Hsien-Chung Wu

Fuzzy-valued integrals based on a constructive methodology


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FUZZY-VALUED INTEGRALS BASED ON A
CONSTRUCTIVE METHODOLOGY

HSIEN-CHUNG WU, Kaohsiung

(Received May 18, 2005, in revised version July 28, 2005)

Abstract. The procedures for constructing a fuzzy number and a fuzzy-valued function from a family of closed intervals and two families of real-valued functions, respectively, are proposed in this paper. The constructive methodology follows from the form of the well-known “Resolution Identity” (decomposition theorem) in fuzzy sets theory. The fuzzy-valued measure is also proposed by introducing the notion of convergence for a sequence of fuzzy numbers. Under this setting, we develop the fuzzy-valued integral of fuzzy-valued function with respect to fuzzy-valued measure. Finally, we provide a Dominated Convergence Theorem for fuzzy-valued integrals.

Keywords: dominated convergence theorem, fuzzy number, fuzzy-valued function, fuzzy-valued integral, resolution identity

MSC 2000: 28E10, 03E72

1. Introduction

The concept of fuzzy integrals was first introduced by Sugeno [14]. After that, many subsequent formulations for fuzzy integrals have also been developed. Sim and Wang [11] gave a good review in the subject of fuzzy integrals. Some other interesting approaches are the fuzzy measures assuming values in the set of all fuzzy numbers by Klement [4] and Stojaković [12], the integration of fuzzy-valued functions by Klement [5] and Puri & Ralescu [8], and the fuzzy integrals on product spaces by Suárez-Díaz and Suárez-García [13]. In this paper, we are concerned with a more general setting, the fuzzy-valued integrals of fuzzy-valued measurable functions with respect to fuzzy-valued measures.

We propose a constructive methodology to obtain a fuzzy-valued function from two families of real-valued functions based on a well-known “Resolution Identity” in fuzzy sets theory. In order to propose the fuzzy-valued measures, we invoke the
Hausdorff metric which was proposed by Puri and Ralescu [8] to come up with the convergence of a sequence of fuzzy numbers. Under the settings of fuzzy-valued measures and fuzzy-valued functions, we are able to discuss the integrations of fuzzy-valued measurable functions with respect to fuzzy-valued measures.

In Sections 2 and 3, we first propose the methodology for constructing a fuzzy number from a family of closed intervals, and then we extend the methodology to construct a fuzzy-valued function from two families of real-valued functions. In Section 4, we introduce the notion of limit for a sequence of fuzzy numbers by invoking the Hausdorff metric in order to propose the fuzzy-valued measures. In Section 5, we are concerned with the integration of fuzzy-valued measurable function with respect to fuzzy-valued measure, where the fuzzy-valued measurable function is constructed from two families of real-valued measurable functions. In the final Section 6, we derive the main theorem, the Dominated Convergence Theorem for fuzzy-valued integrals.

2. Construction of fuzzy numbers

Let $U$ be a topological vector space. The fuzzy subset $\tilde{a}$ of $U$ is defined by its membership function $\xi_{\tilde{a}} : U \rightarrow [0, 1]$. The $\alpha$-level set of $\tilde{a}$, denoted by $\tilde{a}_\alpha$, is defined by $\tilde{a}_\alpha = \{x \in U : \xi_{\tilde{a}}(x) \geq \alpha\}$ for all $0 < \alpha \leq 1$. The 0-level set $\tilde{a}_0$ is defined as $\tilde{a}_0 = cl(\{x \in U : \xi_{\tilde{a}}(x) > 0\})$. Let $\tilde{a}$ be a fuzzy subset of $U$. We say that $\tilde{a}$ is normal if there exists an $x \in U$ such that $\xi_{\tilde{a}}(x) = 1$, and that $\tilde{a}$ is convex if its membership function $\xi_{\tilde{a}}$ is quasi-concave, i.e., $\xi_{\tilde{a}}(\lambda x + (1-\lambda)y) \geq \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{a}}(y)\}$ for all $\lambda \in [0, 1]$.

We denote by $F(U)$ the set of all fuzzy subsets $\tilde{a}$ of $U$ with membership function $\xi_{\tilde{a}}$ satisfying the following conditions:

(i) $\tilde{a}$ is normal and convex.

(ii) $\xi_{\tilde{a}}$ is upper semicontinuous, i.e., $\{x \in U : \xi_{\tilde{a}}(x) \geq \alpha\}$ is a closed subset of $U$ for all $\alpha \in (0, 1]$.

(iii) The 0-level set $\tilde{a}_0$ is a compact subset of $U$.

Throughout this paper, the universal set $U$ is assumed as the real number system $\mathbb{R}$ which is endowed with the usual topology. The member $\tilde{a}$ in $F(\mathbb{R})$ is then called a fuzzy number. It is not hard to see that if $\tilde{a}$ is a fuzzy number then $\tilde{a}_\alpha$ is a closed interval in $\mathbb{R}$ for $\alpha \in [0, 1]$. In this case, we write $\tilde{a}_\alpha = [\tilde{a}_\alpha^L, \tilde{a}_\alpha^U]$. The following easy consequence will be used frequently in this paper.

**Proposition 2.1.** Let $\tilde{a}$ be a fuzzy number. Then $\tilde{a}_\beta \subseteq \tilde{a}_\alpha$ for $\alpha < \beta$, i.e., $\tilde{a}_\alpha^L \leq \tilde{a}_\beta^L$ and $\tilde{a}_\alpha^U \geq \tilde{a}_\beta^U$ for $\alpha < \beta$. 

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Let \( \tilde{a} \) be a fuzzy number. Then \( \tilde{a} \) is called a nonnegative fuzzy number if \( \xi_{\tilde{a}}(x) = 0 \) for all \( x < 0 \), and called a nonpositive fuzzy number if \( \xi_{\tilde{a}}(x) = 0 \) for all \( x > 0 \). We say that \( \tilde{a} \) is a crisp number with value \( m \) if its membership function is given by

\[
\xi_{\tilde{a}}(r) = \begin{cases} 
1 & \text{if } r = m, \\
0 & \text{otherwise.}
\end{cases}
\]

We also use the notation \( \tilde{1}_{\{m\}} \) to represent the crisp number with value \( m \). It is easy to see that \((\tilde{1}_{\{m\}})^L_\alpha = (\tilde{1}_{\{m\}})^U_\alpha = m\) for all \( \alpha \in [0,1] \). In other words, each real number \( m \) can be regarded as a crisp number \( \tilde{1}_{\{m\}} \).

Let \( \oplus \) be an addition between two fuzzy numbers \( \tilde{a} \) and \( \tilde{b} \). The membership function of \( \tilde{a} \oplus \tilde{b} \) is defined by

\[
\xi_{\tilde{a} \oplus \tilde{b}}(z) = \sup_{x+y=z} \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\}
\]

using the extension principle in Zadeh [16]. Applying the results in Klir and Yuan [3, Chapter 4], we can show the following useful result for further discussions.

**Proposition 2.2.** Let \( \tilde{a} \) and \( \tilde{b} \) be two fuzzy numbers. Then \( \tilde{a} \oplus \tilde{b} \) is also a fuzzy number. Furthermore, we have

\[
(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha].
\]

Let \( \tilde{a} \) be a fuzzy number. We define the membership functions of \( \tilde{a}^+ \) and \( \tilde{a}^- \) as

\[
\xi_{\tilde{a}^+}(r) = \begin{cases} 
\xi_{\tilde{a}}(r) & \text{if } r > 0, \\
1 & \text{if } r = 0 \text{ and } \xi_{\tilde{a}}(r) < 1 \text{ for all } r > 0, \\
\xi_{\tilde{a}}(0) & \text{if } r = 0 \text{ and there exists an } r > 0 \text{ such that } \xi_{\tilde{a}}(r) = 1, \\
0 & \text{otherwise}
\end{cases}
\]

and

\[
\xi_{\tilde{a}^-}(r) = \begin{cases} 
\xi_{\tilde{a}}(r) & \text{if } r < 0, \\
1 & \text{if } r = 0 \text{ and } \xi_{\tilde{a}}(r) < 1 \text{ for all } r < 0, \\
\xi_{\tilde{a}}(0) & \text{if } r = 0 \text{ and there exists an } r < 0 \text{ such that } \xi_{\tilde{a}}(r) = 1, \\
0 & \text{otherwise.}
\end{cases}
\]

From Proposition 2.2, it is not hard to see that

\[(1) \quad \tilde{a} = \tilde{a}^+ \oplus \tilde{a}^-.
\]

We call \( \tilde{a}^+ \) and \( \tilde{a}^- \) the positive part and negative part of \( \tilde{a} \), respectively.

We rephrase the following well-known results for motivating the construction of a fuzzy number from a family of closed intervals.
Proposition 2.3.

(i) (Zadeh [16]) (Resolution Identity) Let \( \tilde{A} \) be a fuzzy set with membership function \( \xi_{\tilde{A}} \) and \( \tilde{A}_\alpha \) be the \( \alpha \)-level set of \( \tilde{A} \) for \( \alpha \in [0,1] \). Then the membership function \( \xi_{\tilde{A}} \) can be expressed as

\[
\xi_{\tilde{A}}(x) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{\tilde{A}_\alpha}(x),
\]

where \( 1_{\tilde{A}_\alpha} \) is the characteristic function of set \( \tilde{A}_\alpha \) (note that the \( \alpha \)-level set \( \tilde{A}_\alpha \) is a usual set).

(ii) (Negoita and Ralescu [6]) Let \( A \) be a set and \( \{A_\alpha : \alpha \in [0,1]\} \) be a family of subsets of \( A \) such that the following conditions are satisfied:

(a) \( A_0 = A \);
(b) \( A_\beta \subseteq A_\alpha \) for \( \alpha < \beta \);
(c) \( A_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha n} \) for \( \alpha_n \uparrow \alpha \).

Then the function \( \xi: A \to [0,1] \) defined by

\[
\xi(x) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{A_\alpha}(x)
\]

has the property that

\[
A_\alpha = \{x \in A : \xi(x) \geq \alpha\} \text{ for all } \alpha \in [0,1].
\]

Let \( \{A_\alpha = [l_\alpha, u_\alpha] : \alpha \in [0,1]\} \) be a family of closed intervals in \( \mathbb{R} \). Then we can induce a fuzzy subset \( \tilde{a} \) of \( \mathbb{R} \) with membership function defined by

\[
\xi_{\tilde{a}}(r) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{A_\alpha}(r)
\]

via the form of Resolution Identity in Proposition 2.3. Note that, in general, this fuzzy subset \( \tilde{a} \) of \( \mathbb{R} \) is not necessarily a fuzzy number. We say that \( \{A_\alpha\} \) is decreasing with respect to \( \alpha \) if \( A_\beta \subseteq A_\alpha \) for \( \alpha < \beta \). Let us further regard \( l_\alpha \) and \( u_\alpha \) as the functions of \( \alpha \) and assume that \( l_\alpha \) and \( u_\alpha \) are left-continuous with respect to \( \alpha \). Therefore if \( \{A_\alpha\} \) is decreasing with respect to \( \alpha \), thus we see that \( \{A_\alpha\} \) is continuously decreasing with respect to \( \alpha \), since \( l_\alpha \) and \( u_\alpha \) are left-continuous with respect to \( \alpha \). It also says that \( A_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha n} \) for \( \alpha_n \uparrow \alpha \). Using routine arguments, we can show the following interesting result.
**Proposition 2.4.** Let \( \{ A_\alpha = [l_\alpha, u_\alpha] : \alpha \in [0, 1] \} \) be a family of closed intervals. Suppose that the following conditions are satisfied:

(i) \( A_1 \neq \emptyset \);

(ii) \( \{ A_\alpha \} \) is decreasing with respect to \( \alpha \);

(iii) \( l_\alpha \) and \( u_\alpha \) are left-continuous with respect to \( \alpha \).

