) \quad \begin{aligned} &\int_{V_\lambda(G)} \left(\int_{\|Y' - X'\| < h} u^2\left(Y', -\frac{3}{4}\lambda\right) dY' \right) dX' \\ &= \int_{a-3\lambda/4}^{b+3\lambda/4} \left(\int_{x_1-h}^{x_1+h} u^2\left(y_1, -\frac{3}{4}\lambda\right) dy_1 \right) dx_1. \end{aligned}$$

Using the mean value theorem we obtain (owing to the sufficient smoothness of functions considered)

$$\begin{aligned} \int_{a-3\lambda/4}^{b+3\lambda/4} \left(\int_{x_1-h}^{x_1+h} u^2\left(y_1, -\frac{3}{4}\lambda\right) dy_1 \right) dx_1 &= \int_{a-3\lambda/4}^{b+3\lambda/4} u^2\left(\tilde{y}_1, -\frac{3}{4}\lambda\right) dx_1 \cdot \int_{x_1-h}^{x_1+h} dy_1 \\ &= 2h \int_{a-3\lambda/4}^{b+3\lambda/4} u^2\left(\tilde{y}_1, -\frac{3}{4}\lambda\right) dx_1, \end{aligned}$$

where

$$(55) \quad \tilde{y}_1 = x_1 + \eta, \quad \eta \in (-h, h).$$

The transformation $x_1 + \eta = t$ with (55) yields

$$\begin{aligned} 2h \int_{a-3\lambda/4}^{b+3\lambda/4} u^2\left(\tilde{y}_1, -\frac{3}{4}\lambda\right) dx_1 &= 2h \int_{a-3\lambda/4+\eta}^{b+3\lambda/4+\eta} u^2\left(t, -\frac{3}{4}\lambda\right) dt \\ &\leq 2h \int_{a-\lambda}^{b+\lambda} u^2\left(t, -\frac{3}{4}\lambda\right) dt. \end{aligned}$$

Using (53) and combining the result just obtained with (54) we obtain (in the case $N = 2$)

$$(56) \quad \int_{V_\lambda(G)} \left(\int_{\|Y'-X'\|<h} u^2\left(Y', -\frac{3}{4}\lambda\right) dY' \right) dX' \leq 2h \int_{U_\lambda(G)} u^2\left(Y', -\frac{3}{4}\lambda\right) dY'$$

and thus we have (in the case of $N = 2$), due to (51) and (52),

$$(57) \quad \|\psi_{N,3}\|_{0,P_\lambda}^2 \leq C_2 h^{-1} \int_{U_\lambda(G)} u^2\left(Y', -\frac{3}{4}\lambda\right) dY'.$$

b) $N = 3$: Our task is now to prove relation (57) in the case $N = 3$ (i.e., to prove relation (62)). Now the domains $U_\lambda(G)$ and $V_\lambda(G)$ are given by $(4)_1$ and $(4)_2$, respectively. Using the mean value theorem we obtain

$$\begin{aligned} (58) \quad &\int_{V_\lambda(G)} \left(\int_{\|Y'-X'\|<h} u^2\left(Y', -\frac{3}{4}\lambda\right) dY' \right) dX' \\ &= \int_{V_\lambda(G)} u^2\left(\tilde{Y}', -\frac{3}{4}\lambda\right) dX' \cdot \int_{\|Y'-X'\|<h} dY' \\ &= \pi h^2 \int_{V_\lambda(G)} u^2\left(\tilde{Y}', -\frac{3}{4}\lambda\right) dX', \end{aligned}$$

where

$$(59) \quad \tilde{Y}' = X' + [\eta_1, \eta_2], \quad [\eta_1, \eta_2] \in \{\|Y' - X'\| < h\}.$$

This means that

$$(60) \quad \eta_1 \in (-h \cos \alpha, h \cos \alpha), \quad \eta_2 \in (-h \sin \alpha, h \sin \alpha), \quad \alpha \in (0, \pi).$$

The transformation

$$x_1 + \eta_1 = s_1, \quad x_2 + \eta_2 = s_2$$

with (59) and (60) yields

$$\pi h^2 \int_{V_\lambda(G)} u^2\left(\tilde{Y}', -\frac{3}{4}\lambda\right) dX' \leq \pi h^2 \int_{U_\lambda(G)} u^2\left(S', -\frac{3}{4}\lambda\right) dS'.$$

Using (4)₁, (4)₂ and combining the result just obtained with (58) we obtain (in the case of $N = 3$)

$$(61) \quad \int_{V_\lambda(G)} \left(\int_{\|Y' - X'\| < h} u^2\left(Y', -\frac{3}{4}\lambda\right) dY' \right) dX' \leq \pi h^2 \int_{U_\lambda(G)} u^2\left(Y', -\frac{3}{4}\lambda\right) dY'$$

and thus we have (if $N = 3$), due to (51) and (52),

$$(62) \quad \|\psi_{N,3}\|_{0,P_\lambda}^2 \leq C_2 h^{-1} \int_{U_\lambda(G)} u^2\left(Y', -\frac{3}{4}\lambda\right) dY'.$$

