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FUZZY-VALUED INTEGRALS BASED ON A CONSTRUCTIVE METHODOLOGY

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Abstract. The procedures for constructing a fuzzy number and a fuzzy-valued function from a family of closed intervals and two families of real-valued functions, respectively, are proposed in this paper. The constructive methodology follows from the form of the well-known "Resolution Identity" (decomposition theorem) in fuzzy sets theory. The fuzzy-valued measure is also proposed by introducing the notion of convergence for a sequence of fuzzy numbers. Under this setting, we develop the fuzzy-valued integral of fuzzy-valued function with respect to fuzzy-valued measure. Finally, we provide a Dominated Convergence Theorem for fuzzy-valued integrals.

Keywords: dominated convergence theorem, fuzzy number, fuzzy-valued function, fuzzy-valued integral, resolution identity

MSC 2000: 28E10, 03E72

1. Introduction

The concept of fuzzy integrals was first introduced by Sugeno [14]. After that, many subsequent formulations for fuzzy integrals have also been developed. Sim and Wang [11] gave a good review in the subject of fuzzy integrals. Some other interesting approaches are the fuzzy measures assuming values in the set of all fuzzy numbers by Klement [4] and Stojaković [12], the integration of fuzzy-valued functions by Klement [5] and Puri & Ralescu [8], and the fuzzy integrals on product spaces by Suárez-Díaz and Suárez-García [13]. In this paper, we are concerned with a more general setting, the fuzzy-valued integrals of fuzzy-valued measurable functions with respect to fuzzy-valued measures.

We propose a constructive methodology to obtain a fuzzy-valued function from two families of real-valued functions based on a well-known "Resolution Identity" in fuzzy sets theory. In order to propose the fuzzy-valued measures, we invoke the

Hausdorff metric which was proposed by Puri and Ralescu [8] to come up with the convergence of a sequence of fuzzy numbers. Under the settings of fuzzy-valued measures and fuzzy-valued functions, we are able to discuss the integrations of fuzzy-valued measurable functions with respect to fuzzy-valued measures.

In Sections 2 and 3, we first propose the methodology for constructing a fuzzy number from a family of closed intervals, and then we extend the methodology to construct a fuzzy-valued function from two families of real-valued functions. In Section 4, we introduce the notion of limit for a sequence of fuzzy numbers by invoking the Hausdorff metric in order to propose the fuzzy-valued measures. In Section 5, we are concerned with the integration of fuzzy-valued measurable function with respect to fuzzy-valued measure, where the fuzzy-valued measurable function is constructed from two families of real-valued measurable functions. In the final Section 6, we derive the main theorem, the Dominated Convergence Theorem for fuzzy-valued integrals.

2. Construction of fuzzy numbers

Let U be a topological vector space. The fuzzy subset $\tilde{a}$ of U is defined by its membership function $\xi_{\tilde{a}}\colon U \to [0,1]$. The α-level set of $\tilde{a}$, denoted by $\tilde{a}_\alpha$, is defined by $\tilde{a}_\alpha = \{x \in U\colon \xi_{\tilde{a}}(x) \geqslant \alpha\}$ for all $0 < \alpha \leqslant 1$. The 0-level set $\tilde{a}_0$ is defined as $\tilde{a}_0 = \operatorname{cl}(\{x \in U\colon \xi_{\tilde{a}}(x) > 0\})$. Let $\tilde{a}$ be a fuzzy subset of U. We say that $\tilde{a}$ is normal if there exists an $x \in U$ such that $\xi_{\tilde{a}}(x) = 1$, and that $\tilde{a}$ is convex if its membership function $\xi_{\tilde{a}}$ is quasi-concave, i.e., $\xi_{\tilde{a}}(\lambda x + (1-\lambda)y) \geqslant \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{a}}(y)\}$ for all $\lambda \in [0,1]$.

We denote by $\mathcal{F}(U)$ the set of all fuzzy subsets $\tilde{a}$ of U with membership function $\xi_{\tilde{a}}$ satisfying the following conditions:

(i) $\tilde{a}$ is normal and convex.

(ii) $\xi_{\tilde{a}}$ is upper semicontinuous, i.e., $\{x \in U\colon \xi_{\tilde{a}}(x) \geqslant \alpha\}$ is a closed subset of U for all $\alpha \in (0,1]$.

(iii) The 0-level set $\tilde{a}_0$ is a compact subset of U.

Throughout this paper, the universal set U is assumed as the real number system $\mathbb{R}$ which is endowed with the usual topology. The member $\tilde{a}$ in $\mathcal{F}(\mathbb{R})$ is then called a fuzzy number. It is not hard to see that if $\tilde{a}$ is a fuzzy number then $\tilde{a}_\alpha$ is a closed interval in $\mathbb{R}$ for $\alpha \in [0,1]$. In this case, we write $\tilde{a}_\alpha = [\tilde{a}^L_\alpha, \tilde{a}^U_\alpha]$. The following easy consequence will be used frequently in this paper.

Proposition 2.1. *Let $\tilde{a}$ be a fuzzy number. Then $\tilde{a}_\beta \subseteq \tilde{a}_\alpha$ for $\alpha < \beta$, i.e., $\tilde{a}^L_\alpha \leqslant \tilde{a}^L_\beta$ and $\tilde{a}^U_\alpha \geqslant \tilde{a}^U_\beta$ for $\alpha < \beta$.*

Let $\tilde{a}$ be a fuzzy number. Then $\tilde{a}$ is called a nonnegative fuzzy number if $\xi_{\tilde{a}}(x) = 0$ for all $x < 0$, and called a nonpositive fuzzy number if $\xi_{\tilde{a}}(x) = 0$ for all $x > 0$. We say that $\tilde{a}$ is a crisp number with value m if its membership function is given by

$$\xi_{\tilde{a}}(r) = \begin{cases} 1 & \text{if } r = m, \\ 0 & \text{otherwise.} \end{cases}$$

We also use the notation $\tilde{1}_{\{m\}}$ to represent the crisp number with value m. It is easy to see that $(\tilde{1}_{\{m\}})^L_\alpha = (\tilde{1}_{\{m\}})^U_\alpha = m$ for all $\alpha \in [0, 1]$. In other words, each real number m can be regarded as a crisp number $\tilde{1}_{\{m\}}$.

Let "$\oplus$" be an addition between two fuzzy numbers $\tilde{a}$ and $\tilde{b}$. The membership function of $\tilde{a} \oplus \tilde{b}$ is defined by

$$\xi_{\tilde{a}\oplus\tilde{b}}(z) = \sup_{x+y=z} \min\{\xi_{\tilde{a}}(x), \xi_{\tilde{b}}(y)\}$$

using the extension principle in Zadeh [16]. Applying the results in Klir and Yuan [3, Chapter 4], we can show the following useful result for further discussions.

Proposition 2.2. *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. Then $\tilde{a} \oplus \tilde{b}$ is also a fuzzy number. Furthermore, we have*

$$(\tilde{a} \oplus \tilde{b})_\alpha = [\tilde{a}^L_\alpha + \tilde{b}^L_\alpha, \tilde{a}^U_\alpha + \tilde{b}^U_\alpha].$$

Let $\tilde{a}$ be a fuzzy number. We define the membership functions of $\tilde{a}^+$ and $\tilde{a}^-$ as

$$\xi_{\tilde{a}^+}(r) = \begin{cases} \xi_{\tilde{a}}(r) & \text{if } r > 0, \\ 1 & \text{if } r = 0 \text{ and } \xi_{\tilde{a}}(r) < 1 \text{ for all } r > 0, \\ \xi_{\tilde{a}}(0) & \text{if } r = 0 \text{ and there exists an } r > 0 \text{ such that } \xi_{\tilde{a}}(r) = 1, \\ 0 & \text{otherwise} \end{cases}$$

and

$$\xi_{\tilde{a}^-}(r) = \begin{cases} \xi_{\tilde{a}}(r) & \text{if } r < 0, \\ 1 & \text{if } r = 0 \text{ and } \xi_{\tilde{a}}(r) < 1 \text{ for all } r < 0, \\ \xi_{\tilde{a}}(0) & \text{if } r = 0 \text{ and there exists an } r < 0 \text{ such that } \xi_{\tilde{a}}(r) = 1, \\ 0 & \text{otherwise.} \end{cases}$$

From Proposition 2.2, it is not hard to see that

$$\tilde{a} = \tilde{a}^+ \oplus \tilde{a}^-. \tag{1}$$

We call $\tilde{a}^+$ and $\tilde{a}^-$ the positive part and negative part of $\tilde{a}$, respectively.

We rephrase the following well-known results for motivating the construction of a fuzzy number from a family of closed intervals.

Proposition 2.3.

(i) (*Zadeh* [16]) (*Resolution Identity*) *Let* $\tilde{A}$ *be a fuzzy set with membership function* $\xi_{\tilde{A}}$ *and* $\tilde{A}_\alpha$ *be the* α*-level set of* $\tilde{A}$ *for* $\alpha \in [0,1]$. *Then the membership function* $\xi_{\tilde{A}}$ *can be expressed as*

$$\xi_{\tilde{A}}(x) = \sup_{\alpha\in[0,1]} \alpha \cdot 1_{\tilde{A}_\alpha}(x),$$

where $1_{\tilde{A}_\alpha}$ *is the characteristic function of set* $\tilde{A}_\alpha$ *(note that the* α*-level set* $\tilde{A}_\alpha$ *is a usual set).*

(ii) (*Negoita and Ralescu* [6]) *Let* A *be a set and* $\{A_\alpha\colon\ \alpha \in [0,1]\}$ *be a family of subsets of* A *such that the following conditions are satisfied:*

(a) $A_0 = A$;

(b) $A_\beta \subseteq A_\alpha$ *for* $\alpha < \beta$;

(c) $A_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha_n}$ *for* $\alpha_n \uparrow \alpha$.

Then the function $\xi\colon\ A \to [0,1]$ *defined by*

$$\xi(x) = \sup_{\alpha\in[0,1]} \alpha \cdot 1_{A_\alpha}(x)$$

has the property that

$$A_\alpha = \{x \in A\colon\ \xi(x) \geqslant \alpha\} \ \text{for all}\ \alpha \in [0,1].$$

Let $\{A_\alpha = [l_\alpha, u_\alpha]\colon\ \alpha \in [0,1]\}$ be a family of closed intervals in $\mathbb{R}$. Then we can induce a fuzzy subset $\tilde{a}$ of $\mathbb{R}$ with membership function defined by

$$\xi_{\tilde{a}}(r) = \sup_{\alpha\in[0,1]} \alpha \cdot 1_{A_\alpha}(r)$$

via the form of Resolution Identity in Proposition 2.3. Note that, in general, this fuzzy subset $\tilde{a}$ of $\mathbb{R}$ is not necessarily a fuzzy number. We say that $\{A_\alpha\}$ is decreasing with respect to α if $A_\beta \subseteq A_\alpha$ for $\alpha < \beta$. Let us further regard l_α and u_α as the functions of α and assume that l_α and u_α are left-continuous with respect to α. Therefore if $\{A_\alpha\}$ is decreasing with respect to α, thus we see that $\{A_\alpha\}$ is continuously decreasing with respect to α, since l_α and u_α are left-continuous with respect to α. It also says that $A_\alpha = \bigcap_{n=1}^{\infty} A_{\alpha_n}$ for $\alpha_n \uparrow \alpha$. Using routine arguments, we can show the following interesting result.