Then \( \{ A_\alpha \} \) induces a fuzzy number \( \tilde{a} \) with \( \tilde{a}_\alpha = A_\alpha \).

Conversely, we also have the following results.

**Proposition 2.5.**

(i) Let \( A_\alpha = \{ x \in \mathbb{R} : \xi(x) \geq \alpha \} \). Then \( \bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha \) for \( \alpha_n \uparrow \alpha \).

(ii) If \( \tilde{a} \) is a fuzzy number then \( \tilde{a}_\alpha^L \uparrow \tilde{a}_\alpha^L \) and \( \tilde{a}_\alpha^U \downarrow \tilde{a}_\alpha^U \) for \( \alpha_n \uparrow \alpha \), i.e., \( \tilde{a}_\alpha^L \) and \( \tilde{a}_\alpha^U \) are left-continuous with respect to \( \alpha \).

Let \( A = [a^L, a^U] \) and \( B = [b^L, b^U] \) be two closed intervals in \( \mathbb{R} \). Then the addition of two closed intervals is denoted and given by

\[
A \oplus_{\text{int}} B \equiv \{ z \in \mathbb{R} : z = x + y \text{ for } x \in A \text{ and } y \in B \} = [a^L + b^L, a^U + b^U].
\]

Let \( A = [l, u] \) be a closed interval in \( \mathbb{R} \). If \( l \geq 0 \) then \( A \) is called a nonnegative closed interval, and if \( u \leq 0 \) then \( A \) is called a nonpositive closed interval. If \( l \leq 0 \) and \( u \geq 0 \) then we let \( A^+ = [0, u] \) and \( A^- = [l, 0] \). We call \( A^+ \) the positive part of \( A \) and \( A^- \) the negative part of \( A \). It is obvious that \( A = A^+ \oplus_{\text{int}} A^- \).

Let the family of closed intervals \( \{ A_\alpha = [l_\alpha, u_\alpha] : \alpha \in [0, 1] \} \) be decreasing with respect to \( \alpha \) and \( A_1 \neq \emptyset \). Then we have \( A_\alpha = A_\alpha^+ \oplus_{\text{int}} A_\alpha^- \) for \( \alpha \in [0, 1] \). Now \( \{ A_\alpha \} \), \( \{ A_\alpha^+ \} \) and \( \{ A_\alpha^- \} \) can induce three respective fuzzy sets \( \tilde{a} \), \( \tilde{b} \) and \( \tilde{c} \) with membership functions defined by

\[
\xi_{\tilde{a}}(r) = \sup_{\alpha \in [0, 1]} \alpha \cdot 1_{A_\alpha}(r),
\]

\[
\xi_{\tilde{b}}(r) = \begin{cases} 
\sup_{\alpha \in [0, 1]} \alpha \cdot 1_{A_\alpha^+}(r) & \text{if } r > 0, \\
1 & \text{if } r = 0 \text{ and } A_1^+ = \emptyset, \\
\sup_{\alpha \in [0, 1]} \alpha \cdot 1_{A_\alpha^+}(0) & \text{if } r = 0 \text{ and } A_1^+ \neq \emptyset, \\
0 & \text{if } r < 0
\end{cases}
\]
and

\[
\xi_{\tilde{c}}(r) = \begin{cases} 
\sup_{\alpha \in [0,1]} \alpha \cdot 1_{A^+_{\alpha}}(r) & \text{if } r < 0, \\
1 & \text{if } r = 0 \text{ and } A^+_1 = \emptyset, \\
\sup_{\alpha \in [0,1]} \alpha \cdot 1_{A^-_{\alpha}}(0) & \text{if } r = 0 \text{ and } A^-_1 \neq \emptyset, \\
0 & \text{if } r > 0.
\end{cases}
\]

Now, for \( r > 0 \), \( r \in A_\alpha \) if and only if \( r \in A^+_\alpha \). Thus \( \xi_{\tilde{a}}+(r) = \xi_{\tilde{b}}(r) \). From the definition of the membership function of \( \tilde{a}^+ \), it is easy to see that \( \xi_{\tilde{a}}+(0) = \xi_{\tilde{b}}(0) \). We conclude that \( \tilde{a}^+ = \tilde{b} \). Similarly, we can conclude that \( \tilde{a}^- = \tilde{c} \). This shows the following result.

**Proposition 2.6.** Let the family of closed intervals \( \{ A_\alpha = [l_\alpha, u_\alpha] : \alpha \in [0,1] \} \) be decreasing with respect to \( \alpha \) and satisfy the conditions in Proposition 2.4. Let \( \tilde{a} \) be a fuzzy number induced by \( \{ A_\alpha \} \). Then \( \tilde{a}^+ \) is a fuzzy number induced by \( \{ A^+_\alpha \} \) and \( \tilde{a}^- \) is a fuzzy number induced by \( \{ A^-_\alpha \} \), where \( \tilde{a} = \tilde{a}^+ \oplus \tilde{a}^- \) and \( A_\alpha = A^+_\alpha \oplus \text{int} \ A^-_\alpha \) for \( \alpha \in [0,1] \).

**Proposition 2.7.** Let the family of closed intervals \( \{ A_\alpha = [l_\alpha, u_\alpha] : \alpha \in [0,1] \} \) and \( \{ \tilde{A}_\alpha = [\tilde{l}_\alpha, \tilde{u}_\alpha] : \alpha \in [0,1] \} \) be decreasing with respect to \( \alpha \) and satisfy the conditions in Proposition 2.4. Suppose that \( \{ A_\alpha \} \) and \( \{ \tilde{A}_\alpha \} \) induce two fuzzy numbers \( \tilde{a} \) and \( \tilde{b} \), respectively, and that \( \{ A_\alpha \oplus \text{int} \ \tilde{A}_\alpha : \alpha \in [0,1] \} \) induces a fuzzy number \( \tilde{c} \). Then \( \tilde{c} = \tilde{a} \oplus \tilde{b} \).

**Proof.** Let \( \tilde{c}_1 \) be induced by \( \{ \hat{A}_\alpha \equiv A_\alpha \oplus \text{int} \ \tilde{A}_\alpha \} \) and \( \tilde{c}_2 = \tilde{a} \oplus \tilde{b} \). By definition, the membership functions of \( \tilde{c}_1 \) and \( \tilde{c}_2 \) are given by

\[
\xi_{\tilde{c}_1}(r) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{\hat{A}_\alpha}(r)
\]

and

\[
\xi_{\tilde{c}_2}(r) = \sup_{r = r_1 + r_2} \min \left\{ \sup_{\alpha \in [0,1]} \alpha \cdot 1_{A_\alpha}(r_1), \sup_{\alpha \in [0,1]} \alpha \cdot 1_{\tilde{A}_\alpha}(r_2) \right\}.
\]

It is not hard to show that \( \xi_{\tilde{c}_1}(r) = \xi_{\tilde{c}_2}(r) \) for all \( r \).
3. Construction of fuzzy-valued functions

In this section, we shall discuss the construction of fuzzy-valued functions from two families of functions.

Let $\tilde{f}$ be a function defined on $X$ by $\tilde{f} : X \rightarrow \mathcal{F}(\mathbb{R})$. Then we say that $\tilde{f}$ is a fuzzy-valued function. We also denote by $\tilde{f}^L_\alpha(x) = (\tilde{f}(x))^L_\alpha$ and $\tilde{f}^U_\alpha(x) = (\tilde{f}(x))^U_\alpha$ for $x \in X$. Therefore the fuzzy-valued function $\tilde{f}$ induces the real-valued functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ for $\alpha \in [0, 1]$.

Let $\mathcal{L}(x) = \{l_\alpha(x) : \alpha \in [0, 1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x) : \alpha \in [0, 1]\}$ be two families of functions, where $l_\alpha$ and $u_\alpha$ are real-valued functions defined on $X$ for $\alpha \in [0, 1]$. Let

$$B_\alpha(x) = [\min\{l_\alpha(x), u_\alpha(x)\}, \max\{l_\alpha(x), u_\alpha(x)\}]$$

for $\alpha \in [0, 1]$. Then we can induce a function $\tilde{f}$ which assumes values in the family of all fuzzy subsets of $\mathbb{R}$; that is to say, for any fixed $x \in X$, $\tilde{f}(x)$ is a fuzzy subset of $\mathbb{R}$ with membership function defined by

$$\xi_{\tilde{f}(x)}(r) = \sup_{\alpha \in [0, 1]} \alpha \cdot 1_{B_\alpha(x)}(r)$$

via the form of Resolution Identity in Proposition 2.3. In the sequel, we are going to construct a subset of $X$ such that $\tilde{f}(x)$ is a fuzzy number for each $x$ in this subset of $X$.

For $\alpha < \beta$ and $\alpha, \beta \in [0, 1]$, we adopt the following notations

$$E_{ll,\alpha,\beta} = \{x \in X : l_\alpha(x) \leq l_\beta(x)\},$$

$$E_{uu,\alpha,\beta} = \{x \in X : u_\beta(x) \leq u_\alpha(x)\},$$

$$E_{lu,\alpha} = \{x \in X : l_\alpha(x) \leq u_\alpha(x)\}.$$ We assume $E_{lu,1} = \{x \in X : l_1(x) \leq u_1(x)\} \neq \emptyset$. We also let

$$\begin{align*}
E_{ll} &= \bigcap_{0 \leq \alpha < \beta \leq 1} E_{ll,\alpha,\beta}, \\
E_{uu} &= \bigcap_{0 \leq \alpha < \beta \leq 1} E_{uu,\alpha,\beta}, \\
E_{lu} &= \bigcap_{\alpha \in [0, 1]} E_{lu,\alpha}
\end{align*}$$

and

$$E_{\mathcal{L}\mathcal{U}} = E_{ll} \cap E_{uu} \cap E_{lu}.$$ Then, for each $x \in E_{\mathcal{L}\mathcal{U}}$, we have a family of decreasing closed intervals $\{A_\alpha(x) = [l_\alpha(x), u_\alpha(x)] : \alpha \in [0, 1]\}$ induced from $\{\mathcal{L}(x), \mathcal{U}(x)\}$. Then the membership function of $\tilde{f}(x)$, for $x \in E_{\mathcal{L}\mathcal{U}}$, is given by

$$\xi_{\tilde{f}(x)}(r) = \sup_{\alpha \in [0, 1]} \alpha \cdot 1_{A_\alpha(x)}(r)$$
from (2). Let us also adopt the following notations

\[ F_{\alpha;A}^L = \{ x \in X : l_{\alpha_n}(x) \to l_{\alpha}(x) \text{ for } \alpha_n \uparrow \alpha \} , \]
\[ F_{\alpha;A}^U = \{ x \in X : u_{\alpha_n}(x) \to u_{\alpha}(x) \text{ for } \alpha_n \uparrow \alpha \} . \]

Let \( F_{\alpha;A} = F_{\alpha;A}^L \cap F_{\alpha;A}^U \) and \( G_{\alpha;A} = F_{\alpha;A} \cap E_{\mathcal{L} \mathcal{U}} \). Then, for each \( x \in G_{\alpha;A} \), we see that \( A_\alpha(x) = \bigcap_{n=1}^{\infty} A_{\alpha_n}(x) \) for \( \alpha_n \uparrow \alpha \). Let \( F_A = \bigcap_{\alpha \in [0,1]} F_{\alpha;A} \) and \( G_A = \bigcap_{\alpha \in [0,1]} G_{\alpha;A} \).