Let us consider the integral on the right-hand side of (62) or on the right-hand side of (57). (Note that relation (62) has the same form as relation (57) which is written in the case of $N = 2$.) We obtain for $Y' \in U_\lambda(G)$ (let us point out that we assume u to be *smooth* enough and extend then the result by the density argument (see [5, Theorem 5.5.9])):

$$u\left(Y', -\frac{3}{4}\lambda\right) = u(Y', 0) + \int_0^{-3\lambda/4} \frac{\partial u}{\partial y_N}(Y', \xi) d\xi;$$

now we square this relation, use the inequality $(a + b)^2 \leq 2(a^2 + b^2)$ and then apply the Schwarz inequality; finally, we extend the interval of integration $(-\frac{3}{4}\lambda, 0)$ to $(-\lambda, 0)$ arriving at

$$\begin{aligned} u^2\left(Y', -\frac{3}{4}\lambda\right) &\leq 2 \left[u^2(Y', 0) + \left(\int_0^{-3\lambda/4} \frac{\partial u}{\partial y_N}(Y', \xi) d\xi \right)^2 \right] \\ &= 2 \left[u^2(Y', 0) + \left(- \int_{-3\lambda/4}^0 \frac{\partial u}{\partial y_N}(Y', \xi) d\xi \right)^2 \right] \\ &\leq 2 \left[u^2(Y', 0) + \frac{3}{4}\lambda \int_{-3\lambda/4}^0 \left(\frac{\partial u}{\partial y_N}(Y', \xi) \right)^2 d\xi \right] \\ &\leq 2 \left[u^2(Y', 0) + \frac{3}{4}\lambda \int_{-\lambda}^0 \left(\frac{\partial u}{\partial y_N}(Y', \xi) \right)^2 d\xi \right]. \end{aligned}$$

Hence ($h = \frac{1}{4}\lambda$)

$$(63) \quad h^{-1} \int_{U_\lambda(G)} u^2\left(Y', -\frac{3}{4}\lambda\right) dY' \leq C_3 \left[h^{-1} \int_{U_\lambda(G)} u^2(Y', 0) dY' + \int_{U_\lambda(G)} \left(\frac{\partial u}{\partial y_N}(Y', \xi) \right)^2 dY' d\xi \right].$$

The theorem on the absolute continuity of an integral implies

$$(64) \quad \int_{U_\lambda(G)} \left(\frac{\partial u}{\partial y_N}(Y', \xi) \right)^2 dY' d\xi \rightarrow 0 \quad (\text{with } \lambda \rightarrow 0)$$

and further we have (owing to the fact that $u|_{\Gamma} = 0$, $\Gamma = G \times \{0\}$)

$$(65) \quad \int_{U_{\lambda}(G)} u^2(Y', 0) dY' = \int_{U_{\lambda}(G) \setminus G} u^2(Y', 0) dY'.$$

When analyzing the integral on the right-hand side of (65) we shall distinguish between the cases $N = 2$ and $N \geq 3$ (in applications it is sufficient to consider the case $N = 3$).

1) The case $N = 2$. Again, let for simplicity $G = (a, b)$ (see Fig. 1). Then

$$\int_{U_{\lambda}(G) \setminus G} u^2(Y', 0) dY' = \int_{a-\lambda}^a u^2(y_1, 0) dy_1 + \int_b^{b+\lambda} u^2(y_1, 0) dy_1 = I_1 + I_2.$$

The integral I_1 will be written in the form

$$I_1 = - \int_a^{a-\lambda} u^2(y_1, 0) dy_1$$

and the substitution $y_1 = a - t$, where $t \in (0, \lambda)$, will be used. We obtain, according to the theorem on substitution in a simple integral,

$$I_1 = \int_0^{\lambda} u^2(a - t, 0) dt.$$

Similarly, using in the integral I_2 the substitution $y_1 = b + t$ where $t \in (0, \lambda)$, we obtain

$$I_2 = \int_0^{\lambda} u^2(b + t, 0) dt.$$

Hence, according to the theorem on differentiation of an integral as a function of the upper limit and by the l'Hospital rule (and taking into account that the first integral on the right-hand side of (63) is multiplied by h^{-1}), we obtain

$$(66) \quad \begin{aligned} & \lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} \int_{U_{\lambda}(G) \setminus G} u^2(Y', 0) dY' \\ &= \lim_{\lambda \rightarrow 0+} \frac{1}{\lambda} \left(\int_0^{\lambda} u^2(a - t, 0) dt + \int_0^{\lambda} u^2(b + t, 0) dt \right) \\ &= \lim_{\lambda \rightarrow 0+} (u^2(a - \lambda, 0) + u^2(b + \lambda, 0)) \\ &= u^2(a, 0) + u^2(b, 0) = 0, \end{aligned}$$

because $u(a, 0) = 0$, $u(b, 0) = 0$ (u is, according to our assumption, a *smooth* function). All these results yield $\|\psi_N\|_{0, P_{\lambda}} \rightarrow 0$ with $\lambda \rightarrow 0$ (when $N = 2$).

2) In this part of the proof we use, beside the Trace Theorem in the form of [5, Theorem 6.8.13] (which concerns the mapping $\mathfrak{R}: H^{1,p} \rightarrow H^{1-\frac{1}{p},p}(\partial\Omega)$), the following theorem (we cite it from [1], where it is introduced by the following words: “We shall not attempt any proof” and references to the works of Besov [2], Uspenskij [9] and Lizorkin [6] are given. Adams further writes: “The theorem is stated for \mathbb{R}^N but can obviously be extended to domains with sufficient regularity” (as, for example, domains $\Omega \in \tilde{C}^{0,1}$).