Proposition 2.4. *Let $\{A_\alpha = [l_\alpha, u_\alpha]\colon\ \alpha \in [0,1]\}$ be a family of closed intervals. Suppose that the following conditions are satisfied:*

(i) $A_1 \neq \emptyset$;

(ii) $\{A_\alpha\}$ *is decreasing with respect to* α;

(iii) l_α *and* u_α *are left-continuous with respect to* α.

Then $\{A_\alpha\}$ *induces a fuzzy number* $\tilde{a}$ *with* $\tilde{a}_\alpha = A_\alpha$.

Conversely, we also have the following results.

Proposition 2.5.

(i) *Let* $A_\alpha = \{x \in \mathbb{R}\colon\ \xi(x) \geqslant \alpha\}$. *Then* $\bigcap_{n=1}^{\infty} A_{\alpha_n} = A_\alpha$ *for* $\alpha_n \uparrow \alpha$.

(ii) *If* $\tilde{a}$ *is a fuzzy number then* $\tilde{a}^L_{\alpha_n} \uparrow \tilde{a}^L_\alpha$ *and* $\tilde{a}^U_{\alpha_n} \downarrow \tilde{a}^U_\alpha$ *for* $\alpha_n \uparrow \alpha$, *i.e.,* $\tilde{a}^L_\alpha$ *and* $\tilde{a}^U_\alpha$ *are left-continuous with respect to* α.

Let $A = [a^L, a^U]$ and $B = [b^L, b^U]$ be two closed intervals in $\mathbb{R}$. Then the addition of two closed intervals is denoted and given by

$$A \oplus_{\text{int}} B \equiv \{z \in \mathbb{R}\colon\ z = x + y \ \text{ for } \ x \in A \ \text{ and } \ y \in B\} = [a^L + b^L, a^U + b^U].$$

Let $A = [l, u]$ be a closed interval in $\mathbb{R}$. If $l \geqslant 0$ then A is called a nonnegative closed interval, and if $u \leqslant 0$ then A is called a nonpositive closed interval. If $l \leqslant 0$ and $u \geqslant 0$ then we let $A^+ = [0, u]$ and $A^- = [l, 0]$. We call A^+ the positive part of A and A^- the negative part of A. It is obvious that $A = A^+ \oplus_{\text{int}} A^-$.

Let the family of closed intervals $\{A_\alpha = [l_\alpha, u_\alpha]\colon\ \alpha \in [0,1]\}$ be decreasing with respect to α and $A_1 \neq \emptyset$. Then we have $A_\alpha = A^+_\alpha \oplus_{\text{int}} A^-_\alpha$ for $\alpha \in [0,1]$. Now $\{A_\alpha\}$, $\{A^+_\alpha\}$ and $\{A^-_\alpha\}$ can induce three respective fuzzy sets $\tilde{a}$, $\tilde{b}$ and $\tilde{c}$ with membership functions defined by

$$\xi_{\tilde{a}}(r) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{A_\alpha}(r),$$

$$\xi_{\tilde{b}}(r) = \begin{cases} \sup\limits_{\alpha \in [0,1]} \alpha \cdot 1_{A^+_\alpha}(r) & \text{if } r > 0, \\ 1 & \text{if } r = 0 \text{ and } A^+_1 = \emptyset, \\ \sup\limits_{\alpha \in [0,1]} \alpha \cdot 1_{A^+_\alpha}(0) & \text{if } r = 0 \text{ and } A^+_1 \neq \emptyset, \\ 0 & \text{if } r < 0 \end{cases}$$

and

$$\xi_{\tilde{c}}(r) = \begin{cases} \sup\limits_{\alpha\in[0,1]} \alpha \cdot 1_{A_\alpha^-}(r) & \text{if } r < 0, \\ 1 & \text{if } r = 0 \text{ and } A_1^- = \emptyset, \\ \sup\limits_{\alpha\in[0,1]} \alpha \cdot 1_{A_\alpha^-}(0) & \text{if } r = 0 \text{ and } A_1^- \neq \emptyset, \\ 0 & \text{if } r > 0. \end{cases}$$

Now, for $r > 0$, $r \in A_\alpha$ if and only if $r \in A_\alpha^+$. Thus $\xi_{\tilde{a}^+}(r) = \xi_{\tilde{a}}(r) = \xi_{\tilde{b}}(r)$. From the definition of the membership function of $\tilde{a}^+$, it is easy to see that $\xi_{\tilde{a}^+}(0) = \xi_{\tilde{b}}(0)$. We conclude that $\tilde{a}^+ = \tilde{b}$. Similarly, we can conclude that $\tilde{a}^- = \tilde{c}$. This shows the following result.

Proposition 2.6. *Let the family of closed intervals* $\{A_\alpha = [l_\alpha, u_\alpha]\colon\ \alpha \in [0,1]\}$ *be decreasing with respect to* α *and satisfy the conditions in Proposition* 2.4. *Let* $\tilde{a}$ *be a fuzzy number induced by* $\{A_\alpha\}$. *Then* $\tilde{a}^+$ *is a fuzzy number induced by* $\{A_\alpha^+\}$ *and* $\tilde{a}^-$ *is a fuzzy number induced by* $\{A_\alpha^-\}$, *where* $\tilde{a} = \tilde{a}^+ \oplus \tilde{a}^-$ *and* $A_\alpha = A_\alpha^+ \oplus_{\text{int}} A_\alpha^-$ *for* $\alpha \in [0,1]$.

Proposition 2.7. *Let the family of closed intervals* $\{A_\alpha = [l_\alpha, u_\alpha]\colon\ \alpha \in [0,1]\}$ *and* $\{\bar{A}_\alpha = [\bar{l}_\alpha, \bar{u}_\alpha]\colon\ \alpha \in [0,1]\}$ *be decreasing with respect to* α *and satisfy the conditions in Proposition* 2.4. *Suppose that* $\{A_\alpha\}$ *and* $\{\bar{A}_\alpha\}$ *induce two fuzzy numbers* $\tilde{a}$ *and* $\tilde{b}$, *respectively, and that* $\{A_\alpha \oplus_{\text{int}} \bar{A}_\alpha\colon\ \alpha \in [0,1]\}$ *induces a fuzzy number* $\tilde{c}$. *Then* $\tilde{c} = \tilde{a} \oplus \tilde{b}$.

P r o o f. Let $\tilde{c}_1$ be induced by $\{\hat{A}_\alpha \equiv A_\alpha \oplus_{\text{int}} \bar{A}_\alpha\}$ and $\tilde{c}_2 = \tilde{a} \oplus \tilde{b}$. By definition, the membership functions of $\tilde{c}_1$ and $\tilde{c}_2$ are given by

$$\xi_{\tilde{c}_1}(r) = \sup_{\alpha\in[0,1]} \alpha \cdot 1_{\hat{A}_\alpha}(r)$$

and

$$\xi_{\tilde{c}_2}(r) = \sup_{r=r_1+r_2} \min\Big\{ \sup_{\alpha\in[0,1]} \alpha \cdot 1_{A_\alpha}(r_1),\ \sup_{\alpha\in[0,1]} \alpha \cdot 1_{\bar{A}_\alpha}(r_2) \Big\}.$$

It is not hard to show that $\xi_{\tilde{c}_1}(r) = \xi_{\tilde{c}_2}(r)$ for all r. □

3. Construction of fuzzy-valued functions

In this section, we shall discuss the construction of fuzzy-valued functions from two families of functions.

Let $\tilde{f}$ be a function defined on X by $\tilde{f}\colon X \to \mathcal{F}(\mathbb{R})$. Then we say that $\tilde{f}$ is a fuzzy-valued function. We also denote by $\tilde{f}^L_\alpha(x) = (\tilde{f}(x))^L_\alpha$ and $\tilde{f}^U_\alpha(x) = (\tilde{f}(x))^U_\alpha$ for $x \in X$. Therefore the fuzzy-valued function $\tilde{f}$ induces the real-valued functions $\tilde{f}^L_\alpha$ and $\tilde{f}^U_\alpha$ for $\alpha \in [0,1]$.

Let $\mathcal{L}(x) = \{l_\alpha(x)\colon \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x)\colon \alpha \in [0,1]\}$ be two families of functions, where l_α and u_α are real-valued functions defined on X for $\alpha \in [0,1]$. Let

$$B_\alpha(x) = [\min\{l_\alpha(x), u_\alpha(x)\}, \max\{l_\alpha(x), u_\alpha(x)\}]$$

for $\alpha \in [0,1]$. Then we can induce a function $\tilde{f}$ which assumes values in the family of all fuzzy subsets of $\mathbb{R}$; that is to say, for any fixed $x \in X$, $\tilde{f}(x)$ is a fuzzy subset of $\mathbb{R}$ with membership function defined by

$$\xi_{\tilde{f}(x)}(r) = \sup_{\alpha\in[0,1]} \alpha \cdot 1_{B_\alpha(x)}(r) \tag{2}$$

via the form of Resolution Identity in Proposition 2.3. In the sequel, we are going to construct a subset of X such that $\tilde{f}(x)$ is a fuzzy number for each x in this subset of X.

For $\alpha < \beta$ and $\alpha, \beta \in [0,1]$, we adopt the following notations

$$\begin{aligned} E_{ll,\alpha,\beta} &= \{x \in X\colon\ l_\alpha(x) \leqslant l_\beta(x)\},\\ E_{uu,\alpha,\beta} &= \{x \in X\colon\ u_\beta(x) \leqslant u_\alpha(x)\},\\ E_{lu,\alpha} &= \{x \in X\colon\ l_\alpha(x) \leqslant u_\alpha(x)\}. \end{aligned}$$

We assume $E_{lu,1} = \{x \in X\colon\ l_1(x) \leqslant u_1(x)\} \neq \emptyset$. We also let

$$E_{ll} = \bigcap_{0\leqslant\alpha<\beta\leqslant1} E_{ll,\alpha,\beta},\quad E_{uu} = \bigcap_{0\leqslant\alpha<\beta\leqslant1} E_{uu,\alpha,\beta},\quad E_{lu} = \bigcap_{\alpha\in[0,1]} E_{lu,\alpha}$$

and

$$E_{\mathcal{L}\mathcal{U}} = E_{ll} \cap E_{uu} \cap E_{lu}.$$

Then, for each $x \in E_{\mathcal{L}\mathcal{U}}$, we have a family of decreasing closed intervals $\{A_\alpha(x) = [l_\alpha(x), u_\alpha(x)]\colon\ \alpha \in [0,1]\}$ induced from $\{\mathcal{L}(x), \mathcal{U}(x)\}$. Then the membership function of $\tilde{f}(x)$, for $x \in E_{\mathcal{L}\mathcal{U}}$, is given by

$$\xi_{\tilde{f}(x)}(r) = \sup_{\alpha\in[0,1]} \alpha \cdot 1_{A_\alpha(x)}(r)$$

from (2). Let us also adopt the following notations

$$
\begin{aligned}
F^L_{\alpha;A} &= \{x \in X\colon\ l_{\alpha_n}(x) \to l_\alpha(x) \ \text{ for } \ \alpha_n \uparrow \alpha\},\\
F^U_{\alpha;A} &= \{x \in X\colon\ u_{\alpha_n}(x) \to u_\alpha(x) \ \text{ for } \ \alpha_n \uparrow \alpha\}.
\end{aligned} \tag{3}
$$