Then we see that \( G_A = F_A \cap E_{\mathcal{L} \mathcal{U}} \). Now, from Proposition 2.4, \( \tilde{f}(x) \) is a fuzzy number for \( x \in G_A \), i.e., \( \tilde{f} \) is a fuzzy-valued function defined on \( G_A \) and \( \tilde{f}_\alpha(x) = A_\alpha(x) = [l_{\alpha}(x), u_{\alpha}(x)] \) for \( x \in G_A \) and \( \alpha \in [0,1] \). We call \( \tilde{f} \) the pseudo-fuzzy-valued function induced by \( \{ \mathcal{L}, \mathcal{U} \} \). The reason why we call \( \tilde{f} \) the pseudo-fuzzy-valued function is that \( \tilde{f}(x) \) is just a fuzzy subset of \( \mathbb{R} \), not a fuzzy number, for \( x \in X \setminus G_A \). The following proposition is useful for defining the fuzzy-valued integrals.

**Proposition 3.1.**

(i) If there exists a countable dense subset \( \{ \alpha_n \} \) of \([0,1]\) such that \( E_{lu,\alpha_n} \subseteq F_A \) for all \( n \), then \( E_{lu} \) can be expressed as countable intersections.

(ii) If there exists a countable dense subset \( \{ \beta_n \} \) of \([0,1]\), such that \( E_{ll,\alpha,\beta_n} \subseteq F_A \) and \( E_{uu,\alpha,\beta_n} \subseteq F_A \) for all \( \alpha \in [0,\beta_n] \) and all \( n \), then \( E_{ll} \) and \( E_{uu} \) can be expressed as countable intersections.

**Proof.** It will be enough to just prove case \( E_{ll} \). We now have

\[ E_{ll} = \bigcap_{\{ \beta : 0 \leq \beta \leq 1 \}} \bigcap_{\{ \alpha : 0 \leq \alpha < \beta \leq 1 \}} E_{ll,\alpha,\beta} \equiv \bigcap_{\{ \beta : 0 \leq \beta \leq 1 \}} H_\beta \subseteq \bigcap_{n=1}^{\infty} H_{\beta_n}, \]

where \( H_\beta = \bigcap_{\{ \alpha : 0 \leq \alpha < \beta \leq 1 \}} E_{ll,\alpha,\beta} \). Given any \( \beta \in [0,1] \), there exists a subsequence \( \{ \beta_{n_k} \} \subseteq \{ \beta_n \} \) such that \( \beta_{n_k} \uparrow \beta \). If \( \alpha < \beta \) then we have \( l_\alpha(x) \leq l_{\beta_{n_k}}(x) \) for some \( K > 0 \), \( \alpha < \beta_{n_k} \) and \( k > K \). Therefore, we have \( l_\alpha(x) \leq l_\beta(x) \) for \( \alpha < \beta \) by taking limit, i.e, \( x \in \bigcap_{0 \leq \beta \leq 1} H_\beta \). Thus \( E_{ll} = \bigcap_{n=1}^{\infty} H_{\beta_n} \). For fixed \( \beta_n \), let \( \{ \alpha_m^{(n)} \}_{m=1}^{\infty} \) be any countable dense subset of \([0,\beta_n]\). Similarly, we can show that

\[ H_{\beta_n} = \bigcap_{\{ \alpha : 0 \leq \alpha < \beta_n \leq 1 \}} E_{ll,\alpha,\beta_n} = \bigcap_{m=1}^{\infty} \bigcap_{\{ \alpha_m^{(n)} : 0 < \alpha_m^{(n)} < \beta_n \}} E_{ll,\alpha_m^{(n)},\beta_n}. \]

This completes the proof. \( \square \)
Let $\tilde{f}$ and $\tilde{g}$ be two pseudo-fuzzy-valued functions induced by $\{L, U\}$ and $\{\hat{L}, \hat{U}\}$, respectively. At the same time, we also have two corresponding families of decreasing closed intervals

$$\{A_\alpha(x) = [l_\alpha(x), u_\alpha(x)]: \alpha \in [0, 1] \text{ and } x \in E_{\mathcal{L}U}\}$$

and

$$\{A_\alpha(x) = [\bar{l}_\alpha(x), \bar{u}_\alpha(x)]: \alpha \in [0, 1] \text{ and } x \in E_{\mathcal{L}\overline{U}}\}$$

from $\{\mathcal{L}, \mathcal{U}\}$ and $\{\mathcal{L}, \mathcal{U}\}$, respectively. Let

$$\hat{\mathcal{L}}(x) \equiv \{\hat{l}_\alpha(x) = l_\alpha(x) + \bar{l}_\alpha(x): \alpha \in [0, 1]\}$$

and

$$\hat{\mathcal{U}}(x) \equiv \{\hat{u}_\alpha(x) = u_\alpha(x) + \bar{u}_\alpha(x): \alpha \in [0, 1]\}.$$

We denote by $\hat{\mathcal{L}} = \mathcal{L} \oplus_{\text{fct}} \hat{\mathcal{L}}$ and $\hat{\mathcal{U}} = \mathcal{U} \oplus_{\text{fct}} \hat{\mathcal{U}}$. Then we also have a family of decreasing closed intervals

$$\{\hat{A}_\alpha(x) = [\hat{l}_\alpha(x), \hat{u}_\alpha(x)]: \alpha \in [0, 1] \text{ and } x \in E_{\hat{\mathcal{L}}\hat{U}}\}$$

from $\{\hat{\mathcal{L}}, \hat{\mathcal{U}}\}$. Therefore $\{\hat{\mathcal{L}}, \hat{\mathcal{U}}\}$ can induce a pseudo-fuzzy-valued function $\hat{h}$ such that $\hat{h}$ is a fuzzy-valued function on $G_{\hat{\mathcal{L}}}$.

Proposition 3.2.

(i) Let $\tilde{f}$ be a fuzzy-valued function defined on $X$. We consider the families $\mathcal{L}(x) = \{\tilde{f}_\alpha^L(x): \alpha \in [0, 1]\}$ and $\mathcal{U}(x) = \{\tilde{f}_\alpha^U(x): \alpha \in [0, 1]\}$. Then $\{\mathcal{L}, \mathcal{U}\}$ induces $\tilde{f}$ and $E_{\mathcal{L}\mathcal{U}} = F_A = X$, i.e., $G_A = X$.

(ii) Let $\tilde{f}$ and $\tilde{g}$ be two fuzzy-valued functions defined on the same set $X$. Let $\mathcal{L}(x) = \{\tilde{f}_\alpha^L(x)\}$, $\hat{\mathcal{L}}(x) = \{\hat{g}_\alpha^L(x)\}$, $\mathcal{U}(x) = \{\tilde{f}_\alpha^U(x)\}$ and $\hat{\mathcal{U}}(x) = \{\hat{g}_\alpha^U(x)\}$. Suppose that $\tilde{f}_0$ and $\tilde{g}_0$ are induced by $\{\mathcal{L}, \mathcal{U}\}$ and $\{\hat{\mathcal{L}}, \hat{\mathcal{U}}\}$, respectively, and $\hat{h}$ is
induced by \( \hat{\mathcal{L}} = \mathcal{L} \oplus \text{fct} \hat{\mathcal{L}}, \hat{\mathcal{U}} = \mathcal{U} \oplus \text{fct} \hat{\mathcal{U}} \). Then \( \tilde{h} \approx \tilde{f}_0 \oplus \tilde{g}_0, \tilde{f}_0 = \tilde{f}, \tilde{g}_0 = \tilde{g} \) and\( \tilde{h}(x) = \tilde{f}(x) \oplus \tilde{g}(x) \) for all \( x \in X \), i.e., \( \tilde{h}_\alpha(x) = \tilde{f}_\alpha(x) \oplus \text{int} \tilde{g}_\alpha(x) \) for all \( x \in X \).

**Definition 3.1.** Let \( \mathcal{L}(x) = \{ l_\alpha(x) : \alpha \in [0, 1] \} \) and \( \mathcal{U}(x) = \{ u_\alpha(x) : \alpha \in [0, 1] \} \) be two families of real-valued functions defined on \( X \). We say that \( \{ \mathcal{L}, \mathcal{U} \} \) is a standard family if \( E_{lu, \alpha} \subseteq F_A, E_{ll, \alpha, \beta} \subseteq F_A \) and \( E_{uu, \alpha, \beta} \subseteq F_A \) for all \( \alpha < \beta \) and \( \alpha, \beta \in [0, 1] \).

**Proposition 3.3.** Let \( \tilde{f} \) be a pseudo-fuzzy-valued function induced by a standard family \( \{ \mathcal{L}, \mathcal{U} \} \). Then \( G_A = E_{\mathcal{L} \mathcal{U}} \), and \( G_A \) can be expressed as countable intersections.

**Proof.** By the definition of standard family, we see that \( E_{\mathcal{L} \mathcal{U}} \subseteq F_A \). This means that \( G_A = E_{\mathcal{L} \mathcal{U}} \) since \( G_A = E_{\mathcal{L} \mathcal{U}} \cap F_A \). The countable intersections of \( G_A \) follow from Proposition 3.1 immediately. \( \square \)

### 4. The fuzzy-valued measures

In order to define the fuzzy-valued measure, we need to consider the limit of a sequence of fuzzy numbers. Thus we first introduce a metric on the set of all fuzzy numbers \( F(\mathbb{R}) \).

Let \( A \subseteq \mathbb{R}^n \) and \( B \subseteq \mathbb{R}^n \). The **Hausdorff metric** is defined as

\[
 d_H(A, B) = \max \left\{ \sup_{a \in A} \inf_{b \in B} \| a - b \|, \sup_{b \in B} \inf_{a \in A} \| a - b \| \right\}.
\]

According to Puri and Ralescu [8], we define the metric \( d_\mathcal{F} \) in \( F(\mathbb{R}) \) as

\[
 d_\mathcal{F}(\tilde{a}, \tilde{b}) = \sup_{\alpha \in [0, 1]} d_H(\tilde{a}_\alpha, \tilde{b}_\alpha),
\]

since \( \tilde{a}_\alpha \) and \( \tilde{b}_\alpha \) are bounded closed intervals for all \( \alpha \in [0, 1] \). We can see that \( (F(\mathbb{R}), d_\mathcal{F}) \) is a complete metric space. The following result is obvious.