Theorem A. *Let $s > 0$, $1 < p \leq q < \infty$, and $1 \leq k \leq n$. Let $\chi = s - n/p + k/q$. If*

- (1) $\chi \geq 0$ and $p < q$, or
 - (2) $\chi > 0$ and χ is not an integer, or
 - (3) $\chi \geq 0$ and $1 < p \leq 2$,
- then (direct imbedding theorem)

$$(67) \quad W^{s,p}(\mathbb{R}^n) \rightarrow W^{\chi,q}(\mathbb{R}^k).$$

Imbedding (67) does not necessarily hold for $p = q > 2$ and χ a nonnegative integer.

In this part of the proof we restrict ourselves to the case $N = 3$. Our considerations start now again from relation (65). Using the Trace Theorem in the form of [5, Theorem 6.8.13] we see that

$$(68) \quad u(Y', 0) \in H^{\frac{1}{2}}(\Delta).$$

Now we use Theorem A: We have (using the notation of Theorem A)

$$s = \frac{1}{2}, \quad p = q = 2, \quad n = N - 1, \quad k = n - 1 = N - 2.$$

Hence

$$\chi = s - \frac{n}{p} + \frac{k}{q} = \frac{1}{2} - \frac{N-1}{2} + \frac{N-2}{2} = 0.$$

Thus, according to the assertion of Theorem A,

$$H^{\frac{1}{2}}(\mathbb{R}^{N-1}) = W^{\frac{1}{2},2}(\mathbb{R}^{N-1}) \rightarrow H^0(\mathbb{R}^{N-2}) = L_2(\mathbb{R}^{N-2}),$$

and consequently

$$H^{\frac{1}{2}}(\Delta) \rightarrow L_2(\partial\Delta)$$

and also (again according to Theorem A, because $G \subset \Delta$)

$$H^{\frac{1}{2}}(\Delta) \rightarrow L_2(\partial G).$$

Thus (as $u(Y', 0) = u|_G$)

$$u(Y', 0)|_{\partial G} \in L_2(\partial G).$$

Then the convergence

$$\frac{1}{\lambda} \int_{U_\lambda(G) \setminus G} u^2(Y', 0) dY' \rightarrow 0 \quad (\text{with } \lambda \rightarrow 0)$$

follows from the properties of traces: Of course, *locally* we have

$$\int_{\mathcal{R}} u^2(Y', 0) dY' \leq \int_{\mathcal{R}'} \left(\int_{a(z_1) - K\lambda}^{a(z_1)} u^2(z_1, z_2, 0) dz_2 \right) dz_1,$$

where \mathcal{R} is the intersection of $U_\lambda(G) \setminus G$ with some suitable neighbourhood

$$\mathcal{R}_1 = \{[z_1, z_2]: z_1 \in \mathcal{R}', z_2 \in (a(z_1) - \gamma, a(z_1) + \gamma)\}$$

of any fixed point of ∂G (see Fig. 4) and $a(z_1)$ is the function which represents ∂G

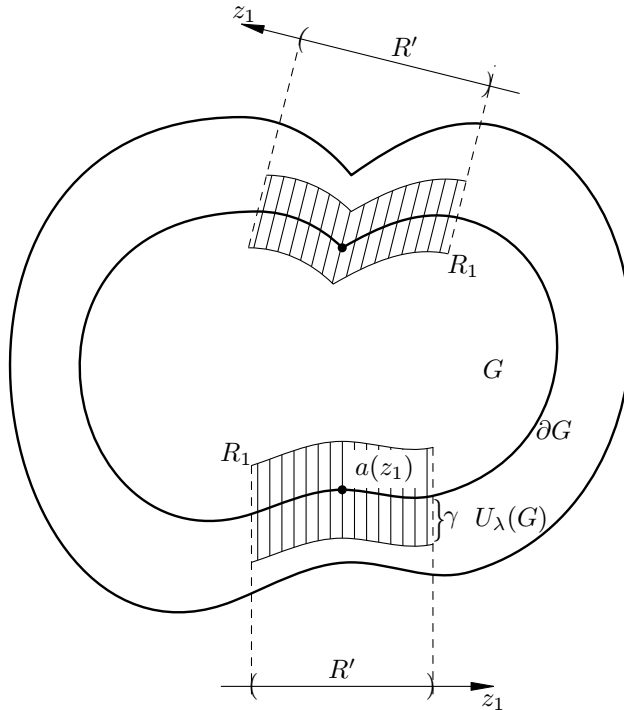


Figure 4. Concerning the case $N = 3$ (see 1 b) of part D1.)

with respect to a local Cartesian coordinate system of axes (z_1, z_2) with a Lipschitz constant L and $K \leq \sqrt{L^2 + 1}$. The function

$$(69) \quad \Phi(\eta) = \int_{\mathcal{R}'} u^2(z_1, a(z_1) - \eta, 0) dz_1$$

is a continuous function of $\eta \in \langle -\gamma, 0 \rangle$: indeed, the function $u(z_1, z_2, 0)$ is assumed sufficiently smooth on $\mathcal{R}' \times \langle -\gamma, \gamma \rangle$, hence it is continuous on $\overline{\mathcal{R}'} \times \langle -\gamma, \gamma \rangle$; by Cantor's theorem it is there uniformly continuous. As $a(z_1)$ is Lipschitz continuous on $\overline{\mathcal{R}'}$, we have $|a(z_1) - a(z'_1)| < L|z_1 - z'_1|$. All these facts imply that (according to the definition of uniform continuity of a function of two variables)

$$|u(z_1, a(z_1) - \eta, 0) - u(z'_1, a(z'_1) - \eta', 0)| < \varepsilon \quad \text{for } |z_1 - z'_1| < \frac{\delta}{2L}, \quad |\eta - \eta'| < \frac{\delta}{2}.$$

It suffices now to use the theorem on continuity of an integral with respect to a parameter.