Let $F_{\alpha;A} = F^L_{\alpha;A} \cap F^U_{\alpha;A}$ and $G_{\alpha;A} = F_{\alpha;A} \cap E_{\mathcal{LU}}$. Then, for each $x \in G_{\alpha;A}$, we see that $A_\alpha(x) = \bigcap_{n=1}^{\infty} A_{\alpha_n}(x)$ for $\alpha_n \uparrow \alpha$. Let $F_A = \bigcap_{\alpha\in[0,1]} F_{\alpha;A}$ and $G_A = \bigcap_{\alpha\in[0,1]} G_{\alpha;A}$. Then we see that $G_A = F_A \cap E_{\mathcal{LU}}$. Now, from Proposition 2.4, $\tilde{f}(x)$ is a fuzzy number for $x \in G_A$, i.e., $\tilde{f}$ is a fuzzy-valued function defined on G_A and $\tilde{f}_\alpha(x) = A_\alpha(x) = [l_\alpha(x), u_\alpha(x)]$ for $x \in G_A$ and $\alpha \in [0,1]$. We call $\tilde{f}$ the pseudo-fuzzy-valued function induced by $\{\mathcal{L},\mathcal{U}\}$. The reason why we call $\tilde{f}$ the pseudo-fuzzy-valued funtion is that $\tilde{f}(x)$ is just a fuzzy subset of $\mathbb{R}$, not a fuzzy number, for $x \in X \setminus G_A$. The following proposition is useful for defining the fuzzy-valued integrals.

Proposition 3.1.

(i) *If there exists a countable dense subset $\{\alpha_n\}$ of $[0,1]$ such that $E_{lu,\alpha_n} \subseteq F_A$ for all n, then E_{lu} can be expressed as countable intersections.*

(ii) *If there exists a countable dense subset $\{\beta_n\}$ of $[0,1]$, such that $E_{ll,\alpha,\beta_n} \subseteq F_A$ and $E_{uu,\alpha,\beta_n} \subseteq F_A$ for all $\alpha \in [0,\beta_n)$ and all n, then E_{ll} and E_{uu} can be expressed as countable intersections.*

Proof. It will be enough to just prove case E_{ll}. We now have

$$
E_{ll} = \bigcap_{\{\beta\colon\, 0\leqslant\beta\leqslant1\}} \ \bigcap_{\{\alpha\colon\, 0\leqslant\alpha<\beta\leqslant1\}} E_{ll,\alpha,\beta} \equiv \bigcap_{\{\beta\colon\, 0\leqslant\beta\leqslant1\}} H_\beta \subseteq \bigcap_{n=1}^{\infty} H_{\beta_n}, \tag{4}
$$

where $H_\beta = \bigcap_{\{\alpha\colon\, 0\leqslant\alpha<\beta\leqslant1\}} E_{ll,\alpha,\beta}$. Given any $\beta \in [0,1]$, there exists a subsequence $\{\beta_{n_k}\} \subseteq \{\beta_n\}$ such that $\beta_{n_k} \uparrow \beta$. If $\alpha < \beta$ then we have $l_\alpha(x) \leqslant l_{\beta_{n_k}}(x)$ for some $K > 0$, $\alpha < \beta_{n_k}$ and $k > K$. Therefore, we have $l_\alpha(x) \leqslant l_\beta(x)$ for $\alpha < \beta$ by taking limit, i.e, $x \in \bigcap_{0\leqslant\beta\leqslant1} H_\beta$. Thus $E_{ll} = \bigcap_{n=1}^{\infty} H_{\beta_n}$. For fixed β_n, let $\{\alpha^{(n)}_m\}_{m=1}^{\infty}$ be any countable dense subset of $[0,\beta_n]$. Similarly, we can show that

$$
H_{\beta_n} = \bigcap_{\{\alpha\colon\, 0\leqslant\alpha<\beta_n\leqslant1\}} E_{ll,\alpha,\beta_n} = \bigcap_{m=1,\, \alpha^{(n)}_m<\beta_n}^{\infty} E_{ll,\alpha^{(n)}_m,\beta_n}. \tag{5}
$$

This completes the proof. □

Let $\tilde{f}$ and $\tilde{g}$ be two pseudo-fuzzy-valued functions induced by $\{\mathcal{L},\mathcal{U}\}$ and $\{\bar{\mathcal{L}},\overline{\mathcal{U}}\}$, respectively. At the same time, we also have two corresponding families of decreasing closed intervals

$$\{A_\alpha(x) = [l_\alpha(x), u_\alpha(x)]\colon\ \alpha \in [0,1] \text{ and } x \in E_{\mathcal{L}\mathcal{U}}\}$$

and

$$\{\bar{A}_\alpha(x) = [\bar{l}_\alpha(x), \bar{u}_\alpha(x)]\colon\ \alpha \in [0,1] \text{ and } x \in E_{\bar{\mathcal{L}}\overline{\mathcal{U}}}\}$$

from $\{\mathcal{L},\mathcal{U}\}$ and $\{\bar{\mathcal{L}},\overline{\mathcal{U}}\}$, respectively. Let

$$\widehat{\mathcal{L}}(x) \equiv \{\hat{l}_\alpha(x) = l_\alpha(x) + \bar{l}_\alpha(x)\colon\ \alpha \in [0,1]\}$$

and

$$\widehat{\mathcal{U}}(x) \equiv \{\hat{u}_\alpha(x) = u_\alpha(x) + \bar{u}_\alpha(x)\colon\ \alpha \in [0,1]\}.$$

We denote by $\widehat{\mathcal{L}} = \mathcal{L} \oplus_{\mathrm{fct}} \bar{\mathcal{L}}$ and $\widehat{\mathcal{U}} = \mathcal{U} \oplus_{\mathrm{fct}} \overline{\mathcal{U}}$. Then we also have a family of decreasing closed intervals

$$\{\hat{A}_\alpha(x) = [\hat{l}_\alpha(x), \hat{u}_\alpha(x)]\colon\ \alpha \in [0,1] \text{ and } x \in E_{\hat{\mathcal{L}}\hat{\mathcal{U}}}\}$$

from $\{\widehat{\mathcal{L}},\widehat{\mathcal{U}}\}$. Therefore $\{\widehat{\mathcal{L}},\widehat{\mathcal{U}}\}$ can induce a pseudo-fuzzy-valued function $\tilde{h}$ such that $\tilde{h}$ is a fuzzy-valued function on $G_{\hat{A}}$. Now, we see that $x \in E_{ll,\alpha,\beta} \cap E_{\overline{l}\overline{l},\alpha,\beta}$ implies $\hat{l}_\alpha(x) = l_\alpha(x) + \bar{l}_\alpha(x) \leqslant l_\beta(x) + \bar{l}_\beta(x) = \hat{l}_\beta(x)$ for $\alpha < \beta$, i.e., $(E_{ll,\alpha,\beta} \cap E_{\overline{l}\overline{l},\alpha,\beta}) \subseteq E_{\hat{l}\hat{l},\alpha,\beta}$. Similarly, we also have $(E_{uu,\alpha,\beta} \cap E_{\bar{u}\bar{u},\alpha,\beta}) \subseteq E_{\hat{u}\hat{u},\alpha,\beta}$ and $(E_{lu,\alpha} \cap E_{\bar{l}\bar{u},\alpha}) \subseteq E_{\hat{l}\hat{u},\alpha}$ for $\alpha < \beta$. Suppose that $x \in F^L_{\alpha;A} \cap F^L_{\alpha;\bar{A}}$. Then, for $\alpha_n \uparrow \alpha$, we have $\lim_{n\to\infty} \hat{l}_{\alpha_n}(x) = \hat{l}_\alpha(x)$, i.e., $(F^L_{\alpha;A} \cap F^L_{\alpha;\bar{A}}) \subseteq F^L_{\alpha;\widehat{A}}$. Similarly, we also have $(F^U_{\alpha;A} \cap F^U_{\alpha;\bar{A}}) \subseteq F^U_{\alpha;\widehat{A}}$. Therefore we write $\tilde{h} \approx \tilde{f} \oplus \tilde{g}$ if $(E_{ll,\alpha,\beta} \cap E_{\overline{l}\overline{l},\alpha,\beta}) = E_{\hat{l}\hat{l},\alpha,\beta}$, $(E_{uu,\alpha,\beta} \cap E_{\bar{u}\bar{u},\alpha,\beta}) = E_{\hat{u}\hat{u},\alpha,\beta}$, $(E_{lu,\alpha} \cap E_{\bar{l}\bar{u},\alpha}) = E_{\hat{l}\hat{u},\alpha}$, $(F^L_{\alpha;A} \cap F^L_{\alpha;\bar{A}}) = F^L_{\alpha;\widehat{A}}$ and $(F^U_{\alpha;A} \cap F^U_{\alpha;\bar{A}}) = F^U_{\alpha;\widehat{A}}$ for $\alpha < \beta$. In this case, we conclude that $(E_{\mathcal{L}\mathcal{U}} \cap E_{\bar{\mathcal{L}}\overline{\mathcal{U}}}) = E_{\widehat{\mathcal{L}}\widehat{\mathcal{U}}}$ and $(F_A \cap F_{\bar{A}}) = F_{\widehat{A}}$, i.e., $(G_A \cap G_{\bar{A}}) = G_{\widehat{A}}$. From Propositions 2.1, 2.5 (ii) and 2.3 (ii), we can show the following results for later use.

Proposition 3.2.

(i) *Let $\tilde{f}$ be a fuzzy-valued function defined on X. We consider the families $\mathcal{L}(x) = \{\tilde{f}^L_\alpha(x)\colon\ \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{\tilde{f}^U_\alpha(x)\colon\ \alpha \in [0,1]\}$. Then $\{\mathcal{L},\mathcal{U}\}$ induces $\tilde{f}$ and $E_{\mathcal{L}\mathcal{U}} = F_A = X$, i.e., $G_A = X$.*

(ii) *Let $\tilde{f}$ and $\tilde{g}$ be two fuzzy-valued functions defined on the same set X. Let $\mathcal{L}(x) = \{\tilde{f}^L_\alpha(x)\}$, $\bar{\mathcal{L}}(x) = \{\tilde{g}^L_\alpha(x)\}$, $\mathcal{U}(x) = \{\tilde{f}^U_\alpha(x)\}$ and $\overline{\mathcal{U}}(x) = \{\tilde{g}^U_\alpha(x)\}$. Suppose that $\tilde{f}_0$ and $\tilde{g}_0$ are induced by $\{\mathcal{L},\mathcal{U}\}$ and $\{\bar{\mathcal{L}},\overline{\mathcal{U}}\}$, respectively, and $\tilde{h}$ is* *induced by* $\{\widehat{\mathcal{L}} = \mathcal{L} \oplus_{\rm fct} \bar{\mathcal{L}}, \widehat{\mathcal{U}} = \mathcal{U} \oplus_{\rm fct} \overline{\mathcal{U}}\}$. *Then* $\tilde{h} \approx \tilde{f}_0 \oplus \tilde{g}_0$, $\tilde{f}_0 = \tilde{f}$, $\tilde{g}_0 = \tilde{g}$ *and* $\tilde{h}(x) = \tilde{f}(x) \oplus \tilde{g}(x)$ *for all* $x \in X$, *i.e.,* $\tilde{h}_\alpha(x) = \tilde{f}_\alpha(x) \oplus_{\rm int} \tilde{g}_\alpha(x)$ *for all* $x \in X$.