**Proposition 4.1.** Let \( \tilde{a} \) and \( \tilde{b} \) be two fuzzy numbers. Then we have

\[
 d_H(\tilde{a}_\alpha, \tilde{b}_\alpha) = \max \{ |\tilde{a}_\alpha^L - \tilde{b}_\alpha^L|, |\tilde{a}_\alpha^U - \tilde{b}_\alpha^U| \}.
\]

**Definition 4.1.** Let \( \{ \tilde{a}_n \} \) be a sequence of fuzzy numbers. Then \( \{ \tilde{a}_n \} \) is said to converge if there is a fuzzy number \( \tilde{a} \) with the following property: \( \forall \varepsilon > 0, \exists N > 0 \) such that \( d_\mathcal{F}(\tilde{a}_n, \tilde{a}) < \varepsilon \) for \( n > N \). In this case, we also say that the sequence \( \{ \tilde{a}_n \} \) converges to \( \tilde{a} \), and it is denoted by

\[
 \lim_{n \to \infty} \tilde{a}_n = \tilde{a}.
\]

If there is no such \( \tilde{a} \), the sequence \( \{ \tilde{a}_n \} \) is said to diverge.
**Proposition 4.2.** Let \( \{\tilde{a}_n\} \) be a sequence of fuzzy numbers. If the limit of the sequence \( \{\tilde{a}_n\} \) exists, then it is unique and
\[
(\lim_{n \to \infty} \tilde{a}_n)_\alpha = \left[ \lim_{n \to \infty} (\tilde{a}_n)_\alpha^L, \lim_{n \to \infty} (\tilde{a}_n)_\alpha^U \right]
\]
for all \( \alpha \in [0, 1] \). Moreover, \( \{(\tilde{a}_n)_\alpha^L\} \) and \( \{(\tilde{a}_n)_\alpha^U\} \) converge uniformly with respect to \( \alpha \) on \([0, 1]\).

**Proof.** The result follows from Proposition 4.1 immediately. \(\square\)

**Definition 4.2.** Let \( \{\tilde{a}_n\} \) be a sequence of fuzzy numbers. Let \( \tilde{s}_n = \bigoplus_{i=1}^n \tilde{a}_i \) be the partial sum of the sequence \( \{\tilde{a}_n\} \). If the limit of the sequence \( \{\tilde{s}_n\} \) exists, then the infinite (fuzzy) sum of the sequence \( \{\tilde{a}_n\} \) is said to converge, and we also write
\[
\bigoplus_{n=1}^\infty \tilde{a}_n = \lim_{n \to \infty} \tilde{s}_n = \lim_{n \to \infty} \bigoplus_{i=1}^n \tilde{a}_i,
\]
otherwise the infinite (fuzzy) sum of the sequence \( \{\tilde{a}_n\} \) is said to diverge.

**Proposition 4.3.** If \( \{\tilde{a}_n\} \) is a sequence of fuzzy numbers, and the infinite sum of the sequence \( \{\tilde{a}_n\} \) exists, then we have
\[
\left( \bigoplus_{n=1}^\infty \tilde{a}_n \right)_\alpha = \left[ \sum_{n=1}^\infty (\tilde{a}_n)_\alpha^L, \sum_{n=1}^\infty (\tilde{a}_n)_\alpha^U \right].
\]

**Proof.** The result follows from Propositions 4.2 and 2.2 immediately. \(\square\)

We denote by \( \tilde{0} \) a crisp number with value 0. Then we are in a position to consider the fuzzy-valued measures.

**Definition 4.3.** By a fuzzy-valued measure \( \tilde{\mu} \) on a measurable space \((X, \mathcal{M})\), we mean a nonnegative fuzzy-valued set function defined on all sets in \( \mathcal{M} \) which satisfies the following two conditions:

(i) \( \tilde{\mu}(\emptyset) = \tilde{0} \);

(ii) \( \tilde{\mu}\left( \bigcup_{i=1}^\infty E_i \right) = \bigoplus_{i=1}^\infty \tilde{\mu}(E_i) \) for any sequence \( \{E_i\} \) of disjoint measurable sets.

Let \( \tilde{\mu} \) be a fuzzy-valued measure on a measurable space \((X, \mathcal{M})\). Then \( \tilde{\mu}(E) \) is a fuzzy number for \( E \in \mathcal{M} \). Therefore, we can define the set functions \( \tilde{\mu}_\alpha^L(E) = (\tilde{\mu}(E))_\alpha^L \) and \( \tilde{\mu}_\alpha^U(E) = (\tilde{\mu}(E))_\alpha^U \) on \((X, \mathcal{M})\) for each \( \alpha \in [0, 1] \). Then, from Proposition 4.3, we see that if \( \tilde{\mu} \) is a fuzzy-valued measure on a measurable space \((X, \mathcal{M})\), then \( \tilde{\mu}_\alpha^L \) and \( \tilde{\mu}_\alpha^U \) are the traditional measures on the same measurable space \((X, \mathcal{M})\).

Let \( \mu_1 \) and \( \mu_2 \) be two measures on the same measurable space \((X, \mathcal{M})\). Recall that \( \mu_1 \) is absolutely continuous with respect to \( \mu_2 \), denoted as \( \mu_1 \ll \mu_2 \), if \( \mu_2(E) = 0 \) implies \( \mu_1(E) = 0 \) for each set \( E \).
Definition 4.4. Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. Then $\tilde{\mu}_L^\alpha$ and $\tilde{\mu}_U^\alpha$ are the traditional measures on $(X, \mathcal{M})$ for all $\alpha \in [0, 1]$. We say that $\tilde{\mu}$ is a canonical fuzzy-valued measure if the conditions $\tilde{\mu}_L^\alpha \ll \tilde{\mu}_L^\beta$, $\tilde{\mu}_U^\alpha \ll \tilde{\mu}_U^\beta$ and $\tilde{\mu}_L^\alpha \ll \tilde{\mu}_L^\beta$ are satisfied for all $\alpha < \beta$ and $\alpha, \beta \in [0, 1]$.

Let $\nu$ and $\mu$ be two measures on the same measurable space $(X, \mathcal{M})$. Recall that $\mu$ and $\nu$ are equivalent measures if $\mu \ll \nu$ and $\nu \ll \mu$. Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. We denote by $\Xi = \{\tilde{\mu}_L^\alpha, \tilde{\mu}_U^\alpha : \alpha \in [0, 1]\}$ a family of measures which are all on the same measurable space $(X, \mathcal{M})$.

Proposition 4.4. If $\tilde{\mu}$ is a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$, then all measures in $\Xi$ are equivalent.

Proof. The result follows from Proposition 2.1 and the definition of canonical fuzzy-valued measure immediately.$\square$

5. The fuzzy-valued integrals

In this section, we shall discuss the fuzzy-valued integral of fuzzy-valued measurable function which is constructed from two families of measurable functions.

Definition 5.1. Let $(X, \mathcal{M})$ be a measurable space. Let $\mathcal{L}(x) = \{l_\alpha(x) : \alpha \in [0, 1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x) : \alpha \in [0, 1]\}$ be two families of real-valued functions defined on $X$. Let $\tilde{f}$ be a pseudo-fuzzy-valued function induced by $\{\mathcal{L}, \mathcal{U}\}$. If $l_\alpha$ and $u_\alpha$ are measurable functions for all $\alpha \in [0, 1]$, then we say that $\tilde{f}$ is measurable.

We denote by $\mathcal{F}$ the family of all fuzzy subsets of $\mathbb{R}$. Recall that $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers. Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X, \mathcal{M})$ and $\mu$ be a traditional measure on a measurable space $(X, \mathcal{M})$. We consider a function $\tilde{f} : X \to \mathcal{F}$ which assumes values in $\mathcal{F}$, not in $\mathcal{F}(\mathbb{R})$. Then we say that $\tilde{f}$ is a fuzzy-valued function a.e. $[\mu]$ if the set $Z = \{x \in X : \tilde{f}(x) \in \mathcal{F}(\mathbb{R})\}$ satisfies $\mu(Z^c) = 0$, and that $\tilde{f}$ is a fuzzy-valued function a.e. $[\tilde{\mu}]$ if $\tilde{\mu}(Z^c) = 0$, i.e., $\tilde{\mu}_L^\alpha(Z^c) = 0 = \tilde{\mu}_U^\alpha(Z^c)$ for all $\alpha \in [0, 1]$.

Definition 5.2. Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. Let $\mathcal{L}(x) = \{l_\alpha(x) : \alpha \in [0, 1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x) : \alpha \in [0, 1]\}$ be two families of real-valued measurable functions defined on $X$. Then $\{\mathcal{L}, \mathcal{U}\}$ is said to be a canonical family with respect to $\tilde{\mu}$ if $\{\mathcal{L}, \mathcal{U}\}$ is a standard family and there exists a measure $\mu \in \Xi$ such that the following conditions are satisfied:

(i) $l_\alpha \leq l_\beta$ a.e. $[\mu]$, $u_\beta \leq u_\alpha$ a.e. $[\mu]$ and $l_\alpha \leq u_\alpha$ a.e. $[\mu]$ for all $\alpha < \beta$ and $\alpha, \beta \in [0, 1]$.

(ii) $l_{\alpha_n} \uparrow l_\alpha$ a.e. $[\mu]$ and $u_{\alpha_n} \downarrow u_\alpha$ a.e. $[\mu]$ for $\alpha_n \uparrow \alpha$. 

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Proposition 5.1. Let \( \mathcal{L}(x) = \{ l_\alpha(x) : \alpha \in [0, 1] \} \) and \( \mathcal{U}(x) = \{ u_\alpha(x) : \alpha \in [0, 1] \} \) be two families of real-valued measurable functions defined on \( X \). Let \( \tilde{f} \) be a pseudo-fuzzy-valued measurable function induced by \( \{ \mathcal{L}, \mathcal{U} \} \). Then the following statements hold true.

(i) Suppose that \( \{ \mathcal{L}, \mathcal{U} \} \) is a standard family. If \( \mu \) is a measure on a measurable space \( (X, \mathcal{M}) \) such that conditions (i) and (ii) in Definition 5.2 are satisfied, then \( \mu(G_\alpha^\alpha) = 0 \). That is to say, \( \tilde{f} \) is a fuzzy-valued measurable function a.e. \([\mu]\).

(ii) Suppose that \( \{ \mathcal{L}, \mathcal{U} \} \) is a canonical family with respect to \( \tilde{\mu} \), where \( \tilde{\mu} \) is a canonical fuzzy-valued measure on a measurable space \( (X, \mathcal{M}) \). Then \( \tilde{\mu}(G_\alpha^\alpha) = 0 \), i.e., \( \tilde{f} \) is a fuzzy-valued measurable function a.e. \([\tilde{\mu}]\).

Proof. From condition (i) in Definition 5.2, Eqs. (4) and (5) in the proof of Proposition 3.1, we see that

\[
0 \leq \mu(E_{\ell}^c) \leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \mu(E_{c,\alpha}^c(n), \beta_n) = 0.
\]

Similarly, we also have \( \mu(E_{uu}^c) = 0 = \mu(E_{lv}^c) \). Thus we conclude that \( \mu(E_{\ell uv}^c) = 0 \).