Thus, using the Fubini theorem and making the change of variables in the form $\eta = a(z_1) - z_2$, we obtain by means of the l'Hospital rule and the theorem on differentiation of the integral as a function of the upper limit

$$\lim_{\lambda \rightarrow 0^+} \frac{1}{\lambda} \int_{\mathcal{R}} u^2(Y', 0) dY' \leq K \lim_{\lambda \rightarrow 0^+} \frac{1}{K\lambda} \int_{-K\lambda}^0 \Phi(\eta) d\eta = -K\Phi(0) = 0,$$

because $u|_{\Gamma} = 0$ and the function $\Phi(\eta)$ is given by relation (69); we must keep in mind that $u^2(z_1, a(z_1), 0)|_{z_1 \in \mathcal{R}'} = u^2|_{\partial G} = u^2|_G = u^2|_{\Gamma} = 0$.

D2.1. $N = 2$. First we shall discuss the case $N = 2$. Let, for simplicity, the set G be an open interval on the axis x_1 :

$$G = (a, b).$$

Then

$$\begin{aligned} U_\lambda(G) &= (a - \lambda, b + \lambda), & V_\lambda(G) &= \left(a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda\right), \\ W_\lambda(G) &= \left(a - \frac{1}{2}\lambda, b + \frac{1}{2}\lambda\right), & Z_\lambda(G) &= \left(a - \frac{1}{4}\lambda, b + \frac{1}{4}\lambda\right), \end{aligned}$$

and

$$\begin{aligned} U_\lambda(\Gamma) &= (a - \lambda, b + \lambda) \times \left(-\lambda, \frac{1}{4}\lambda\right), \\ V_\lambda(\Gamma) &= \left(a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda\right) \times \left(-\frac{3}{4}\lambda, \frac{1}{4}\lambda\right), \\ W_\lambda(\Gamma) &= \left(a - \frac{1}{2}\lambda, b + \frac{1}{2}\lambda\right) \times \left(-\frac{1}{2}\lambda, \frac{1}{4}\lambda\right), \\ Z_\lambda(\Gamma) &= \left(a - \frac{1}{4}\lambda, b + \frac{1}{4}\lambda\right) \times \left(-\frac{1}{4}\lambda, \frac{1}{4}\lambda\right). \end{aligned}$$

By (32), Fig. 3 and the Green-Gauss-Ostrogradskij theorem we have

$$\begin{aligned}
 (70) \quad \psi_1(X) &= \frac{\partial w_\lambda}{\partial x_1}(X) = \int_{Q_\lambda} \frac{\partial \omega_h}{\partial x_1}(X - Y) u_\lambda(Y) \, dY \\
 &= - \int_{Q_\lambda} \frac{\partial \omega_h}{y_1}(X - Y) u_\lambda(Y) \, dY = \\
 &= \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_1}(Y) \, dY \\
 &\quad - \int_{-\lambda}^{\lambda/4} \omega_h(X - [b + \lambda, y_2]) u_\lambda(b + \lambda, y_2) \, dy_2 \\
 &\quad + \int_{-\lambda/2}^{\lambda/4} \omega_h\left(X - \left[b + \frac{1}{2}\lambda, y_2\right]\right) u_\lambda\left(b + \frac{1}{2}\lambda, y_2\right) \, dy_2 \\
 &\quad - \int_{-\lambda/2}^{\lambda/4} \omega_h\left(X - \left[a - \frac{1}{2}\lambda, y_2\right]\right) u_\lambda\left(a - \frac{1}{2}\lambda, y_2\right) \, dy_2 \\
 &\quad + \int_{-\lambda}^{\lambda/4} \omega_h(X - [a - \lambda, y_2]) u_\lambda(a - \lambda, y_2) \, dy_2.
 \end{aligned}$$

Similarly to Section D1 the first integral on the right-hand side of (70) tends to zero in $L_2(P_\lambda)$; this means

$$(71) \quad \lim_{\lambda \rightarrow 0} \int_{P_\lambda} \left(\int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_1}(Y) \, dY \right)^2 \, dX = 0,$$

where (see Fig. 3)

$$P_\lambda = V_\lambda(\Gamma) \cap \Theta, \quad Q_\lambda = U_\lambda(\Gamma) \setminus \overline{W_\lambda(\Gamma)}.$$