Definition 3.1. Let $\mathcal{L}(x) = \{l_\alpha(x)\colon \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x)\colon \alpha \in [0,1]\}$ be two families of real-valued functions defined on X. We say that $\{\mathcal{L},\mathcal{U}\}$ is a standard family if $E_{lu,\alpha} \subseteq F_A$, $E_{ll,\alpha,\beta} \subseteq F_A$ and $E_{uu,\alpha,\beta} \subseteq F_A$ for all $\alpha < \beta$ and $\alpha, \beta \in [0,1]$.

Proposition 3.3. *Let* $\tilde{f}$ *be a pseudo-fuzzy-valued function induced by a standard family* $\{\mathcal{L},\mathcal{U}\}$. *Then* $G_A = E_{\mathcal{L}\mathcal{U}}$, *and* G_A *can be expressed as countable intersections.*

Proof. By the definition of standard family, we see that $E_{\mathcal{L}\mathcal{U}} \subseteq F_A$. This means that $G_A = E_{\mathcal{L}\mathcal{U}}$ since $G_A = E_{\mathcal{L}\mathcal{U}} \cap F_A$. The countable intersections of G_A follow from Proposition 3.1 immediately. □

4. The fuzzy-valued measures

In order to define the fuzzy-valued measure, we need to consider the limit of a sequence of fuzzy numbers. Thus we first introduce a metric on the set of all fuzzy numbers $\mathcal{F}(\mathbb{R})$.

Let $A \subseteq \mathbb{R}^n$ and $B \subseteq \mathbb{R}^n$. The *Hausdorff metric* is defined as

$$d_H(A,B) = \max\Big\{\sup_{a\in A}\inf_{b\in B}\|a-b\|, \sup_{b\in B}\inf_{a\in A}\|a-b\|\Big\}.$$

According to Puri and Ralescu [8], we define the metric $d_{\mathcal{F}}$ in $\mathcal{F}(\mathbb{R})$ as

$$d_{\mathcal{F}}(\tilde{a},\tilde{b}) = \sup_{\alpha\in[0,1]} d_H(\tilde{a}_\alpha, \tilde{b}_\alpha),$$

since $\tilde{a}_\alpha$ and $\tilde{b}_\alpha$ are bounded closed intervals for all $\alpha \in [0,1]$. We can see that $(\mathcal{F}(\mathbb{R}), d_{\mathcal{F}})$ is a complete metric space. The following result is obvious.

Proposition 4.1. *Let* $\tilde{a}$ *and* $\tilde{b}$ *be two fuzzy numbers. Then we have*

$$d_H(\tilde{a}_\alpha, \tilde{b}_\alpha) = \max\big\{|\tilde{a}_\alpha^L - \tilde{b}_\alpha^L|, |\tilde{a}_\alpha^U - \tilde{b}_\alpha^U|\big\}.$$

Definition 4.1. Let $\{\tilde{a}_n\}$ be a sequence of fuzzy numbers. Then $\{\tilde{a}_n\}$ is said to converge if there is a fuzzy number $\tilde{a}$ with the following property: $\forall\, \varepsilon > 0$, $\exists\, N > 0$ such that $d_{\mathcal{F}}(\tilde{a}_n, \tilde{a}) < \varepsilon$ for $n > N$. In this case, we also say that the sequence $\{\tilde{a}_n\}$ converges to $\tilde{a}$, and it is denoted by

$$\lim_{n\to\infty} \tilde{a}_n = \tilde{a}.$$

If there is no such $\tilde{a}$, the sequence $\{\tilde{a}_n\}$ is said to diverge.

Proposition 4.2. *Let $\{\tilde{a}_n\}$ be a sequence of fuzzy numbers. If the limit of the sequence $\{\tilde{a}_n\}$ exists, then it is unique and*

$$\big(\lim_{n\to\infty} \tilde{a}_n\big)_\alpha = \big[\lim_{n\to\infty} (\tilde{a}_n)^L_\alpha, \lim_{n\to\infty} (\tilde{a}_n)^U_\alpha\big]$$

for all $\alpha \in [0,1]$. Moreover, $\{(\tilde{a}_n)^L_\alpha\}$ and $\{(\tilde{a}_n)^U_\alpha\}$ converge uniformly with respect to α on $[0,1]$.

Proof. The result follows from Proposition 4.1 immediately. □

Definition 4.2. Let $\{\tilde{a}_n\}$ be a sequence of fuzzy numbers. Let $\tilde{s}_n = \bigoplus_{i=1}^{n} \tilde{a}_i$ be the partial sum of the sequence $\{\tilde{a}_n\}$. If the limit of the sequence $\{\tilde{s}_n\}$ exists, then the infinite (fuzzy) sum of the sequence $\{\tilde{a}_n\}$ is said to converge, and we also write

$$\bigoplus_{n=1}^{\infty} \tilde{a}_n = \lim_{n\to\infty} \tilde{s}_n = \lim_{n\to\infty} \bigoplus_{i=1}^{n} \tilde{a}_i,$$

otherwise the infinite (fuzzy) sum of the sequence $\{\tilde{a}_n\}$ is said to diverge.

Proposition 4.3. *If $\{\tilde{a}_n\}$ is a sequence of fuzzy numbers, and the infinite sum of the sequence $\{\tilde{a}_n\}$ exists, then we have*

$$\left(\bigoplus_{n=1}^{\infty} \tilde{a}_n\right)_\alpha = \left[\sum_{n=1}^{\infty} (\tilde{a}_n)^L_\alpha, \sum_{n=1}^{\infty} (\tilde{a}_n)^U_\alpha\right].$$

Proof. The result follows from Propositions 4.2 and 2.2 immediately. □

We denote by $\tilde{0}$ a crisp number with value 0. Then we are in a position to consider the fuzzy-valued measures.

Definition 4.3. By a fuzzy-valued measure $\tilde{\mu}$ on a measurable space $(X, \mathcal{M})$, we mean a nonnegative fuzzy-valued set function defined on all sets in $\mathcal{M}$ which satisfies the following two conditions:

(i) $\tilde{\mu}(\emptyset) = \tilde{0}$;

(ii) $\tilde{\mu}\Big(\bigcup_{i=1}^{\infty} E_i\Big) = \bigoplus_{i=1}^{\infty} \tilde{\mu}(E_i)$ for any sequence $\{E_i\}$ of disjoint measurable sets.

Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. Then $\tilde{\mu}(E)$ is a fuzzy number for $E \in \mathcal{M}$. Therefore, we can define the set functions $\tilde{\mu}^L_\alpha(E) = (\tilde{\mu}(E))^L_\alpha$ and $\tilde{\mu}^U_\alpha(E) = (\tilde{\mu}(E))^U_\alpha$ on $(X, \mathcal{M})$ for each $\alpha \in [0,1]$. Then, from Proposition 4.3, we see that if $\tilde{\mu}$ is a fuzzy-valued measure on a measurable space $(X, \mathcal{M})$, then $\tilde{\mu}^L_\alpha$ and $\tilde{\mu}^U_\alpha$ are the traditional measures on the same measurable space $(X, \mathcal{M})$.

Let μ_1 and μ_2 be two measures on the same measurable space $(X, \mathcal{M})$. Recall that μ_1 is absolutely continuous with respect to μ_2, denoted as $\mu_1 \ll \mu_2$, if $\mu_2(E) = 0$ implies $\mu_1(E) = 0$ for each set E.

Definition 4.4. Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X,\mathcal{M})$. Then $\tilde{\mu}^L_\alpha$ and $\tilde{\mu}^U_\alpha$ are the traditional measures on $(X,\mathcal{M})$ for all $\alpha \in [0,1]$. We say that $\tilde{\mu}$ is a canonical fuzzy-valued measure if the conditions $\tilde{\mu}^L_\beta \ll \tilde{\mu}^L_\alpha$, $\tilde{\mu}^U_\alpha \ll \tilde{\mu}^U_\beta$ and $\tilde{\mu}^U_\alpha \ll \tilde{\mu}^L_\alpha$ are satisfied for all $\alpha < \beta$ and $\alpha,\beta \in [0,1]$.

Let ν and μ be two measures on the same measurable space $(X,\mathcal{M})$. Recall that μ and ν are equivalent measures if $\mu \ll \nu$ and $\nu \ll \mu$. Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X,\mathcal{M})$. We denote by $\Xi = \{\tilde{\mu}^L_\alpha, \tilde{\mu}^U_\alpha \colon \alpha \in [0,1]\}$ a family of measures which are all on the same measurable space $(X,\mathcal{M})$.

Proposition 4.4. *If $\tilde{\mu}$ is a canonical fuzzy-valued measure on a measurable space $(X,\mathcal{M})$, then all measures in Ξ are equivalent.*

Proof. The result follows from Proposition 2.1 and the definition of canonical fuzzy-valued measure immediately. □

5. The fuzzy-valued integrals

In this section, we shall discuss the fuzzy-valued integral of fuzzy-valued measurable function which is constructed from two families of measurable functions.

Definition 5.1. Let $(X,\mathcal{M})$ be a measurable space. Let $\mathcal{L}(x) = \{l_\alpha(x)\colon \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x)\colon \alpha \in [0,1]\}$ be two families of real-valued functions defined on X. Let $\tilde{f}$ be a pseudo-fuzzy-valued function induced by $\{\mathcal{L},\mathcal{U}\}$. If l_α and u_α are measurable functions for all $\alpha \in [0,1]$, then we say that $\tilde{f}$ is measurable.

We denote by $\mathcal{F}$ the family of all fuzzy subsets of $\mathbb{R}$. Recall that $\mathcal{F}(\mathbb{R})$ denotes the set of all fuzzy numbers. Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X,\mathcal{M})$ and μ be a traditional measure on a measurable space $(X,\mathcal{M})$. We consider a function $\tilde{f}\colon X \to \mathcal{F}$ which assumes values in $\mathcal{F}$, not in $\mathcal{F}(\mathbb{R})$. Then we say that $\tilde{f}$ is a fuzzy-valued function a.e. $[\mu]$ if the set $Z = \{x \in X\colon \tilde{f}(x) \in \mathcal{F}(\mathbb{R})\}$ satisfies $\mu(Z^c) = 0$, and that $\tilde{f}$ is a fuzzy-valued function a.e. $[\tilde{\mu}]$ if $\tilde{\mu}(Z^c) = \tilde{0}$, i.e., $\tilde{\mu}^L_\alpha(Z^c) = 0 = \tilde{\mu}^U_\alpha(Z^c)$ for all $\alpha \in [0,1]$.