From Proposition 3.3, we also see that \( \mu(G_\alpha^\alpha) = 0 \). Since \( \tilde{f}(x) \in \mathcal{F}(\mathbb{R}) \) for \( x \in G_A \), \( \tilde{f} \) is a fuzzy-valued measurable function a.e. \([\tilde{\mu}]\). Now, if \( \mu \in \Xi \), then, from Proposition 4.4, we have \( \tilde{\mu}_\alpha^L(G_\alpha^\alpha) = 0 = \tilde{\mu}_\alpha^U(G_\alpha^\alpha) \) for all \( \alpha \in [0, 1] \). It follows that \( \tilde{\mu}(G_\alpha^\alpha) = 0 \). This completes the proof. \( \square \)

Definition 5.3. Let \( \tilde{\mu} \) be a fuzzy-valued measure on a measurable space \( (X, \mathcal{M}) \). Let \( \mathcal{L}(x) = \{ l_\alpha(x) : \alpha \in [0, 1] \} \) and \( \mathcal{U}(x) = \{ u_\alpha(x) : \alpha \in [0, 1] \} \) be two families of real-valued functions defined on \( X \). We say that \( \{ \mathcal{L}, \mathcal{U} \} \) is nonnegative (resp. nonpositive) a.e. \([\tilde{\mu}]\) if \( l_\alpha \geq 0 \) (resp. \( \leq 0 \)) a.e. \([\tilde{\mu}_\alpha^U]\) and \( u_\alpha \geq 0 \) (resp. \( \leq 0 \)) a.e. \([\tilde{\mu}_\alpha^L]\).

Definition 5.4. Let \( \tilde{\mu} \) be a canonical fuzzy-valued measure on a measurable space \( (X, \mathcal{M}) \). Let \( \mathcal{L}(x) = \{ l_\alpha(x) : \alpha \in [0, 1] \} \) and \( \mathcal{U}(x) = \{ u_\alpha(x) : \alpha \in [0, 1] \} \) be two families of real-valued measurable functions defined on \( X \), and \( \{ \mathcal{L}, \mathcal{U} \} \) be a canonical family with respect to \( \tilde{\mu} \). Let \( \tilde{f} \) be a pseudo-fuzzy-valued measurable function induced by \( \{ \mathcal{L}, \mathcal{U} \} \). Suppose that \( l_\alpha \in L^1(\tilde{\mu}_\alpha^L) \) (i.e., Lebesgue integrable with respect to \( \tilde{\mu}_\alpha^L \)) and \( u_\alpha \in L^1(\tilde{\mu}_\alpha^U) \) (i.e., Lebesgue integrable with respect to \( \tilde{\mu}_\alpha^U \)) for all \( \alpha \in [0, 1] \). Then we consider the following two cases.

(i) If \( \{ \mathcal{L}, \mathcal{U} \} \) is nonnegative a.e. \([\tilde{\mu}]\), then, from Proposition 4.4 and condition (i) in Definition 5.2, we have \( \int_E l_\alpha d\tilde{\mu}_\alpha^L \leq \int_E u_\alpha d\tilde{\mu}_\alpha^L \leq \int_E u_\alpha d\tilde{\mu}_\alpha^U \) since \( l_\alpha \leq u_\alpha \) a.e. \([\tilde{\mu}_\alpha^L]\) and \( \tilde{\mu}_\alpha^L \leq \tilde{\mu}_\alpha^U \). Therefore we consider the closed interval \( C_\alpha \) as

\[
C_\alpha = \left[ \int_E l_\alpha d\tilde{\mu}_\alpha^L, \int_E u_\alpha d\tilde{\mu}_\alpha^U \right]
\]

for \( \alpha \in [0, 1] \).
(ii) If \{L, U\} is nonpositive a.e. [\[\tilde{\mu}\]] then, similarly, we consider the closed interval \(C_\alpha\) as

\[
C_\alpha = \left[ \int_E l_\alpha \, d\tilde{\mu}_\alpha^U, \int_E u_\alpha \, d\tilde{\mu}_\alpha^L \right]
\]

for \(\alpha \in [0, 1]\). The membership function of the fuzzy-valued integral \(\int_E \tilde{f} \, d\tilde{\mu}\) is defined by

\[
\xi_{\int_E \tilde{f} \, d\tilde{\mu}}(r) = \sup_{\alpha \in [0, 1]} \alpha \cdot 1_{C_\alpha}(r)
\]

via the form of Resolution Identity in Proposition 2.3, and we say that \(\tilde{f}\) is integrable with respect to \(\tilde{\mu}\) on \(E\).

Now we want to explain that Definition 5.4 is well-defined. It will be enough to just justify the nonnegative case. Let \(\tilde{f}\) be a pseudo-fuzzy-valued measurable function induced by a canonical family \{\(L, U\)\}. Suppose that \(\tilde{f}\) is also induced by another canonical family \{\(L', U'\)\}. Then we can induce decreasing closed intervals \(\{A_\alpha(x): \alpha \in [0, 1]\}\) from \{\(L, U\)\} for \(x \in E_{L'U'}\) and decreasing closed intervals \(\{A'_\alpha(x): \alpha \in [0, 1]\}\) from \{\(L', U'\)\} for \(x \in E_{L'U'}\). Since \(\{A_\alpha(x): \alpha \in [0, 1]\}\) and \(\{A'_\alpha(x): \alpha \in [0, 1]\}\) induce the same fuzzy number \(\tilde{f}(x)\) for \(x \in E_{L'U'}\), it is not hard to see that \(A_\alpha(x) = A'_\alpha(x)\) for \(x \in E_{L'U'}\) and all \(\alpha \in [0, 1]\). It follows that \(l_\alpha(x) = l'_\alpha(x)\) and \(u_\alpha(x) = u'_\alpha(x)\) for \(x \in E_{L'U'}\) and all \(\alpha \in [0, 1]\). Using Proposition 4.4 and similar arguments as in the proof of Proposition 5.1, we see that

\[
\tilde{\mu}_\alpha^L(E_{L'U'}) = \tilde{\mu}_\alpha^L(E_{L'U'}^c) = \tilde{\mu}_\alpha^U(E_{L'U'}) = \tilde{\mu}_\alpha^U(E_{L'U'}^c) = 0
\]

for all \(\alpha \in [0, 1]\). It follows that \(l_\alpha = l'_\alpha\) a.e. [\(\mu_\alpha^L\)] and \(u_\alpha = u'_\alpha\) a.e. [\(\mu_\alpha^U\)] for all \(\alpha \in [0, 1]\), i.e., for the nonnegative case

\[
\int_E l_\alpha \, d\tilde{\mu}_\alpha^L = \int_E l'_\alpha \, d\tilde{\mu}_\alpha^L \quad \text{and} \quad \int_E u_\alpha \, d\tilde{\mu}_\alpha^U = \int_E u'_\alpha \, d\tilde{\mu}_\alpha^U
\]

for all \(\alpha \in [0, 1]\). This means that Definition 5.4 is well-defined.

In order to make the fuzzy-valued integrals more tractable mathematically, we need the following results.

**Proposition 5.2.** Let \(\{f_n\}\) be a sequence of nonnegative measurable functions on \((X, \mathcal{M})\) and \(\{\mu_n\}\) be a sequence of measures on \((X, \mathcal{M})\).

(i) If \(f_n \uparrow f\) a.e. [\(\mu\)] and \(\mu_n \uparrow \mu\) then

\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu_n.
\]

(ii) If \(f_n \downarrow f\) a.e. [\(\mu_1\)] and \(\mu_n \downarrow \mu\) with \(f_1 \in L^1(\mu_1)\) and \(\mu_1(X) < \infty\) then

\[
\int_X f \, d\mu = \lim_{n \to \infty} \int_X f_n \, d\mu_n.
\]
Proof. Using the routine arguments in real analysis, the results follow from the Generalized Fatou’s Lemma and Generalized Dominated Convergence Theorem in Royden [9]. □

Let \( \tilde{\mu} \) be a fuzzy-valued measure on a measurable space \((X, \mathcal{M})\). We write \( \tilde{\mu}(E) \prec \infty \) if and only if \( \tilde{\mu}_L^\alpha(E) < \infty \) and \( \tilde{\mu}_U^\alpha(E) < \infty \) for \( E \in \mathcal{M} \) and all \( \alpha \in [0, 1] \).

**Theorem 5.1.** Let \( \tilde{\mu} \) be a canonical fuzzy-valued measure on a measurable space \((X, \mathcal{M})\). Let \( L(x) = \{l_\alpha(x): \alpha \in [0, 1]\} \) and \( U(x) = \{u_\alpha(x): \alpha \in [0, 1]\} \) be two families of real-valued functions defined on \( X \), and \( \{L, U\} \) be also a canonical family with respect to \( \tilde{\mu} \). Let \( \tilde{f} \) be induced by \( \{L, U\} \). If \( \tilde{f} \) is integrable on \( E \) and \( \tilde{\mu}(E) \prec \infty \), then we have the following results.

(i) If \( \{L, U\} \) is nonnegative a.e. \([\tilde{\mu}]\) then

\[
\left( \int_E \tilde{f} \, d\tilde{\mu} \right)_\alpha = \left[ \int_E l_\alpha \, d\tilde{\mu}_L^\alpha, \int_E u_\alpha \, d\tilde{\mu}_U^\alpha \right]
\]

for all \( \alpha \in [0, 1] \).

(ii) If \( \{L, U\} \) is nonpositive a.e. \([\tilde{\mu}]\) then

\[
\left( \int_E \tilde{f} \, d\tilde{\mu} \right)_\alpha = \left[ \int_E l_\alpha \, d\tilde{\mu}_U^\alpha, \int_E u_\alpha \, d\tilde{\mu}_L^\alpha \right]
\]

for all \( \alpha \in [0, 1] \). Furthermore, the fuzzy-valued integral \( \int_E \tilde{f} \, d\tilde{\mu} \) is a fuzzy number.

Proof. Let \( C_\alpha \) be the closed interval given in Definition 5.4. From conditions in Definition 5.2, Propositions 4.4 and 5.2, we see that the family of closed intervals \( \{C_\alpha\} \) is continuously decreasing with respect to \( \alpha \). That is to say, \( \{C_\alpha\} \) satisfies all conditions in Proposition 2.3 (ii). Therefore, using Proposition 2.3 (ii), we have \( \left( \int_E \tilde{f} \, d\tilde{\mu} \right)_\alpha = C_\alpha \). It is also not hard to show that the fuzzy-valued integral \( \int_E \tilde{f} \, d\tilde{\mu} \) is a fuzzy number. □

**Theorem 5.2.** Let \( \tilde{\mu} \) be a canonical fuzzy-valued measure on a measurable space \((X, \mathcal{M})\), and \( \tilde{f} \) be a nonnegative or nonpositive fuzzy-valued function defined on \( X \). Suppose that \( \tilde{f}_L^\alpha \in L^1(\tilde{\mu}_L^\alpha) \) and \( \tilde{f}_U^\alpha \in L^1(\tilde{\mu}_U^\alpha) \) for all \( \alpha \in [0, 1] \). Then \( \tilde{f} \) is integrable on \( E \). We also have that

(i) if \( \tilde{f} \) is nonnegative then

\[
\left( \int_E \tilde{f} \, d\tilde{\mu} \right)_\alpha = \left[ \int_E \tilde{f}_L^\alpha \, d\tilde{\mu}_L^\alpha, \int_E \tilde{f}_U^\alpha \, d\tilde{\mu}_U^\alpha \right]
\]

for all \( \alpha \in [0, 1] \);
(ii) if $\tilde{f}$ is nonpositive then
\[
\left(\int_E \tilde{f} \, d\tilde{\mu}\right)_\alpha = \left[\int_E \tilde{f}^L_\alpha \, d\tilde{\mu}^L_\alpha, \int_E \tilde{f}^U_\alpha \, d\tilde{\mu}^U_\alpha\right]
\]
for all $\alpha \in [0, 1]$. Furthermore, the fuzzy-valued integral $\int_E \tilde{f} \, d\tilde{\mu}$ is a fuzzy number.