Now we prove that the second and fifth integrals on the right-hand side of (70) are equal to zero. We have

$$\begin{aligned}
 \omega_h(X - [b + \lambda, y_2]) &= \omega_h([x_1, x_2] - [b + \lambda, y_2]) \\
 &= \omega_h(x_1 - b - \lambda, x_2 - y_2) \\
 &= \omega_h(x_1 - b - 4h, x_2 - y_2) = \omega_h(Z),
 \end{aligned}$$

where we set

$$Z = [x_1 - b - 4h, x_2 - y_2].$$

Thus

$$\|Z\| = \sqrt{(x_1 - b - 4h)^2 + (x_2 - y_2)^2}.$$

The most inconvenient situation occurs when

$$(x_2 - y_2)^2 = 0.$$

As $X \in P_\lambda = V_\lambda(\Gamma) \cap \Theta$ we have

$$(72) \quad x_1 \in \left(a - \frac{3}{4}\lambda, b + \frac{3}{4}\lambda\right) = (a - 3h, b + 3h).$$

In such a situation the inequality

$$(73) \quad \|Z\| \geq h$$

holds because for every x_1 satisfying (72) we have

$$(x_1 - b - 4h)^2 > [(b + 3h) - b - 4h]^2 = h^2.$$

Hence the *second* integral on the right-hand side of (70) is equal to zero.

A similar situation arises in the case of the *fifth* integral on the right-hand side of (70) because

$$\omega_h(X - [a - \lambda, y_2]) = \omega_h(x_1 - a + 4h, x_2 - y_2) = \omega_h(Z)$$

and again we have

$$(x_1 - a + 4h)^2 > [(a - 3h) - a + 4h]^2 = h^2$$

for every x_1 satisfying (72), hence relation (73) holds again.

As to the *third* and *fourth* integrals on the right-hand side of (70) they can be different from zero. We see it (in the case of the third integral) from the relation

$$\omega_h\left(X - \left[b + \frac{1}{2}\lambda, y_2\right]\right) = \omega_h(x_1 - b - 2h, x_2 - y_2) = \omega_h(Z)$$

where, due to (72), we can have $x_1 = b + 2h$, and thus it is possible that

$$\|Z\| < h.$$

Thus, let us consider

$$(74) \quad \psi_{1,3}(X) = \int_{-\lambda/2}^{\lambda/4} \omega_h\left(X - \left[b + \frac{1}{2}\lambda, y_2\right]\right) u_\lambda\left(b + \frac{1}{2}\lambda, y_2\right) dy_2,$$

$$(75) \quad \psi_{1,4}(X) = \int_{-\lambda/2}^{\lambda/4} \omega_h\left(X - \left[a - \frac{1}{2}\lambda, y_2\right]\right) u_\lambda\left(a - \frac{1}{2}\lambda, y_2\right) dy_2.$$

Taking into account the preceding text and relation (70) we can write

$$(76) \quad |\psi_1(X)|^2 = (\psi_{1,3}(X) + \psi_{1,4}(X))^2 \leq 2(|\psi_{1,3}(X)|^2 + |\psi_{1,4}(X)|^2);$$

thus it suffices to consider the case of relation (74); the considerations in the case of (75) follow the same lines.

Using the Schwarz inequality and relation (74) we obtain

$$(77) \quad |\psi_{1,3}(X)|^2 \leq \int_{-\lambda/2}^{\lambda/4} \omega_h^2 \left(X - \left[b + \frac{1}{2}\lambda, y_2 \right] \right) dy_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left(b + \frac{1}{2}\lambda, y_2 \right) dy_2.$$

Inequality (77) will be rewritten to the form

$$(78) \quad |\psi_{1,3}(X)|^2 \leq \int_{-\lambda/2}^{\lambda/4} \omega_h^2 \left(x_1 - b - \frac{1}{2}\lambda, x_2 - y_2 \right) dy_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left(b + \frac{1}{2}\lambda, y_2 \right) dy_2 \\ \leq \int_{-\infty}^{\infty} \omega_h^2 \left(x_1 - b - \frac{1}{2}\lambda, y_2 - x_2 \right) dy_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left(b + \frac{1}{2}\lambda, y_2 \right) dy_2.$$

In the first integral on the right-hand side of (78) we transform the variables in the form

$$y_2 - x_2 = z_2 \Rightarrow dy_2 = dz_2$$

and in the second integral we change the notation of the integration variable. We obtain

$$(79) \quad |\psi_{1,3}(X)|^2 \leq \int_{-\infty}^{\infty} \omega_h^2 \left(x_1 - b - \frac{1}{2}\lambda, z_2 \right) dz_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left(b + \frac{1}{2}\lambda, s \right) ds.$$

Let us integrate inequality (79) over P_λ :

$$(80) \quad \int_{P_\lambda} |\psi_{1,3}(X)|^2 dX \\ \leq \int_{P_\lambda} \left\{ \int_{-\infty}^{\infty} \omega_h^2 \left(x_1 - b - \frac{1}{2}\lambda, z_2 \right) dz_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda^2 \left(b + \frac{1}{2}\lambda, s \right) ds \right\} dX \\ = \int_{a-3\lambda/4}^{b+3\lambda/4} \left(\int_{-\infty}^{\infty} \omega_h^2 \left(x_1 - \frac{1}{2}\lambda - b, z_2 \right) dz_2 \right) dx_1 \\ \times \int_{-3\lambda/4}^0 dx_2 \int_{-\lambda/2}^{\lambda/4} u_\lambda \left(b + \frac{1}{2}\lambda, s \right) ds \\ \leq \int_{-\infty}^{\infty} \left(\int_{-\infty}^{\infty} \omega_h^2 \left(x_1 - b - \frac{1}{2}\lambda, z_2 \right) dz_2 \right) dx_1 \cdot 3h \int_{-\lambda/2}^{\lambda/4} u_\lambda \left(b + \frac{1}{2}\lambda, s \right) ds.$$