Definition 5.2. Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X,\mathcal{M})$. Let $\mathcal{L}(x) = \{l_\alpha(x)\colon \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x)\colon \alpha \in [0,1]\}$ be two families of real-valued measurable functions defined on X. Then $\{\mathcal{L},\mathcal{U}\}$ is said to be a canonical family with respect to $\tilde{\mu}$ if $\{\mathcal{L},\mathcal{U}\}$ is a standard family and there exists a measure $\mu \in \Xi$ such that the following conditions are satisfied:

(i) $l_\alpha \leqslant l_\beta$ a.e. $[\mu]$, $u_\beta \leqslant u_\alpha$ a.e. $[\mu]$ and $l_\alpha \leqslant u_\alpha$ a.e. $[\mu]$ for all $\alpha < \beta$ and $\alpha,\beta \in [0,1]$.

(ii) $l_{\alpha_n} \uparrow l_\alpha$ a.e. $[\mu]$ and $u_{\alpha_n} \downarrow u_\alpha$ a.e. $[\mu]$ for $\alpha_n \uparrow \alpha$.

Proposition 5.1. *Let $\mathcal{L}(x) = \{l_\alpha(x)\colon\ \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x)\colon\ \alpha \in [0,1]\}$ be two families of real-valued measurable functions defined on X. Let $\tilde{f}$ be a pseudo-fuzzy-valued measurable function induced by $\{\mathcal{L},\mathcal{U}\}$. Then the following statements hold true.*

(i) *Suppose that $\{\mathcal{L},\mathcal{U}\}$ is a standard family. If μ is a measure on a measurable space $(X,\mathcal{M})$ such that conditions* (i) *and* (ii) *in Definition* 5.2 *are satisfied, then $\mu(G_A^c) = 0$. That is to say, $\tilde{f}$ is a fuzzy-valued measurable function a.e. $[\mu]$.*

(ii) *Suppose that $\{\mathcal{L},\mathcal{U}\}$ is a canonical family with respect to $\tilde{\mu}$, where $\tilde{\mu}$ is a canonical fuzzy-valued measure on a measurable space $(X,\mathcal{M})$. Then $\tilde{\mu}(G_A^c) = \tilde{0}$, i.e., $\tilde{f}$ is a fuzzy-valued measurable function a.e. $[\tilde{\mu}]$.*

P r o o f. From condition (i) in Definition 5.2, Eqs. (4) and (5) in the proof of Proposition 3.1, we see that

$$0 \leqslant \mu(E_{ll}^c) \leqslant \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \mu\big(E^c_{ll,\alpha_m^{(n)},\beta_n}\big) = 0.$$

Similarly, we also have $\mu(E_{uu}^c) = 0 = \mu(E_{lu}^c)$. Thus we conclude that $\mu(E_{\mathcal{LU}}^c) = 0$. From Proposition 3.3, we also see that $\mu(G_A^c) = 0$. Since $\tilde{f}(x) \in \mathcal{F}(\mathbb{R})$ for $x \in G_A$, $\tilde{f}$ is a fuzzy-valued measurable function a.e. $[\mu]$. Now, if $\mu \in \Xi$, then, from Proposition 4.4, we have $\tilde{\mu}_\alpha^L(G_A^c) = 0 = \tilde{\mu}_\alpha^U(G_A^c)$ for all $\alpha \in [0,1]$. It follows that $\tilde{\mu}(G_A^c) = \tilde{0}$. This completes the proof. □

Definition 5.3. Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X,\mathcal{M})$. Let $\mathcal{L}(x) = \{l_\alpha(x)\colon\ \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x)\colon\ \alpha \in [0,1]\}$ be two families of real-valued functions defined on X. We say that $\{\mathcal{L},\mathcal{U}\}$ is nonnegative (resp. nonpositive) a.e. $[\tilde{\mu}]$ if $l_\alpha \geqslant 0$ (resp. $\leqslant 0$) a.e. $[\tilde{\mu}_\alpha^U]$ and $u_\alpha \geqslant 0$ (resp. $\leqslant 0$) a.e. $[\tilde{\mu}_\alpha^U]$.

Definition 5.4. Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X,\mathcal{M})$. Let $\mathcal{L}(x) = \{l_\alpha(x)\colon\ \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x)\colon\ \alpha \in [0,1]\}$ be two families of real-valued measurable functions defined on X, and $\{\mathcal{L},\mathcal{U}\}$ be a canonical family with respect to $\tilde{\mu}$. Let $\tilde{f}$ be a pseudo-fuzzy-valued measurable function induced by $\{\mathcal{L},\mathcal{U}\}$. Suppose that $l_\alpha \in L^1(\tilde{\mu}_\alpha^L)$ (i.e., Lebesgue integrable with respect to $\tilde{\mu}_\alpha^L$) and $u_\alpha \in L^1(\tilde{\mu}_\alpha^U)$ (i.e., Lebesgue integrable with respect to $\tilde{\mu}_\alpha^U$) for all $\alpha \in [0,1]$. Then we consider the following two cases.

(i) If $\{\mathcal{L},\mathcal{U}\}$ is nonnegative a.e. $[\tilde{\mu}]$, then, from Proposition 4.4 and condition (i) in Definition 5.2, we have $\int_E l_\alpha \,\mathrm{d}\tilde{\mu}_\alpha^L \leqslant \int_E u_\alpha \,\mathrm{d}\tilde{\mu}_\alpha^L \leqslant \int_E u_\alpha \,\mathrm{d}\tilde{\mu}_\alpha^U$ since $l_\alpha \leqslant u_\alpha$ a.e. $[\tilde{\mu}_\alpha^L]$ and $\tilde{\mu}_\alpha^L \leqslant \tilde{\mu}_\alpha^U$. Therefore we consider the closed interval C_α as

$$C_\alpha = \left[\int_E l_\alpha \,\mathrm{d}\tilde{\mu}_\alpha^L, \int_E u_\alpha \,\mathrm{d}\tilde{\mu}_\alpha^U\right]$$

for $\alpha \in [0,1]$.

(ii) If $\{\mathcal{L},\mathcal{U}\}$ is nonpositive a.e. $[\tilde{\mu}]$ then, similarly, we consider the closed interval C_α as

$$C_\alpha = \left[\int_E l_\alpha \, \mathrm{d}\tilde{\mu}_\alpha^U, \int_E u_\alpha \, \mathrm{d}\tilde{\mu}_\alpha^L\right]$$

for $\alpha \in [0,1]$. The membership function of the fuzzy-valued integral $\int_E \tilde{f} \, \mathrm{d}\tilde{\mu}$ is defined by

$$\xi_{\int_E \tilde{f} \, \mathrm{d}\tilde{\mu}}(r) = \sup_{\alpha \in [0,1]} \alpha \cdot 1_{C_\alpha}(r)$$

via the form of Resolution Identity in Proposition 2.3, and we say that $\tilde{f}$ is integrable with respect to $\tilde{\mu}$ on E.

Now we want to explain that Definition 5.4 is well-defined. It will be enough to just justify the nonnegative case. Let $\tilde{f}$ be a pseudo-fuzzy-valued measurable function induced by a canonical family $\{\mathcal{L},\mathcal{U}\}$. Suppose that $\tilde{f}$ is also induced by another canonical family $\{\mathcal{L}',\mathcal{U}'\}$. Then we can induce decreasing closed intervals $\{A_\alpha(x)\colon\ \alpha \in [0,1]\}$ from $\{\mathcal{L},\mathcal{U}\}$ for $x \in E_{\mathcal{L}\mathcal{U}}$ and decreasing closed intervals $\{A'_\alpha(x)\colon\ \alpha \in [0,1]\}$ from $\{\mathcal{L}',\mathcal{U}'\}$ for $x \in E_{\mathcal{L}'\mathcal{U}'}$. Since $\{A_\alpha(x)\colon\ \alpha \in [0,1]\}$ and $\{A'_\alpha(x)\colon\ \alpha \in [0,1]\}$ induce the same fuzzy number $\tilde{f}(x)$ for $x \in E_{\mathcal{L}\mathcal{U}} \cap E_{\mathcal{L}'\mathcal{U}'}$, it is not hard to see that $A_\alpha(x) = A'_\alpha(x)$ for $x \in E_{\mathcal{L}\mathcal{U}} \cap E_{\mathcal{L}'\mathcal{U}'}$ and all $\alpha \in [0,1]$. It follows that $l_\alpha(x) = l'_\alpha(x)$ and $u_\alpha(x) = u'_\alpha(x)$ for $x \in E_{\mathcal{L}\mathcal{U}} \cap E_{\mathcal{L}'\mathcal{U}'}$ and all $\alpha \in [0,1]$. Using Proposition 4.4 and similar arguments as in the proof of Proposition 5.1, we see that $\tilde{\mu}_\alpha^L(E^c_{\mathcal{L}\mathcal{U}}) = \tilde{\mu}_\alpha^L(E^c_{\mathcal{L}'\mathcal{U}'}) = \tilde{\mu}_\alpha^U(E^c_{\mathcal{L}\mathcal{U}}) = \tilde{\mu}_\alpha^U(E^c_{\mathcal{L}'\mathcal{U}'}) = 0$ for all $\alpha \in [0,1]$. It follows that $l_\alpha = l'_\alpha$ a.e. $[\tilde{\mu}_\alpha^L]$ and $u_\alpha = u'_\alpha$ a.e. $[\tilde{\mu}_\alpha^U]$ for all $\alpha \in [0,1]$, i.e., for the nonnegative case

$$\int_E l_\alpha \, \mathrm{d}\tilde{\mu}_\alpha^L = \int_E l'_\alpha \, \mathrm{d}\tilde{\mu}_\alpha^L \quad \text{and} \quad \int_E u_\alpha \, \mathrm{d}\tilde{\mu}_\alpha^U = \int_E u'_\alpha \, \mathrm{d}\tilde{\mu}_\alpha^U$$

for all $\alpha \in [0,1]$. This means that Definition 5.4 is well-defined.

In order to make the fuzzy-valued integrals more tractable mathematically, we need the following results.

Proposition 5.2. *Let $\{f_n\}$ be a sequence of nonnegative measurable functions on $(X,\mathcal{M})$ and $\{\mu_n\}$ be a sequence of measures on $(X,\mathcal{M})$.*

(i) *If $f_n \uparrow f$ a.e. $[\mu]$ and $\mu_n \uparrow \mu$ then*

$$\int_X f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu_n.$$

(ii) *If $f_n \downarrow f$ a.e. $[\mu_1]$ and $\mu_n \downarrow \mu$ with $f_1 \in L^1(\mu_1)$ and $\mu_1(X) < \infty$ then*

$$\int_X f \, \mathrm{d}\mu = \lim_{n\to\infty} \int_X f_n \, \mathrm{d}\mu_n.$$

P r o o f. Using the routine arguments in real analysis, the results follow from the Generalized Fatou's Lemma and Generalized Dominated Convergence Theorem in Royden [9]. □

Let $\tilde{\mu}$ be a fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. We write $\tilde{\mu}(E) \prec \infty$ if and only if $\tilde{\mu}^L_\alpha(E) < \infty$ and $\tilde{\mu}^U_\alpha(E) < \infty$ for $E \in \mathcal{M}$ and all $\alpha \in [0,1]$.