**Proof.** We consider the families $\mathcal{L}(x) = \{\tilde{f}^L_\alpha(x): \alpha \in [0, 1]\}$ and $\mathcal{U}(x) = \{\tilde{f}^U_\alpha(x): \alpha \in [0, 1]\}$. By Proposition 3.2 (i), $\tilde{f}$ is induced by $\{\mathcal{L}, \mathcal{U}\}$ on the whole domain $X$. Since $\tilde{f}^L_\alpha \uparrow \tilde{f}^L_\alpha$, $\tilde{f}^U_\alpha \downarrow \tilde{f}^U_\alpha$, $\tilde{\mu}^L_\alpha \uparrow \tilde{\mu}^L_\alpha$ and $\tilde{\mu}^U_\alpha \downarrow \tilde{\mu}^U_\alpha$ for $\alpha_n \uparrow \alpha$ from Proposition 5.2 (ii), the result follows from Propositions 5.2 and 2.3 (ii) using similar arguments as in the proof of Theorem 5.1. 

**Proposition 5.3.** Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. Let $\tilde{f}$ and $\tilde{g}$ be pseudo-fuzzy-valued measurable functions induced by two canonical families $\{\mathcal{L}, \mathcal{U}\}$ and $\{\tilde{\mathcal{L}}, \tilde{\mathcal{U}}\}$ with respect to $\tilde{\mu}$, respectively. Suppose that $\{\mathcal{L}, \mathcal{U}\}$ and $\{\tilde{\mathcal{L}}, \tilde{\mathcal{U}}\}$ are nonnegative or nonpositive a.e. $[\tilde{\mu}]$ simultaneously, and that $\tilde{h} \approx \tilde{f} \oplus \tilde{g}$. If $\tilde{f}$ and $\tilde{g}$ are integrable on $E$ and $\tilde{\mu}(E) < \infty$, then $\tilde{h}$ is also integrable on $E$, and
\[
\int_E \tilde{h} \, d\tilde{\mu} = \int_E \tilde{f} \, d\tilde{\mu} \oplus \int_E \tilde{g} \, d\tilde{\mu}.
\]

**Proof.** Now $\tilde{\mathcal{L}} = \mathcal{L} \oplus_{\text{fct}} \tilde{\mathcal{L}}$ and $\tilde{\mathcal{U}} = \mathcal{U} \oplus_{\text{fct}} \tilde{\mathcal{U}}$. From Proposition 4.4 and the similar arguments in the proof of Proposition 5.1, it is not hard to show that $\{\tilde{\mathcal{L}}, \tilde{\mathcal{U}}\}$ is a canonical family with respect to $\tilde{\mu}$ which induces $\tilde{h}$. Since $\tilde{f}$ and $\tilde{g}$ are integrable on $E$, using Theorem 5.1 and Proposition 2.2, we see that $\tilde{h}$ is integrable on $E$ and
\[
\left(\int_E \tilde{h} \, d\tilde{\mu}\right)_\alpha = \left(\int_E \tilde{f} \, d\tilde{\mu} \oplus \int_E \tilde{g} \, d\tilde{\mu}\right)_\alpha
\]
for all $\alpha \in [0, 1]$. Similarly for the nonpositive case. This completes the proof. 

**Proposition 5.4.** Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. Let $\tilde{f}$ and $\tilde{g}$ be nonnegative or nonpositive fuzzy-valued functions simultaneously. If $\tilde{f}$ and $\tilde{g}$ are integrable on $E$, then $\tilde{h} = \tilde{f} \oplus \tilde{g}$ is also integrable on $E$ and
\[
\int_E \tilde{h} \, d\tilde{\mu} = \int_E \tilde{f} \, d\tilde{\mu} \oplus \int_E \tilde{g} \, d\tilde{\mu}.
\]

**Proof.** The result follows by using similar arguments as in the proofs of Theorem 5.2 and Proposition 5.3. □
In the sequel, we shall introduce the fuzzy-valued integral of the general case, i.e., the fuzzy-valued function \( \tilde{f} \) is not restricted to nonnegative or nonpositive case. Let \( A(x) = [l(x), u(x)] \), where \( l \) and \( u \) are real-valued functions defined on \( X \) with \( l \leq u \). We define \( A^+(x) = [l^+(x), u^+(x)] \) and \( A^-(x) = [l^-(x), u^-(x)] \), where \( l^+(x) = \max\{l(x), 0\} \), \( u^+(x) = \max\{u(x), 0\} \), \( l^-(x) = \min\{0, l(x)\} \) and \( u^-(x) = \min\{0, u(x)\} \). Then we have \( l(x) = l^+(x) + l^-(x) \) and \( u(x) = u^+(x) + u^-(x) \). Thus \( A(x) = A^+(x) \oplus \text{int} A^-(x) \).

Let \( \mathcal{L}(x) = \{l_\alpha(x): \alpha \in [0, 1]\} \) and \( \mathcal{U}(x) = \{u_\alpha(x): \alpha \in [0, 1]\} \) be two families of real-valued functions defined on \( X \). We have a family of decreasing closed intervals \( \{A_\alpha(x)\} \) from \( \{\mathcal{L}, \mathcal{U}\} \). Let \( \mathcal{L}^+(x) = \{l_\alpha^+(x)\}, \mathcal{L}^-(x) = \{l_\alpha^-(x)\}, \mathcal{U}^+(x) = \{u_\alpha^+(x)\} \) and \( \mathcal{U}^-(x) = \{u_\alpha^-(x)\} \). Then we have the corresponding families of decreasing closed intervals \( \{A^+_\alpha(x)\} \) and \( \{A^-_\alpha(x)\} \) from \( \{\mathcal{L}^+, \mathcal{U}^+\} \) and \( \{\mathcal{L}^-, \mathcal{U}^-\} \), respectively. We can see that \( A_\alpha(x) = A^+_\alpha(x) \oplus \text{int} A^-_\alpha(x) \) for \( x \in E_{\mathcal{LU}} \). Let \( \tilde{f}, \tilde{f}^+, \text{and} \tilde{f}^- \) be induced by \( \{\mathcal{L}, \mathcal{U}\}, \{\mathcal{L}^+, \mathcal{U}^+\} \) and \( \{\mathcal{L}^-, \mathcal{U}^-\} \), respectively, where \( \mathcal{L} = \mathcal{L}^+ \oplus \text{fct} \mathcal{L}^-, \text{and} \mathcal{U} = \mathcal{U}^+ \oplus \text{fct} \mathcal{U}^- \).

Remark 5.1. Since \( \tilde{f}(x) \) is a fuzzy number for any fixed \( x \in X \), we see that \( \tilde{f}^+(x) \) and \( \tilde{f}^-(x) \) are the positive and negative parts of \( \tilde{f}(x) \), respectively, and \( \tilde{f}(x) = \tilde{f}^+(x) \oplus \tilde{f}^-(x) \) for any fixed \( x \in X \) by looking at (1). Therefore, \( \tilde{f} \) can induce two fuzzy-valued functions \( \tilde{f}^+ \) and \( \tilde{f}^- \) such that \( \tilde{f} = \tilde{f}^+ \oplus \tilde{f}^- \). From Proposition 2.6, \( \tilde{f}^+(x) = \tilde{f}^+(x) \) and \( \tilde{f}^-(x) = \tilde{f}^-(x) \) for \( x \in E_{\mathcal{LU}} \), i.e., \( \tilde{f}(x) = \tilde{f}^+(x) \oplus \tilde{f}^-(x) \) for \( x \in E_{\mathcal{LU}} \).

Definition 5.5. Let \( \tilde{\mu} \) be a canonical fuzzy-valued measure on a measurable space \( (X, \mathcal{M}) \). Let \( \mathcal{L}(x) = \{l_\alpha(x): \alpha \in [0, 1]\} \) and \( \mathcal{U}(x) = \{u_\alpha(x): \alpha \in [0, 1]\} \) be two families of real-valued functions defined on \( X \) such that \( \{\mathcal{L}^+, \mathcal{U}^+\} \) and \( \{\mathcal{L}^-, \mathcal{U}^-\} \) are two canonical families with respect to \( \tilde{\mu} \), where \( \{\mathcal{L}^+, \mathcal{U}^+\} \) is nonnegative a.e. \( [\tilde{\mu}] \) and \( \{\mathcal{L}^-, \mathcal{U}^-\} \) is nonpositive a.e. \( [\tilde{\mu}] \). Let \( \tilde{f}, \tilde{f}^{++}, \text{and} \tilde{f}^{--} \) be induced by \( \{\mathcal{L}, \mathcal{U}\}, \{\mathcal{L}^+, \mathcal{U}^+\} \) and \( \{\mathcal{L}^-, \mathcal{U}^-\} \), respectively. If \( \tilde{f}^{++} \) and \( \tilde{f}^{--} \) are integrable on \( E \), then we say that \( \tilde{f} \) is integrable on \( E \), and the fuzzy-valued integral \( \int_E \tilde{f} d\tilde{\mu} \) is defined by
\[
\int_E \tilde{f} d\tilde{\mu} = \int_E \tilde{f}^{++} d\tilde{\mu} \oplus \int_E \tilde{f}^{--} d\tilde{\mu}.
\]

Remark 5.2. From Theorem 5.1 and Proposition 2.2, \( \int_E \tilde{f} d\tilde{\mu} \) is a fuzzy number and
\[
\left( \int_E \tilde{f} d\tilde{\mu} \right)_\alpha = \left[ \int_E l^+_\alpha d\tilde{\mu}_\alpha^L + \int_E l^-_\alpha d\tilde{\mu}_\alpha^U \right] \left[ \int_E u^+_\alpha d\tilde{\mu}_\alpha^U + \int_E u^-_\alpha d\tilde{\mu}_\alpha^L \right].
\]
Theorem 5.3. Let \( \tilde{\mu} \) be a canonical fuzzy-valued measure on a measurable space \((X, \mathcal{M})\). Let \( \tilde{f} \) be a fuzzy-valued function defined on \( X \). If \( \tilde{f}^+ \) and \( \tilde{f}^- \) are integrable on \( E \), then \( \tilde{f} \) is also integrable on \( E \) and

\[
\int_E \tilde{f} \, d\tilde{\mu} = \int_E \tilde{f}^+ \, d\tilde{\mu} \oplus \int_E \tilde{f}^- \, d\tilde{\mu}.
\]

Proof. We consider the families \( \mathcal{L}(x) = \{ \tilde{f}^+_{\alpha}(x) : \alpha \in [0,1] \} \) and \( \mathcal{U}(x) = \{ \tilde{f}^-_{\alpha}(x) : \alpha \in [0,1] \} \). Then \( E_{\mathcal{L}} = X \) (the whole domain) from Proposition 3.2. From Remark 5.1, we see that \( \hat{f}^+ + \hat{f}^- \) and \( \hat{f}^- + \hat{f}^- \) for \( x \in E_{\mathcal{L}} = X \). The result follows from Remark 5.2 and Theorem 5.2 immediately. \( \square \)