In the first integral on the right-hand side of (80) we transform the variables in the form

$$x_1 - b - \frac{1}{2}\lambda = z_1 \Rightarrow dx_1 = dz_1;$$

hence we obtain

$$(81) \quad \|\psi_{1,3}\|_{0,P_\lambda}^2 \leq \int_{\mathbb{R}^2} \omega_h^2(Z) dZ \cdot 3h \int_{-\lambda/2}^{\lambda/4} u_\lambda\left(b + \frac{1}{2}\lambda, s\right) ds.$$

Relations (81) and (52) (with $N = 2$) yield

$$(82) \quad \|\psi_{1,3}\|_{0,P_\lambda}^2 \leq C_2 h^{-1} \int_{-\lambda/2}^{\lambda/4} u_\lambda^2\left(b + \frac{1}{2}\lambda, s\right) ds.$$

Inequality (82) is a one-dimensional analogue of inequalities (57) and (62). In this case we proceed as follows (we use definition (7) of the function u_λ and the definition of h : $h = \frac{1}{4}\lambda$):

$$(83) \quad h^{-1} \int_{-\lambda/2}^{\lambda/4} u_\lambda^2\left(b + \frac{1}{2}\lambda, s\right) ds = h^{-1} \int_{-2h}^h u_\lambda^2\left(b + \frac{1}{2}\lambda, s\right) ds \\ = \frac{1}{h} \int_{-2h}^0 u^2\left(b + \frac{1}{2}\lambda, s - h\right) ds + \frac{1}{h} \int_0^h u^2\left(b + \frac{1}{2}\lambda, s - h\right) ds.$$

Let $h \rightarrow 0$ and let us evaluate each limit on the right-hand side of (83) separately. In both cases we have the limit of type $\frac{0}{0}$; thus we can use the l'Hospital rule and the theorem on differentiation of an integral as a function of the upper limit. Again, as in part D1, we assume the function u to be sufficiently *smooth* and then extend the result by a density argument (see [5, Theorem 5.5.9]). First we consider the limit of the second term on the right-hand side of (83):

$$(84) \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h u^2\left(b + \frac{1}{2}\lambda, s - h\right) ds = \lim_{h \rightarrow 0} u^2(b + 2h, 0) = 0,$$

because $[b, 0] \in \bar{\Gamma} = \overline{G \times \{0\}}$ and $u|_{\Gamma} = 0$. (The function u is smooth.)

As to the first term on the right-hand side of (83), we proceed as follows:

$$(85) \quad -2 \lim_{h \rightarrow 0} \frac{1}{2h} \int_0^{-2h} u^2\left(b + \frac{1}{2}\lambda, s - h\right) ds = -2 \lim_{h \rightarrow 0} u^2(b + 2h, -3h) \\ = -2u^2(b, 0) = 0.$$

Relations (81)–(85) yield

$$(86) \quad \lim_{\lambda \rightarrow 0} \|\psi_{1,3}\|_{0,P_\lambda}^2 = 0.$$

We can prove in the same way that

$$(87) \quad \lim_{\lambda \rightarrow 0} \|\psi_{1,4}\|_{0,P_\lambda}^2 = 0.$$

Relations (76) and (86), (87) imply

$$(88) \quad \lim_{\lambda \rightarrow 0} \|\psi_1\|_{0,P_\lambda}^2 = 0.$$

The proof of Lemma 2 in the case $N = 2$ is complete.

D2.2. $N = 3$. Now we shall consider the situation $i < N$ in the case $N = 3$, this means the cases $i = 1$ and $i = 2$.

For greater simplicity we assume that $G \subset \overline{G} \subset \Delta$ is a simply connected domain. Further we assume that the boundary ∂G consists of a finite number of smooth arcs; this assumption is sufficient for applications. In the case $N = 3$ the sets $U_\lambda(G), \dots, Z_\lambda(G)$ and $U_\lambda(\Gamma), \dots, Z_\lambda(\Gamma)$ are defined by relations (4) and (5), respectively. Domains (5) are three-dimensional cylinders with bases parallel to the coordinate plane (x_1, x_2) and with the lateral area of a cylinder formed by straight-lines parallel to the axis x_3 . The projections of both bases of the cylinder $M_\lambda(\Gamma)$ ($M = U, \dots, Z$) are identical with the two-dimensional domain $M_\lambda(G)$ which lies in the coordinate plane (x_1, x_2) .