Theorem 5.1. *Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. Let $\mathcal{L}(x) = \{l_\alpha(x) \colon \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x) \colon \alpha \in [0,1]\}$ be two families of real-valued functions defined on X, and $\{\mathcal{L}, \mathcal{U}\}$ be also a canonical family with respect to $\tilde{\mu}$. Let $\tilde{f}$ be induced by $\{\mathcal{L}, \mathcal{U}\}$. If $\tilde{f}$ is integrable on E and $\tilde{\mu}(E) \prec \infty$, then we have the following results.*

(i) *If $\{\mathcal{L}, \mathcal{U}\}$ is nonnegative a.e. $[\tilde{\mu}]$ then*

$$\left(\int_E \tilde{f} \, \mathrm{d}\tilde{\mu}\right)_\alpha = \left[\int_E l_\alpha \, \mathrm{d}\tilde{\mu}^L_\alpha, \int_E u_\alpha \, \mathrm{d}\tilde{\mu}^U_\alpha\right]$$

for all $\alpha \in [0,1]$.

(ii) *If $\{\mathcal{L}, \mathcal{U}\}$ is nonpositive a.e. $[\tilde{\mu}]$ then*

$$\left(\int_E \tilde{f} \, \mathrm{d}\tilde{\mu}\right)_\alpha = \left[\int_E l_\alpha \, \mathrm{d}\tilde{\mu}^U_\alpha, \int_E u_\alpha \, \mathrm{d}\tilde{\mu}^L_\alpha\right]$$

for all $\alpha \in [0,1]$. Furthermore, the fuzzy-valued integral $\int_E \tilde{f} \, \mathrm{d}\tilde{\mu}$ is a fuzzy number.

P r o o f. Let C_α be the closed interval given in Definition 5.4. From conditions in Definition 5.2, Propositions 4.4 and 5.2, we see that the family of closed intervals $\{C_\alpha\}$ is continuously decreasing with respect to α. That is to say, $\{C_\alpha\}$ satisfies all conditions in Proposition 2.3 (ii). Therefore, using Proposition 2.3 (ii), we have $\left(\int_E \tilde{f} \, \mathrm{d}\tilde{\mu}\right)_\alpha = C_\alpha$. It is also not hard to show that the fuzzy-valued integral $\int_E \tilde{f} \, \mathrm{d}\tilde{\mu}$ is a fuzzy number. □

Theorem 5.2. *Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$, and $\tilde{f}$ be a nonnegative or nonpositive fuzzy-valued function defined on X. Suppose that $\tilde{f}^L_\alpha \in L^1(\tilde{\mu}^L_\alpha)$ and $\tilde{f}^U_\alpha \in L^1(\tilde{\mu}^U_\alpha)$ for all $\alpha \in [0,1]$. Then $\tilde{f}$ is integrable on E. We also have that*

(i) *if $\tilde{f}$ is nonnegative then*

$$\left(\int_E \tilde{f} \, \mathrm{d}\tilde{\mu}\right)_\alpha = \left[\int_E \tilde{f}^L_\alpha \, \mathrm{d}\tilde{\mu}^L_\alpha, \int_E \tilde{f}^U_\alpha \, \mathrm{d}\tilde{\mu}^U_\alpha\right]$$

for all $\alpha \in [0,1]$;

(ii) *if $\tilde{f}$ is nonpositive then*

$$\left(\int_E \tilde{f}\,\mathrm{d}\tilde{\mu}\right)_\alpha = \left[\int_E \tilde{f}^L_\alpha\,\mathrm{d}\tilde{\mu}^U_\alpha, \int_E \tilde{f}^U_\alpha\,\mathrm{d}\tilde{\mu}^L_\alpha\right]$$

for all $\alpha \in [0,1]$. Furthermore, the fuzzy-valued integral $\int_E \tilde{f}\,\mathrm{d}\tilde{\mu}$ is a fuzzy number.

Proof. We consider the families $\mathcal{L}(x) = \{\tilde{f}^L_\alpha(x)\colon \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{\tilde{f}^U_\alpha(x)\colon \alpha \in [0,1]\}$. By Proposition 3.2 (i), $\tilde{f}$ is induced by $\{\mathcal{L},\mathcal{U}\}$ on the whole domain X. Since $\tilde{f}^L_{\alpha_n} \uparrow \tilde{f}^L_\alpha$, $\tilde{f}^U_{\alpha_n} \downarrow \tilde{f}^U_\alpha$, $\tilde{\mu}^L_{\alpha_n} \uparrow \tilde{\mu}^L_\alpha$ and $\tilde{\mu}^U_{\alpha_n} \downarrow \tilde{\mu}^U_\alpha$ for $\alpha_n \uparrow \alpha$ from Proposition 5.2 (ii), the result follows from Propositions 5.2 and 2.3 (ii) using similar arguments as in the proof of Theorem 5.1. □

Proposition 5.3. *Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X,\mathcal{M})$. Let $\tilde{f}$ and $\tilde{g}$ be pseudo-fuzzy-valued measurable functions induced by two canonical families $\{\mathcal{L},\mathcal{U}\}$ and $\{\bar{\mathcal{L}},\overline{\mathcal{U}}\}$ with respect to $\tilde{\mu}$, respectively. Suppose that $\{\mathcal{L},\mathcal{U}\}$ and $\{\bar{\mathcal{L}},\overline{\mathcal{U}}\}$ are nonnegative or nonpositive a.e. $[\tilde{\mu}]$ simultaneously, and that $\tilde{h} \approx \tilde{f} \oplus \tilde{g}$. If $\tilde{f}$ and $\tilde{g}$ are integrable on E and $\tilde{\mu}(E) \prec \infty$, then $\tilde{h}$ is also integrable on E, and*

$$\int_E \tilde{h}\,\mathrm{d}\tilde{\mu} = \int_E \tilde{f}\,\mathrm{d}\tilde{\mu} \oplus \int_E \tilde{g}\,\mathrm{d}\tilde{\mu}.$$

Proof. Now $\widehat{\mathcal{L}} = \mathcal{L} \oplus_{\mathrm{fct}} \bar{\mathcal{L}}$ and $\widehat{\mathcal{U}} = \mathcal{U} \oplus_{\mathrm{fct}} \overline{\mathcal{U}}$. From Proposition 4.4 and the similar arguments in the proof of Proposition 5.1, it is not hard to show that $\{\widehat{\mathcal{L}},\widehat{\mathcal{U}}\}$ is a canonical family with respect to $\tilde{\mu}$ which induces $\tilde{h}$. Since $\tilde{f}$ and $\tilde{g}$ are integrable on E, using Theorem 5.1 and Proposition 2.2, we see that $\tilde{h}$ is integrable on E and

$$\left(\int_E \tilde{h}\,\mathrm{d}\tilde{\mu}\right)_\alpha = \left(\int_E \tilde{f}\,\mathrm{d}\tilde{\mu} \oplus \int_E \tilde{g}\,\mathrm{d}\tilde{\mu}\right)_\alpha$$

for all $\alpha \in [0,1]$. Similarly for the nonpositive case. This completes the proof. □

Proposition 5.4. *Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X,\mathcal{M})$. Let $\tilde{f}$ and $\tilde{g}$ be nonnegative or nonpositive fuzzy-valued functions simultaneously. If $\tilde{f}$ and $\tilde{g}$ are integrable on E, then $\tilde{h} = \tilde{f} \oplus \tilde{g}$ is also integrable on E and*

$$\int_E \tilde{h} d\tilde{\mu} = \int_E \tilde{f}\,\mathrm{d}\tilde{\mu} \oplus \int_E \tilde{g}\,\mathrm{d}\tilde{\mu}.$$

Proof. The result follows by using similar arguments as in the proofs of Theorem 5.2 and Proposition 5.3. □

In the sequel, we shall introduce the fuzzy-valued intergal of the general case, i.e., the fuzzy-valued function $\tilde{f}$ is not restricted to nonnegative or nonpositive case. Let $A(x) = [l(x), u(x)]$, where l and u are real-valued functions defined on X with $l \leqslant u$. We define $A^+(x) = [l^+(x), u^+(x)]$ and $A^-(x) = [l^-(x), u^-(x)]$, where $l^+(x) = \max\{l(x), 0\}$, $u^+(x) = \max\{u(x), 0\}$, $l^-(x) = \min\{0, l(x)\}$ and $u^-(x) = \min\{0, u(x)\}$. Then we have $l(x) = l^+(x) + l^-(x)$ and $u(x) = u^+(x) + u^-(x)$. Thus $A(x) = A^+(x) \oplus_{\rm int} A^-(x)$.

Let $\mathcal{L}(x) = \{l_\alpha(x)\colon \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x)\colon \alpha \in [0,1]\}$ be two families of real-valued functions defined on X. We have a family of decreasing closed intervals $\{A_\alpha(x)\}$ from $\{\mathcal{L},\mathcal{U}\}$. Let $\mathcal{L}^+(x) = \{l^+_\alpha(x)\}$, $\mathcal{L}^-(x) = \{l^-_\alpha(x)\}$, $\mathcal{U}^+(x) = \{u^+_\alpha(x)\}$ and $\mathcal{U}^-(x) = \{u^-_\alpha(x)\}$. Then we have the corresponding families of decreasing closed intervals $\{A^+_\alpha(x)\}$ and $\{A^-_\alpha(x)\}$ from $\{\mathcal{L}^+,\mathcal{U}^+\}$ and $\{\mathcal{L}^-,\mathcal{U}^-\}$, respectively. We can see that $A_\alpha(x) = A^+_\alpha(x) \oplus_{\rm int} A^-_\alpha(x)$ for $x \in E_{\mathcal{L}\mathcal{U}}$. Let $\tilde{f}$, $\tilde{f}^{++}$ and $\tilde{f}^{--}$ be induced by $\{\mathcal{L},\mathcal{U}\}$, $\{\mathcal{L}^+,\mathcal{U}^+\}$ and $\{\mathcal{L}^-,\mathcal{U}^-\}$, respectively, where $\mathcal{L} = \mathcal{L}^+ \oplus_{\rm fct} \mathcal{L}^-$ and $\mathcal{U} = \mathcal{U}^+ \oplus_{\rm fct} \mathcal{U}^-$.

R e m a r k 5.1. Since $\tilde{f}(x)$ is a fuzzy number for any fixed $x \in X$, we see that $\tilde{f}^+(x)$ and $\tilde{f}^-(x)$ are the positive and negative parts of $\tilde{f}(x)$, respectively, and $\tilde{f}(x) = \tilde{f}^+(x) \oplus \tilde{f}^-(x)$ for any fixed $x \in X$ by looking at (1). Therefore, $\tilde{f}$ can induce two fuzzy-valued functions $\tilde{f}^+$ and $\tilde{f}^-$ such that $\tilde{f} = \tilde{f}^+ \oplus \tilde{f}^-$. From Proposition 2.6, $\tilde{f}^{++}(x) = \tilde{f}^+(x)$ and $\tilde{f}^{--}(x) = \tilde{f}^-(x)$ for $x \in E_{\mathcal{L}\mathcal{U}}$, i.e., $\tilde{f}(x) = \tilde{f}^{++}(x) \oplus \tilde{f}^{--}(x)$ for $x \in E_{\mathcal{L}\mathcal{U}}$.