Proposition 5.5. Let \( \tilde{\mu} \) be a canonical fuzzy-valued measure on a measurable space \((X, \mathcal{M})\). Let \( \hat{f} \) and \( \hat{g} \) be induced by two families \( \{ \mathcal{L}, \mathcal{U} \} \) and \( \{ \mathcal{L}, \mathcal{U} \} \), respectively. Suppose that \( \{ \mathcal{L}^+, \mathcal{U}^+ \}, \{ \mathcal{L}^-, \mathcal{U}^- \} \) and \( \{ \mathcal{L}^+, \mathcal{U}^- \} \) are canonical families with respect to \( \tilde{\mu} \). We further assume that \( l_{\alpha}(x) \) and \( \tilde{l}_{\alpha}(x) \) have the same sign for each \( x \) (i.e., \( l_{\alpha}(x) \cdot \tilde{l}_{\alpha}(x) \geq 0 \)) and for all \( \alpha \in [0,1] \), and \( u_{\alpha}(x) \) and \( \tilde{u}_{\alpha}(x) \) also have the same sign for each \( x \) and for all \( \alpha \in [0,1] \). Suppose that \( \hat{h} \approx \hat{f} \oplus \hat{g} \). If \( \hat{f} \) and \( \hat{g} \) are integrable on \( E \), then \( \hat{h} \) is also integrable on \( E \) and

\[
\int_E \hat{h} \, d\tilde{\mu} = \int_E \hat{f} \, d\tilde{\mu} \oplus \int_E \hat{g} \, d\tilde{\mu}.
\]

Proof. Let \( \hat{\mathcal{L}}^+ = \mathcal{L}^+ \oplus_{\text{fct}} \mathcal{L}^+ \), \( \hat{\mathcal{U}}^+ = \mathcal{U}^+ \oplus_{\text{fct}} \mathcal{U}^+ \), \( \hat{\mathcal{L}}^- = \mathcal{L}^- \oplus_{\text{fct}} \mathcal{L}^- \) and \( \hat{\mathcal{U}}^- = \mathcal{U}^- \oplus_{\text{fct}} \mathcal{U}^- \). Using similar arguments as in the proof of Proposition 5.3, we can see that \( \{ \hat{\mathcal{L}}^+, \hat{\mathcal{U}}^+ \} \) and \( \{ \hat{\mathcal{L}}^-, \hat{\mathcal{U}}^- \} \) are two canonical families with respect to \( \tilde{\mu} \). We also have \( \hat{l}^-_{\alpha} = l^-_{\alpha} \oplus l^-_{\alpha} \) and \( \hat{u}^-_{\alpha} = u^-_{\alpha} \oplus u^-_{\alpha} \) and \( \hat{l}^+_{\alpha} = l^+_{\alpha} \oplus l^+_{\alpha} \) and \( \hat{u}^+_{\alpha} = u^+_{\alpha} \oplus u^+_{\alpha} \) and \( \hat{u}^-_{\alpha} = u^-_{\alpha} \oplus u^-_{\alpha} \). Since \( l_{\alpha}(x) \) and \( \tilde{l}_{\alpha}(x) \) have the same sign for each \( x \), we have \( \hat{l}^+_{\alpha} = l^+_{\alpha} \oplus l^+_{\alpha} \) and \( \hat{l}^-_{\alpha} = l^-_{\alpha} \oplus l^-_{\alpha} \). Similarly, we also have \( \hat{u}^+_{\alpha} = u^+_{\alpha} \oplus u^+_{\alpha} \) and \( \hat{u}^-_{\alpha} = u^-_{\alpha} \oplus u^-_{\alpha} \). Now, from Remark 5.2 and Proposition 2.2, we have

\[
\left( \int_E \hat{h} \, d\mu \right)_{\alpha} = \left( \int_E \hat{f} \, d\mu \oplus \int_E \hat{g} \, d\mu \right)_{\alpha}
\]

for all \( \alpha \in [0,1] \). This completes the proof. \( \square \)
6. Dominated Convergence Theorems

We shall discuss the Dominated Convergence Theorem for the fuzzy-valued integrals with respect to fuzzy-valued measures.

Definition 6.1. Let $\tilde{a}$ be a fuzzy number. We call $\tilde{a}$ a canonical fuzzy number if $\tilde{a}_\alpha^L$ and $\tilde{a}_\alpha^U$ are continuous with respect to $\alpha$ on $[0, 1]$.

We also need the following results for canonical fuzzy numbers.

Proposition 6.1. Let $\tilde{a}$ and $\tilde{b}$ be two canonical fuzzy numbers. Then $d_F(\tilde{a}, \tilde{b}) < \varepsilon$ if and only if $|\tilde{a}_\alpha^L - \tilde{b}_\alpha^L| < \varepsilon$ and $|\tilde{a}_\alpha^U - \tilde{b}_\alpha^U| < \varepsilon$ for all $\alpha \in [0, 1]$.

Proof. For a compact set $S$ in $\mathbb{R}^n$, from Bazaraa et al. [2], if $f$ is upper semi-continuous on $S$ then $f$ assumes maximum over $S$, and if $f$ is lower semicontinuous on $S$ then $f$ assumes minimum over $S$. Therefore the result follows from Propositions 4.1 immediately.

We denote by $\mathcal{F}_c(\mathbb{R})$ the set of all canonical fuzzy numbers. If a function $\tilde{f}$ is given by $\tilde{f} : X \rightarrow \mathcal{F}_c(\mathbb{R})$, then $\tilde{f}$ is called a canonical fuzzy-valued function. Next we are going to discuss the Dominated Convergence Theorem for canonical fuzzy-valued functions.

From Eq. (3), if $F_{\alpha; A}^L$ and $F_{\alpha; A}^U$ are re-defined as follows

$$F_{\alpha; A}^L = \{x \in X : l_{\alpha_n}(x) \rightarrow l_{\alpha}(x) \text{ for } \alpha_n \rightarrow \alpha\}$$

and

$$F_{\alpha; A}^U = \{x \in X : u_{\alpha_n}(x) \rightarrow u_{\alpha}(x) \text{ for } \alpha_n \rightarrow \alpha\}$$

(the difference is considering $\alpha_n \rightarrow \alpha$, not $\alpha_n \uparrow \alpha$), then, from Proposition 2.4 (note that this proposition still holds true for canonical fuzzy number if condition (iii) is replaced by continuity instead of left-continuity), $\tilde{f}(x)$ is a canonical fuzzy number for each $x \in G_A$. In this case, we also call $\tilde{f}$ a canonical pseudo-fuzzy-valued function induced by $\{L, U\}$.

Theorem 6.1 (Dominated Convergence Theorem). Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, M)$ with $\tilde{\mu}(X) < \infty$. For each $n = 1, 2, \ldots$, let $\mathcal{L}_n(x) = \{l_{\alpha}^{(n)}(x) : \alpha \in [0, 1]\}$ and $\mathcal{U}_n(x) = \{u_{\alpha}^{(n)}(x) : \alpha \in [0, 1]\}$ be two families of real-valued functions defined on $X$, and $\{\mathcal{L}_n, \mathcal{U}_n\}$ be two canonical families with respect to $\tilde{\mu}$. Let $\tilde{f}_n$ be a canonical pseudo-fuzzy-valued function induced by $\{\mathcal{L}_n, \mathcal{U}_n\}$ for each $n = 1, 2, \ldots$. We assume that the following conditions are satisfied:

(i) each $\tilde{f}_n$ is integrable on $E$ for $n = 1, 2, \ldots$;
(ii) for \( n \to \infty \), \( (l_\alpha^{(n)})^+(x) \to l^+(x) \), \( (l_\alpha^{(n)})^-(x) \to l^-(x) \), \( (u_\alpha^{(n)})^+(x) \to u^+(x) \) and \( (u_\alpha^{(n)})^-(x) \to u^-(x) \) uniformly with respect to \( \alpha \) on \([0, 1]\) for any fixed \( x \in X \);

(iii) there exist nonnegative functions \( g^L \in L^1(\tilde{\mu}_\alpha^L) \) and \( g^U \in L^1(\tilde{\mu}_\alpha^U) \) for all \( \alpha \in [0, 1] \) such that \( g^L \geq \max\{(l_\alpha^{(n)})^+, |(u_\alpha^{(n)})^-|\} \) and \( g^U \geq \max\{(u_\alpha^{(n)})^+, |(l_\alpha^{(n)})^-|\} \) for each \( n = 1, 2, \ldots \) and all \( \alpha \in [0, 1] \).

Then the canonical pseudo-fuzzy-valued function \( \tilde{f} \) induced by the families \( \mathcal{L}(x) = \{l_\alpha(x) = l^+(x) + l^-(x) : \alpha \in [0, 1]\} \) and \( \mathcal{U}(x) = \{u_\alpha(x) = u^+(x) + u^-(x) : \alpha \in [0, 1]\} \) is integrable on \( E \) and we also have

\[
\lim_{n \to \infty} \int_E \tilde{f}_n \, d\tilde{\mu} = \int_E \tilde{f} \, d\tilde{\mu}.
\]

**Proof.** From condition (ii), we see that \( l_\alpha^{(n)}(x) \to l(x) \) and \( u_\alpha^{(n)}(x) \to u(x) \) uniformly with respect to \( \alpha \) on \([0, 1]\) for any fixed \( x \). Since \( (l_\alpha^{(n)})^+ \leq (l_1^{(n)})^+ \) a.e. \([\tilde{\mu}_1^L]\), we have the inequality \( \int_E (l_\alpha^{(n)})^+ \, d\tilde{\mu}_1^L \leq \int_E (l_1^{(n)})^+ \, d\tilde{\mu}_1^L \). This shows that \( (l_\alpha^{(n)})^+ \in L^1(\tilde{\mu}_1^L) \), since \( \tilde{f}_n \) is integrable, i.e., \( (l_1^{(n)})^+ \in L^1(\tilde{\mu}_1^L) \). Similarly, since \( (u_\alpha^{(n)})^- \in L^1(\tilde{\mu}_1^L) \), \( (u_0^{(n)})^+ \in L^1(\tilde{\mu}_1^U) \), \( (l_\alpha^{(n)})^- \in L^1(\tilde{\mu}_1^U) \) (note that \( (l_\alpha^{(n)})^- \) and \( (u_\alpha^{(n)})^- \) are nonpositive) and \( \int_E (u_\alpha^{(n)})^- \, d\tilde{\mu}_1^L \leq \int_E (u_\alpha^{(n)})^- \, d\tilde{\mu}_1^L \), \( \int_E (u_\alpha^{(n)})^+ \, d\tilde{\mu}_1^U \leq \int_E (u_\alpha^{(n)})^- \, d\tilde{\mu}_1^U \), \( \int_E (l_\alpha^{(n)})^- \, d\tilde{\mu}_1^U \leq \int_E (l_\alpha^{(n)})^- \, d\tilde{\mu}_1^U \), we have \( (u_\alpha^{(n)})^- \in L^1(\tilde{\mu}_1^L) \) and \( (u_\alpha^{(n)})^+, (l_\alpha^{(n)})^- \in L^1(\tilde{\mu}_1^U) \) for each \( n = 1, 2, \ldots \) and all \( \alpha \in [0, 1] \). Since the convergence is independent of \( \alpha \) in condition (ii), \( (l_\alpha^{(n)})^+ \in L^1(\tilde{\mu}_1^L) \) and \( (l_\alpha^{(n)})^- \in L^1(\tilde{\mu}_1^U) \), from condition (iii) and using the Lebesgue Dominated Convergence Theorem, we have