Let $M_\lambda^1(\Gamma)$ and $M_\lambda^2(\Gamma)$ be the lower and upper bases of the cylinder $M_\lambda(\Gamma)$, respectively, and let $M_\lambda^3(\Gamma) = \partial M_\lambda(\Gamma) \setminus (M_\lambda^1(\Gamma) \cup M_\lambda^2(\Gamma))$. Let $X \in \partial M_\lambda(\Gamma)$ be an arbitrary point except for the points at which $\partial M_\lambda(\Gamma)$ is not smooth. Let $n(X)$ be the unit outer normal to $\partial M_\lambda(\Gamma)$ at the point X . Then we have

$$(89) \quad X \in M_\lambda^1(\Gamma) \Rightarrow n(X) = (0, 0, -1),$$

$$(90) \quad X \in M_\lambda^2(\Gamma) \Rightarrow n(X) = (0, 0, 1),$$

$$(91) \quad X \in M_\lambda^3(\Gamma) \Rightarrow n(X) = (n_1(X), n_2(X), 0), \quad n_1^2(X) + n_2^2(X) = 1.$$

Now we shall compute the functions $\psi_i(X)$ given by relation (32) for $i = 1$ and $i = 2$. By the Green-Gauss-Ostrogradskij theorem and relations (89)–(91) we have

$$(92) \quad \begin{aligned} \psi_i(X) &= \int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_i}(Y) dY \\ &\quad - \int_{U_\lambda^3(\Gamma)} \omega_h(X - Y) u_\lambda(Y) n_i(Y) d\sigma_Y \\ &\quad + \int_{W_\lambda^3(\Gamma)} \omega_h(X - Y) u_\lambda(Y) n_i(Y) d\sigma_Y. \end{aligned}$$

The last two integrals appearing on the right-hand side of (92) are surface integrals of the first kind. Similarly to Section D1 the first integral on the right-hand side of (92) tends to zero in $L_2(P_\lambda)$; this means

$$(93) \quad \lim_{\lambda \rightarrow 0} \int_{P_\lambda} \left(\int_{Q_\lambda} \omega_h(X - Y) \frac{\partial u_\lambda}{\partial y_1}(Y) dY \right)^2 dX = 0,$$

where (see Fig. 3)

$$P_\lambda = V_\lambda(\Gamma) \cap \Theta, \quad Q_\lambda = U_\lambda(\Gamma) \setminus \overline{W_\lambda(\Gamma)}.$$

Now we prove that the second integral on the right-hand side of (92) is equal to zero. We have

$$\|X - Y\| = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2 + (x_3 - y_3)^2}.$$

The most inconvenient situation occurs when $x_3 - y_3 = 0$ (in this case the points X, Y lie in a plane which is parallel to the coordinate plane (x_1, x_2)) and when the point X is very close to $V_\lambda^3(\Gamma)$ or lies on $V_\lambda^3(\Gamma)$. As $Y \in U_\lambda^3(\Gamma)$ and $X \in P_\lambda$ we see that in *every* case we have $\|X - Y\| \geq h$. Thus $\omega_h(X - Y) = 0$ and the second integral on the right-hand side of (92) is equal to zero. Hence

$$\psi_i(X) = \int_{W_\lambda^3(\Gamma)} \omega_h(X - Y) u_\lambda(Y) n_i(Y) d\sigma_Y.$$

Using the Schwarz inequality and the fact that $|n_i(Y)| \leq 1$ we obtain

$$(94) \quad \begin{aligned} |\psi_i(X)|^2 &\leq \int_{W_\lambda^3(\Gamma)} \omega_h^2(X - Y) d\sigma_Y \int_{W_\lambda^3(\Gamma)} u_\lambda^2(Y) d\sigma_Y \\ &= \int_{W_\lambda^3(\Gamma)} \omega_h^2(X - Y) d\sigma_Y \int_{W_\lambda^3(\Gamma)} u_\lambda^2(Z) d\sigma_Z. \end{aligned}$$

Integration of inequality (94) over P_λ yields

$$(95) \quad \|\psi_i\|_{0, P_\lambda}^2 \leq \int_{P_\lambda} \left\{ \int_{W_\lambda^3(\Gamma)} \omega_h^2(X - Y) d\sigma_Y \right\} dX \cdot \int_{W_\lambda^3(\Gamma)} u_\lambda^2(Z) d\sigma_Z.$$

First let us consider the second surface integral appearing on the right-hand side of (95). As $W_\lambda^3(\Gamma) = \partial W_\lambda(G) \times (-\frac{1}{2}\lambda, \frac{1}{4}\lambda)$ we can write using definition (7) of the function u_λ and the properties of a line integral of the first kind:

$$(96) \quad \begin{aligned} \int_{W_\lambda^3(\Gamma)} u_\lambda^2(Z) d\sigma_Z &= \int_{\partial W_\lambda(G) \times (-\frac{1}{2}\lambda, \frac{1}{4}\lambda)} u^2(z_1, z_2, z_3 - h) d\sigma_Z \\ &= \sum_{k=1}^m \int_{-\lambda/2}^{\lambda/4} \left\{ \int_{R'_k} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) \sqrt{1 + [a'_{k,\lambda}(z_1)]^2} dz_1 \right\} dz_3, \end{aligned}$$

where $a_{k,\lambda}$ is a local representation of the curve $\partial W_\lambda(G)$. (The situation is similar to that which is sketched in Fig. 4.) We have $|a'_{k,\lambda}(z_1)|^2 < C_k$, where the constant C_k depends only on the Lipschitz constant of the function $\alpha_k(z_1)$ which represents locally the curve ∂G (which does not depend on λ ; however, $\partial W_\lambda(G)$ depends, similarly as $W_\lambda(G)$, on λ —see (4)₃; attention: the symbols $a_{k,\lambda}$ and α_k represent two different parallel arcs). The symbol R'_k ($k = 1, \dots, m$) denotes a segment on the z_1 -axis of the k th local coordinate system (z_1, z_2) . (It would be more precise to use the notation $(z_1^{(k)}, z_2^{(k)})$ instead of the notation (z_1, z_2) ; however, it would have been very cumbersome.) Thus