Definition 5.5. Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. Let $\mathcal{L}(x) = \{l_\alpha(x)\colon \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{u_\alpha(x)\colon \alpha \in [0,1]\}$ be two families of real-valued functions defined on X such that $\{\mathcal{L}^+,\mathcal{U}^+\}$ and $\{\mathcal{L}^-,\mathcal{U}^-\}$ are two canonical families with respect to $\tilde{\mu}$, where $\{\mathcal{L}^+,\mathcal{U}^+\}$ is nonnegative a.e. $[\tilde{\mu}]$ and $\{\mathcal{L}^-,\mathcal{U}^-\}$ is nonpositive a.e. $[\tilde{\mu}]$. Let $\tilde{f}$, $\tilde{f}^{++}$ and $\tilde{f}^{--}$ be induced by $\{\mathcal{L},\mathcal{U}\}$, $\{\mathcal{L}^+,\mathcal{U}^+\}$ and $\{\mathcal{L}^-,\mathcal{U}^-\}$, respectively. If $\tilde{f}^{++}$ and $\tilde{f}^{--}$ are integrable on E, then we say that $\tilde{f}$ is integrable on E, and the fuzzy-valued integral $\int_E \tilde{f}\,{\rm d}\tilde{\mu}$ is defined by

$$\int_E \tilde{f}\,{\rm d}\tilde{\mu} = \int_E \tilde{f}^{++}\,{\rm d}\tilde{\mu} \oplus \int_E \tilde{f}^{--}\,{\rm d}\tilde{\mu}.$$

R e m a r k 5.2. From Theorem 5.1 and Proposition 2.2, $\int_E \tilde{f}\,{\rm d}\tilde{\mu}$ is a fuzzy number and

$$\left(\int_E \tilde{f}\,{\rm d}\tilde{\mu}\right)_\alpha = \left[\int_E l^+_\alpha\,{\rm d}\tilde{\mu}^L_\alpha + \int_E l^-_\alpha\,{\rm d}\tilde{\mu}^U_\alpha, \int_E u^+_\alpha\,{\rm d}\tilde{\mu}^U_\alpha + \int_E u^-_\alpha\,{\rm d}\tilde{\mu}^L_\alpha\right].$$

Theorem 5.3. *Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. Let $\tilde{f}$ be a fuzzy-valued function defined on X. If $\tilde{f}^{+}$ and $\tilde{f}^{-}$ are integrable on E, then $\tilde{f}$ is also integrable on E and*

$$\int_E \tilde{f}\,\mathrm{d}\tilde{\mu} = \int_E \tilde{f}^{+}\,\mathrm{d}\tilde{\mu} \oplus \int_E \tilde{f}^{-}\,\mathrm{d}\tilde{\mu}.$$

Proof. We consider the families $\mathcal{L}(x) = \{\tilde{f}_\alpha^L(x)\colon\ \alpha \in [0,1]\}$ and $\mathcal{U}(x) = \{\tilde{f}_\alpha^U(x)\colon\ \alpha \in [0,1]\}$. Then $E_{\mathcal{L}\mathcal{U}} = X$ (the whole domain) from Proposition 3.2. From Remark 5.1, we see that $\tilde{f}^{++}(x) = \tilde{f}^{+}(x)$ and $\tilde{f}^{--}(x) = \tilde{f}^{-}(x)$ for $x \in E_{\mathcal{L}\mathcal{U}} = X$. The result follows from Remark 5.2 and Theorem 5.2 immediately. □

Proposition 5.5. *Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$. Let $\tilde{f}$ and $\tilde{g}$ be induced by two families $\{\mathcal{L},\mathcal{U}\}$ and $\{\bar{\mathcal{L}},\overline{\mathcal{U}}\}$, respectively. Suppose that $\{\mathcal{L}^{+},\mathcal{U}^{+}\}$, $\{\bar{\mathcal{L}}^{+},\overline{\mathcal{U}}^{+}\}$, $\{\mathcal{L}^{-},\mathcal{U}^{-}\}$ and $\{\bar{\mathcal{L}}^{-},\overline{\mathcal{U}}^{-}\}$ are canonical families with respect to $\tilde{\mu}$. We further assume that $l_\alpha(x)$ and $\bar{l}_\alpha(x)$ have the same sign for each x (i.e., $l_\alpha(x) \cdot \bar{l}_\alpha(x) \geqslant 0$) and for all $\alpha \in [0,1]$, and $u_\alpha(x)$ and $\bar{u}_\alpha(x)$ also have the same sign for each x and for all $\alpha \in [0,1]$. Suppose that $\tilde{h} \approx \tilde{f} \oplus \tilde{g}$. If $\tilde{f}$ and $\tilde{g}$ are integrable on E, then $\tilde{h}$ is also integrable on E and*

$$\int_E \tilde{h}\,\mathrm{d}\tilde{\mu} = \int_E \tilde{f}\,\mathrm{d}\tilde{\mu} \oplus \int_E \tilde{g}\,\mathrm{d}\tilde{\mu}.$$

Proof. Let $\hat{\mathcal{L}}^{+} = \mathcal{L}^{+} \oplus_{\mathrm{fct}} \bar{\mathcal{L}}^{+}$, $\hat{\mathcal{U}}^{+} = \mathcal{U}^{+} \oplus_{\mathrm{fct}} \overline{\mathcal{U}}^{+}$, $\hat{\mathcal{L}}^{-} = \mathcal{L}^{-} \oplus_{\mathrm{fct}} \bar{\mathcal{L}}^{-}$ and $\hat{\mathcal{U}}^{-} = \mathcal{U}^{-} \oplus_{\mathrm{fct}} \overline{\mathcal{U}}^{-}$. Using similar arguments as in the proof of Proposition 5.3, we can see that $\{\hat{\mathcal{L}}^{+},\hat{\mathcal{U}}^{+}\}$ and $\{\hat{\mathcal{L}}^{-},\hat{\mathcal{U}}^{-}\}$ are two canonical families with respect to $\tilde{\mu}$. We also have $\hat{l}_\alpha = l_\alpha + \bar{l}_\alpha$ and $\hat{u}_\alpha = u_\alpha + \bar{u}_\alpha$. Thus $\hat{l}_\alpha^{+} + \hat{l}_\alpha^{-} = l_\alpha^{+} + l_\alpha^{-} + \bar{l}_\alpha^{+} + \bar{l}_\alpha^{-}$ and $\hat{u}_\alpha^{+} + \hat{u}_\alpha^{-} = u_\alpha^{+} + u_\alpha^{-} + \bar{u}_\alpha^{+} + \bar{u}_\alpha^{-}$. Since $l_\alpha(x)$ and $\bar{l}_\alpha(x)$ have the same sign for each x, we have $\hat{l}_\alpha^{+} = l_\alpha^{+} + \bar{l}_\alpha^{+}$ and $\hat{l}_\alpha^{-} = l_\alpha^{-} + \bar{l}_\alpha^{-}$. Similarly, we also have $\hat{u}_\alpha^{+} = u_\alpha^{+} + \bar{u}_\alpha^{+}$ and $\hat{u}_\alpha^{-} = u_\alpha^{-} + \bar{u}_\alpha^{-}$. Now, from Remark 5.2 and Proposition 2.2, we have

$$\left(\int_E \tilde{h}\,\mathrm{d}\mu\right)_\alpha = \left(\int_E \tilde{f}\,\mathrm{d}\mu \oplus \int_E \tilde{g}\,\mathrm{d}\mu\right)_\alpha$$

for all $\alpha \in [0,1]$. This completes the proof. □

We shall discuss the Dominated Convergence Theorem for the fuzzy-valued integrals with respect to fuzzy-valued measures.

Definition 6.1. Let $\tilde{a}$ be a fuzzy number. We call $\tilde{a}$ a canonical fuzzy number if $\tilde{a}^L_\alpha$ and $\tilde{a}^U_\alpha$ are continuous with respect to α on $[0,1]$.

We also need the following results for canonical fuzzy numbers.

Proposition 6.1. *Let $\tilde{a}$ and $\tilde{b}$ be two canonical fuzzy numbers. Then $d_{\mathcal{F}}(\tilde{a},\tilde{b}) < \varepsilon$ if and only if $|\tilde{a}^L_\alpha - \tilde{b}^L_\alpha| < \varepsilon$ and $|\tilde{a}^U_\alpha - \tilde{b}^U_\alpha| < \varepsilon$ for all $\alpha \in [0,1]$.*

Proof. For a compact set S in $\mathbb{R}^n$, from Bazaraa et al. [2], if f is upper semicontinuous on S then f assumes maximum over S, and if f is lower semicontinuous on S then f assumes minimum over S. Therefore the result follows from Propositions 4.1 immediately. □

We denote by $\mathcal{F}_c(\mathbb{R})$ the set of all canonical fuzzy numbers. If a function $\tilde{f}$ is given by $\tilde{f}\colon X \to \mathcal{F}_c(\mathbb{R})$, then $\tilde{f}$ is called a canonical fuzzy-valued function. Next we are going to discuss the Dominated Convergence Theorem for canonical fuzzy-valued functions.

From Eq. (3), if $F^L_{\alpha;A}$ and $F^U_{\alpha;A}$ are re-defined as follows

$$F^L_{\alpha;A} = \{x \in X\colon\ l_{\alpha_n}(x) \to l_\alpha(x)\ \text{ for }\ \alpha_n \to \alpha\}$$

and

$$F^U_{\alpha;A} = \{x \in X\colon\ u_{\alpha_n}(x) \to u_\alpha(x)\ \text{ for }\ \alpha_n \to \alpha\}$$

(the difference is considering $\alpha_n \to \alpha$, not $\alpha_n \uparrow \alpha$), then, from Proposition 2.4 (note that this proposition still holds true for canonical fuzzy number if condition (iii) is replaced by continuity instead of left-continuity), $\tilde{f}(x)$ is a canonical fuzzy number for each $x \in G_A$. In this case, we also call $\tilde{f}$ a canonical pseudo-fuzzy-valued function induced by $\{\mathcal{L},\mathcal{U}\}$.