\[
(6) \quad \left| \int_E (l_\alpha^{(n)})^+ \, d\tilde{\mu}_1^L - \int_E l_\alpha^+ \, d\tilde{\mu}_1^L \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \int_E (l_\alpha^{(n)})^- \, d\tilde{\mu}_1^U - \int_E l_\alpha^- \, d\tilde{\mu}_1^U \right| < \frac{\varepsilon}{2}
\]

for all \( \alpha \in [0, 1] \) (i.e., independent of \( \alpha \)) for \( n \) sufficiently large. From Remark 5.2 and (6), we can show that

\[
\left| \left( \int_E \tilde{f}_n \, d\tilde{\mu} \right)^L_{\alpha} - \left( \int_E \tilde{f} \, d\tilde{\mu} \right)^L_{\alpha} \right| < \varepsilon
\]

for \( n \) sufficiently large and all \( \alpha \in [0, 1] \). Similarly, we also have

\[
\left| \left( \int_E \tilde{f}_n \, d\tilde{\mu} \right)^U_{\alpha} - \left( \int_E \tilde{f} \, d\tilde{\mu} \right)^U_{\alpha} \right| < \varepsilon
\]

for \( n \) sufficiently large and all \( \alpha \in [0, 1] \). Thus the result follows from Proposition 6.1 immediately. \( \square \)
In the sequel, we are going to discuss the Dominated Convergence Theorem for fuzzy-valued functions. Let \( \{ \tilde{f}_n \} \) be a sequence of fuzzy-valued functions that are integrable on \( E \) and dominated by a nonnegative integrable fuzzy-valued function such that the limit function of \( \{ \tilde{f}_n \} \) exists. Then we are going to show that
\[
\lim_{n \to \infty} \int_E \tilde{f}_n \, d\tilde{\mu} = \int_E \tilde{f} \, d\tilde{\mu},
\]
where \( \tilde{\mu} \) is a canonical fuzzy-valued measure.

Now we are going to fuzzify a nonfuzzy-valued function. Recall that \( F \) denotes the set of all fuzzy subsets of \( \mathbb{R} \). Let \( f : \mathbb{R}^n \to \mathbb{R} \) be a nonfuzzy-valued function (i.e., a real-valued function defined on \( \mathbb{R}^n \)) and \( \tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n \) be \( n \) fuzzy subsets of \( \mathbb{R} \). By the extension principle in Zadeh [16] and Nguyen [7], we can induce a function \( \tilde{f} : F^n \to F \) from the nonfuzzy-valued function \( f \). That is to say, \( \tilde{f}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) \) is a fuzzy subset of \( \mathbb{R} \). The membership function of \( \tilde{f}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n) \) is defined by
\[
(7) \quad \xi_{\tilde{f}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)}(r) = \sup_{\{x_1, \ldots, x_n\} : r = f(x_1, \ldots, x_n)} \min\{\xi_{\tilde{A}_1}(x_1), \ldots, \xi_{\tilde{A}_n}(x_n)\}.
\]

Now we can define the meaning of the absolute value of a fuzzy number. Let \( \tilde{a} \) be a fuzzy number and \( f(x) = |x| \). Then we can consider the fuzzy subset \( |\tilde{a}| \) induced by the real-valued function \( f(x) = |x| \) using Eq. (7). It is not hard to show that \( |\tilde{a}| \) is a fuzzy number and
\[
(8) \quad |\tilde{a}|_{\alpha} = \{|r| : r \in \tilde{a}_{\alpha}\}
\]
for all \( \alpha \in [0, 1] \). Let \( \tilde{a} \) and \( \tilde{b} \) be two fuzzy numbers. We write \( \tilde{a} \succeq \tilde{b} \) if and only if \( \tilde{a}_{\alpha}^L \geq \tilde{b}_{\alpha}^L \) and \( \tilde{a}_{\alpha}^U \geq \tilde{b}_{\alpha}^U \) for all \( \alpha \in [0, 1] \). Then “\( \succeq \)" is a partial ordering on \( F(\mathbb{R}) \). The following results are not hard to prove by using routine arguments.

**Proposition 6.2.** Let \( \{\tilde{a}_n\} \) be a sequence of fuzzy numbers. Then
\[
\lim_{n \to \infty} \tilde{a}_n = \tilde{a} \quad \text{if and only if} \quad \lim_{n \to \infty} \tilde{a}_n^+ = \tilde{a}^+ \quad \text{and} \quad \lim_{n \to \infty} \tilde{a}_n^- = \tilde{a}^-.
\]

**Proposition 6.3.** Let \( \tilde{a} \) and \( \tilde{b} \) be two fuzzy numbers. If \( \tilde{a} \succeq |\tilde{b}| \), then we have
(i) \( \tilde{a}_{\alpha}^L \geq (\tilde{b}^+)_\alpha^L \) and \( \tilde{a}_{\alpha}^L \geq |(\tilde{b}^-)_\alpha^L| \) for all \( \alpha \in [0, 1] \);
(ii) \( \tilde{a}_{\alpha}^U \geq (\tilde{b}^+)_\alpha^U \) and \( \tilde{a}_{\alpha}^U \geq |(\tilde{b}^-)_\alpha^L| \) for all \( \alpha \in [0, 1] \).

We are going to apply Theorems 5.2 and 5.3 to deduce the following Dominated Convergence Theorem.
Theorem 6.2 (Dominated Convergence Theorem). Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$ with $\tilde{\mu}(X) \prec \infty$ and $\{\tilde{f}_n\}$ be a sequence of integrable fuzzy-valued functions with respect to $\tilde{\mu}$ on $E$ such that the limit function $\lim_{n \to \infty} \tilde{f}_n(x) = \tilde{f}(x)$ exists. If there exists a nonnegative integrable fuzzy-valued function $\tilde{g}(x)$ with respect to $\tilde{\mu}$ on $E$ such that $\tilde{g}(x) \succeq |\tilde{f}_n(x)|$ for all $n = 1, 2, \ldots$, then

$$\lim_{n \to \infty} \int_E \tilde{f}_n \, d\tilde{\mu} = \int_E \tilde{f} \, d\tilde{\mu}.$$ 

Proof. Since $\tilde{g}$ is integrable, we have $\tilde{g}_\alpha^L \in L^1(\tilde{\mu}_\alpha^L)$ and $\tilde{g}_\alpha^U \in L^1(\tilde{\mu}_\alpha^U)$ for all $\alpha \in [0, 1]$. From Propositions 6.3 and 2.1, we have $\tilde{g}_\alpha^L \geq \tilde{g}_\alpha^L \geq (\tilde{f}_n^+)_\alpha^L$ and $\tilde{g}_\alpha^L \geq (\tilde{f}_n^-)_\alpha^L$ for all $\alpha \in [0, 1]$, and $\tilde{g}_0^L \geq (\tilde{f}_n^+)_\alpha^L$ and $\tilde{g}_0^U \geq \tilde{g}_\alpha^U \geq (\tilde{f}_n^-)_\alpha^L$ for all $\alpha \in [0, 1]$ (i.e., independent of $\alpha$). Now we consider the following inequality

$$\int_E (\tilde{f}_n^+)_\alpha^L \, d\tilde{\mu}_\alpha^L \leq \int_E (\tilde{f}_n^+)_1 \, d\tilde{\mu}_1^L.$$ 

Since $\tilde{f}_n^+$ is integrable, i.e., $(\tilde{f}_n^+)_\alpha^L \in L^1(\tilde{\mu}_\alpha^L)$ for all $\alpha \in [0, 1]$, it follows that $(\tilde{f}_n^+)_\alpha^L \in L^1(\tilde{\mu}_\alpha^L)$ from (9). Similarly, since $(\tilde{f}_n^-)_\alpha^L \in L^1(\tilde{\mu}_\alpha^L)$, $(\tilde{f}_n^-)_\alpha^L \in L^1(\tilde{\mu}_\alpha^L)$, $(\tilde{f}_n^+)_\alpha^U \in L^1(\tilde{\mu}_\alpha^U)$ (note that $(\tilde{f}_n^-)_\alpha^L$ and $(\tilde{f}_n^-)_\alpha^U$ are nonpositive) and $\int_E (\tilde{f}_n^-)_\alpha^L \, d\tilde{\mu}_\alpha^L \leq \int_E (\tilde{f}_n^-)_\alpha^U \, d\tilde{\mu}_\alpha^U$, $\int_E (\tilde{f}_n^-)_\alpha^L \, d\tilde{\mu}_\alpha^L \leq \int_E (\tilde{f}_n^-)_\alpha^U \, d\tilde{\mu}_\alpha^U$, $\int_E (\tilde{f}_n^+)_\alpha^L \, d\tilde{\mu}_\alpha^L \leq \int_E (\tilde{f}_n^+)_\alpha^U \, d\tilde{\mu}_\alpha^U$, we have $(\tilde{f}_n^-)_\alpha^L \in L^1(\tilde{\mu}_\alpha^L)$ and $(\tilde{f}_n^+)_\alpha^L \in L^1(\tilde{\mu}_\alpha^L)$ for each $n = 1, 2, \ldots$ and all $\alpha \in [0, 1]$. Since $(\tilde{f}_n^+)_\alpha^L \in L^1(\tilde{\mu}_\alpha^L)$ and $(\tilde{f}_n^-)_\alpha^L \in L^1(\tilde{\mu}_\alpha^L)$ for each $n = 1, 2, \ldots$ and all $\alpha \in [0, 1]$, using Propositions 4.2, 6.2 and the Lebesgue’s Dominated Convergence Theorem, we have

$$\left| \int_E (\tilde{f}_n^+)_\alpha^L \, d\tilde{\mu}_\alpha^L - \int_E (\tilde{f}_n^+)_1 \, d\tilde{\mu}_1^L \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \int_E (\tilde{f}_n^-)_\alpha^L \, d\tilde{\mu}_\alpha^L - \int_E (\tilde{f}_n^-)_1 \, d\tilde{\mu}_1^L \right| < \frac{\varepsilon}{2}$$

for $n$ sufficiently large and all $\alpha \in [0, 1]$ (i.e., independent of $\alpha$). From Theorems 5.2 and 5.3, we can show that

$$\left| \left( \int_E \tilde{f}_n \, d\tilde{\mu} \right)_\alpha^L - \left( \int_E \tilde{f} \, d\tilde{\mu} \right)_\alpha^L \right| < \varepsilon$$

for $n$ sufficiently large and all $\alpha \in [0, 1]$. Similarly, we also have

$$\left| \left( \int_E \tilde{f}_n \, d\tilde{\mu} \right)_\alpha^U - \left( \int_E \tilde{f} \, d\tilde{\mu} \right)_\alpha^U \right| < \varepsilon$$

for $n$ sufficiently large and all $\alpha \in [0, 1]$. The result follows from Proposition 6.1 immediately. \qed
References


Author’s address: Hsien-Chung Wu, Department of Mathematics, National Kaohsiung Normal University, Kaohsiung 802, Taiwan, e-mail: hcwu@nknucc.nknu.edu.tw.