$$(97) \quad \int_{W_\lambda^3(\Gamma)} u_\lambda^2(Z) d\sigma_Z \leq C_0 \sum_{k=1}^m \int_{-\lambda/2}^{\lambda/4} \left\{ \int_{R'_k} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) dz_1 \right\} dz_3,$$

where $C_0 = \max(C_1, \dots, C_m)$. Let us consider the expression

$$(98) \quad \begin{aligned} \frac{1}{h} \int_{-\lambda/2}^{\lambda/4} \left\{ \int_{R'_k} u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) dz_1 \right\} dz_3 \\ = \int_{R'_k} \left\{ \frac{1}{h} \int_{-2h}^0 u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) dz_3 \right. \\ \left. + \frac{1}{h} \int_0^h u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) dz_3 \right\} dz_1, \end{aligned}$$

where we have used the Fubini theorem. Let $h \rightarrow 0$ in (98). First let us consider the second expression appearing on the right-hand side of (98). To this end, the following fact should be noted: if $h \rightarrow 0$ then $\partial W_\lambda(G)$ tends to ∂G . The corresponding k th part of ∂G is described in the local coordinate system (z_1, z_2) by the function $\alpha_k(z_1)$. Hence (as the limit is of the type $\frac{0}{0}$, we use the l'Hospital rule and the theorem on differentiation of an integral as a function of the upper limit)

$$\lim_{h \rightarrow 0} \frac{1}{h} \int_0^h u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) dz_3 = \lim_{h \rightarrow 0} u^2(z_1, a_{k,\lambda}(z_1), 0) = u^2(z_1, \alpha_k(z_1), 0).$$

As $[z_1, \alpha_k(z_1), 0] \in \partial G$ and $u|_\Gamma = 0$, where $\Gamma = G \times \{0\}$, we have by the assumed continuity of the function u

$$u^2(z_1, \alpha_k(z_1), 0) = 0,$$

which yields

$$(99) \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_0^h u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) dz_3 = 0.$$

As to the first expression appearing on the right-hand side of (98) we have by the same argument

$$(100) \quad \lim_{h \rightarrow 0} \frac{1}{h} \int_{-2h}^0 u^2(z_1, a_{k,\lambda}(z_1), z_3 - h) dz_3 = -2 \lim_{h \rightarrow 0} u^2(z_1, a_{k,\lambda}(z_1), -3h) \\ = -2u^2(z_1, \alpha_k(z_1), 0) = 0.$$

We see from relations (95)–(100) that to complete the proof of Lemma 2 in the case $N = 3$ means to prove

$$(101) \quad \int_{P_\lambda} \left\{ \int_{W_\lambda^3(\Gamma)} \omega_h^2(X - Y) d\sigma_Y \right\} dX = Ch^{-1}.$$

The Fubini theorem and the mean value theorem yield

$$(102) \quad \int_{P_\lambda} \left\{ \int_{W_\lambda^3(\Gamma)} \omega_h^2(X - Y) d\sigma_Y \right\} dX = \int_{W_\lambda^3(\Gamma)} \left\{ \int_{P_\lambda} \omega_h^2(X - Y) dX \right\} d\sigma_Y \\ = \int_{W_\lambda^3(\Gamma)} \left\{ \int_{\substack{\|X-Y\| < h \\ Y \in W_\lambda^3(\Gamma)}} \omega_h^2(X - Y) dX \right\} d\sigma_Y \\ = \int_{W_\lambda^3(\Gamma)} \omega_h^2(X_0 - Y) d\sigma_Y \int_{\substack{\|X-Y\| < h \\ Y \in W_\lambda^3(\Gamma)}} dX.$$

We have

$$(103) \quad \int_{\substack{\|X-Y\| < h \\ Y \in W_\lambda^3(\Gamma)}} dX = \frac{4}{3}\pi h^3.$$

As to the first integral on the right-hand side of (102), the case $X_0 \in W_\lambda^3(\Gamma)$ is most inconvenient. As the inequality $\|X_0 - Y\| < h$ must hold for $\omega_h^2(X_0 - Y) > 0$ we find

$$\int_{W_\lambda^3(\Gamma)} \omega_h^2(X_0 - Y) d\sigma_Y = \int_{\sigma(X_0)} \omega_h^2(X_0 - Y) d\sigma_Y,$$

where

$$\sigma(X_0) \subset W_\lambda^3(\Gamma), \quad \text{meas}_2 \sigma(X_0) \leq Ch^2.$$

Hence, taking into account (11) with $N = 3$, we conclude that

$$(104) \quad \int_{W_\lambda^3(\Gamma)} \omega_h^2(X_0 - Y) d\sigma_Y = \omega_h^2(X_0 - Y_0) \int_{\sigma(X_0)} d\sigma_Y \leq Ch^{-4}.$$

Relations (102)–(104) imply the desired result (101). Lemma 2 is completely proved. \square

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