Theorem 6.1 (Dominated Convergence Theorem). *Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X,\mathcal{M})$ with $\tilde{\mu}(X) \prec \infty$. For each $n = 1,2,\ldots$, let $\mathcal{L}_n(x) = \{l^{(n)}_\alpha(x)\colon\ \alpha \in [0,1]\}$ and $\mathcal{U}_n(x) = \{u^{(n)}_\alpha(x)\colon\ \alpha \in [0,1]\}$ be two families of real-valued functions defined on X, and $\{\mathcal{L}_n,\mathcal{U}_n\}$ be two canonical families with respect to $\tilde{\mu}$. Let $\tilde{f}_n$ be a canonical pseudo-fuzzy-valued function induced by $\{\mathcal{L}_n,\mathcal{U}_n\}$ for each $n = 1,2,\ldots$. We assume that the following conditions are satisfied:*

(i) *each $\tilde{f}_n$ is integrable on E for $n = 1,2,\ldots$;*

(ii) *for* $n \to \infty$, $(l_\alpha^{(n)})^+(x) \to l^+(x)$, $(l_\alpha^{(n)})^-(x) \to l^-(x)$, $(u_\alpha^{(n)})^+(x) \to u^+(x)$ *and* $(u_\alpha^{(n)})^-(x) \to u^-(x)$ *uniformly with respect to* α *on* $[0,1]$ *for any fixed* $x \in X$;

(iii) *there exist nonnegative functions* $g^L \in L^1(\tilde{\mu}_\alpha^L)$ *and* $g^U \in L^1(\tilde{\mu}_\alpha^U)$ *for all* $\alpha \in [0,1]$ *such that* $g^L \geqslant \max\{(l_\alpha^{(n)})^+, |(u_\alpha^{(n)})^-|\}$ *and* $g^U \geqslant \max\{(u_\alpha^{(n)})^+, |(l_\alpha^{(n)})^-|\}$ *for each* $n = 1, 2, \ldots$ *and all* $\alpha \in [0,1]$.

Then the canonical pseudo-fuzzy-valued function $\tilde{f}$ *induced by the families* $\mathcal{L}(x) = \{l_\alpha(x) = l^+(x) + l^-(x)\colon\ \alpha \in [0,1]\}$ *and* $\mathcal{U}(x) = \{u_\alpha(x) = u^+(x) + u^-(x)\colon\ \alpha \in [0,1]\}$ *is integrable on* E *and we also have*

$$\lim_{n\to\infty} \int_E \tilde{f}_n \,\mathrm{d}\tilde{\mu} = \int_E \tilde{f} \,\mathrm{d}\tilde{\mu}.$$

Proof. From condition (ii), we see that $l_\alpha^{(n)}(x) \to l(x)$ and $u_\alpha^{(n)}(x) \to u(x)$ uniformly with respect to α on $[0,1]$ for any fixed x. Since $(l_\alpha^{(n)})^+ \leqslant (l_1^{(n)})^+$ a.e. $[\tilde{\mu}_1^L]$, we have the inequality $\int_E (l_\alpha^{(n)})^+ \,\mathrm{d}\tilde{\mu}_1^L \leqslant \int_E (l_1^{(n)})^+ \,\mathrm{d}\tilde{\mu}_1^L$. This shows that $(l_\alpha^{(n)})^+ \in L^1(\tilde{\mu}_1^L)$, since $\tilde{f}_n$ is integrable, i.e., $(l_1^{(n)})^+ \in L^1(\tilde{\mu}_1^L)$. Similarly, since $(u_\alpha^{(n)})^- \in L^1(\tilde{\mu}_\alpha^L)$, $(u_0^{(n)})^+ \in L^1(\tilde{\mu}_0^U)$, $(l_\alpha^{(n)})^- \in L^1(\tilde{\mu}_\alpha^U)$ (note that $(l_\alpha^{(n)})^-$ and $(u_\alpha^{(n)})^-$ are nonpositive) and $\int_E (u_\alpha^{(n)})^- \,\mathrm{d}\tilde{\mu}_1^L \leqslant \int_E (u_\alpha^{(n)})^- \,\mathrm{d}\tilde{\mu}_\alpha^L$, $\int_E (u_\alpha^{(n)})^+ \,\mathrm{d}\tilde{\mu}_0^U \leqslant \int_E (u_0^{(n)})^+ \,\mathrm{d}\tilde{\mu}_0^U$, $\int_E (l_\alpha^{(n)})^- \,\mathrm{d}\tilde{\mu}_0^U \leqslant \int_E (l_\alpha^{(n)})^- \,\mathrm{d}\tilde{\mu}_\alpha^U$, we have $(u_\alpha^{(n)})^- \in L^1(\tilde{\mu}_1^L)$ and $(u_\alpha^{(n)})^+, (l_\alpha^{(n)})^- \in L^1(\tilde{\mu}_0^U)$ for each $n = 1, 2, \ldots$ and all $\alpha \in [0,1]$. Since the convergence is independent of α in condition (ii), $(l_\alpha^{(n)})^+ \in L^1(\tilde{\mu}_1^L)$ and $(l_\alpha^{(n)})^- \in L^1(\tilde{\mu}_0^U)$, from condition (iii) and using the Lebesgue Dominated Convergence Theorem, we have

$$(6) \qquad \left| \int_E (l_\alpha^{(n)})^+ \,\mathrm{d}\tilde{\mu}_1^L - \int_E l_\alpha^+ \,\mathrm{d}\tilde{\mu}_1^L \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \int_E (l_\alpha^{(n)})^- \,\mathrm{d}\tilde{\mu}_0^U - \int_E l_\alpha^- \,\mathrm{d}\tilde{\mu}_0^U \right| < \frac{\varepsilon}{2}$$

for all $\alpha \in [0,1]$ (i.e., independent of α) for n sufficiently large. From Remark 5.2 and (6), we can show that

$$\left| \left(\int_E \tilde{f}_n \,\mathrm{d}\tilde{\mu} \right)_\alpha^L - \left(\int_E \tilde{f} \,\mathrm{d}\tilde{\mu} \right)_\alpha^L \right| < \varepsilon$$

for n sufficiently large and all $\alpha \in [0,1]$. Similarly, we also have

$$\left| \left(\int_E \tilde{f}_n \,\mathrm{d}\tilde{\mu} \right)_\alpha^U - \left(\int_E \tilde{f} \,\mathrm{d}\tilde{\mu} \right)_\alpha^U \right| < \varepsilon$$

for n sufficiently large and all $\alpha \in [0,1]$. Thus the result follows from Proposition 6.1 immediately. □

In the sequel, we are going to discuss the Dominated Convergence Theorem for fuzzy-valued functions. Let $\{\tilde{f}_n\}$ be a sequence of fuzzy-valued functions that are integrable on E and dominated by a nonnegative integrable fuzzy-valued function such that the limit function of $\{\tilde{f}_n\}$ exists. Then we are going to show that

$$\lim_{n\to\infty} \int_E \tilde{f}_n \, \mathrm{d}\tilde{\mu} = \int_E \tilde{f} \, \mathrm{d}\tilde{\mu},$$

where $\tilde{\mu}$ is a canonical fuzzy-valued measure.

Now we are going to fuzzify a nonfuzzy-valued function. Recall that $\mathcal{F}$ denotes the set of all fuzzy subsets of $\mathbb{R}$. Let $f\colon \mathbb{R}^n \to \mathbb{R}$ be a nonfuzzy-valued function (i.e., a real-valued function defined on $\mathbb{R}^n$) and $\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n$ be n fuzzy subsets of $\mathbb{R}$. By the extension principle in Zadeh [16] and Nguyen [7], we can induce a function $\tilde{f}\colon \mathcal{F}^n \to \mathcal{F}$ from the nonfuzzy-valued function f. That is to say, $\tilde{f}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ is a fuzzy subset of $\mathbb{R}$. The membership function of $\tilde{f}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)$ is defined by

$$\xi_{\tilde{f}(\tilde{A}_1, \tilde{A}_2, \ldots, \tilde{A}_n)}(r) = \sup_{\{(x_1,\ldots,x_n)\colon\, r = f(x_1,\ldots,x_n)\}} \min\{\xi_{\tilde{A}_1}(x_1), \ldots, \xi_{\tilde{A}_n}(x_n)\}. \tag{7}$$

Now we can define the meaning of the absolute value of a fuzzy number. Let $\tilde{a}$ be a fuzzy number and $f(x) = |x|$. Then we can consider the fuzzy subset $|\tilde{a}|$ induced by the real-valued function $f(x) = |x|$ using Eq. (7). It is not hard to show that $|\tilde{a}|$ is a fuzzy number and

$$|\tilde{a}|_\alpha = \{|r|\colon\ r \in \tilde{a}_\alpha\} \tag{8}$$

for all $\alpha \in [0,1]$. Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. We write $\tilde{a} \succeq \tilde{b}$ if and only if $\tilde{a}^L_\alpha \geqslant \tilde{b}^L_\alpha$ and $\tilde{a}^U_\alpha \geqslant \tilde{b}^U_\alpha$ for all $\alpha \in [0,1]$. Then "$\succeq$" is a partial ordering on $\mathcal{F}(\mathbb{R})$. The following results are not hard to prove by using routine arguments.

Proposition 6.2. *Let $\{\tilde{a}_n\}$ be a sequence of fuzzy numbers. Then*

$$\lim_{n\to\infty} \tilde{a}_n = \tilde{a} \quad \textit{if and only if} \quad \lim_{n\to\infty} \tilde{a}_n^+ = \tilde{a}^+ \ \textit{and} \ \lim_{n\to\infty} \tilde{a}_n^- = \tilde{a}^-.$$

Proposition 6.3. *Let $\tilde{a}$ and $\tilde{b}$ be two fuzzy numbers. If $\tilde{a} \succeq |\tilde{b}|$, then we have*

(i) $\tilde{a}^L_\alpha \geqslant (\tilde{b}^+)^L_\alpha$ *and* $\tilde{a}^L_\alpha \geqslant |(\tilde{b}^-)^U_\alpha|$ *for all* $\alpha \in [0,1]$;

(ii) $\tilde{a}^U_\alpha \geqslant (\tilde{b}^+)^U_\alpha$ *and* $\tilde{a}^U_\alpha \geqslant |(\tilde{b}^-)^L_\alpha|$ *for all* $\alpha \in [0,1]$.

We are going to apply Theorems 5.2 and 5.3 to deduce the following Dominated Convergence Theorem.

Theorem 6.2 (Dominated Convergence Theorem). *Let $\tilde{\mu}$ be a canonical fuzzy-valued measure on a measurable space $(X, \mathcal{M})$ with $\tilde{\mu}(X) \prec \infty$ and $\{\tilde{f}_n\}$ be a sequence of integrable fuzzy-valued functions with respect to $\tilde{\mu}$ on E such that the limit function $\lim\limits_{n\to\infty} \tilde{f}_n(x) = \tilde{f}(x)$ exists. If there exists a nonnegative integrable fuzzy-valued function $\tilde{g}(x)$ with respect to $\tilde{\mu}$ on E such that $\tilde{g}(x) \succeq |\tilde{f}_n(x)|$ for all $n = 1, 2, \ldots$, then*

$$\lim_{n\to\infty} \int_E \tilde{f}_n \, \mathrm{d}\tilde{\mu} = \int_E \tilde{f} \, \mathrm{d}\tilde{\mu}.$$

Proof. Since $\tilde{g}$ is integrable, we have $\tilde{g}^L_\alpha \in L^1(\tilde{\mu}^L_\alpha)$ and $\tilde{g}^U_\alpha \in L^1(\tilde{\mu}^U_\alpha)$ for all $\alpha \in [0, 1]$. From Propositions 6.3 and 2.1, we have $\tilde{g}^L_1 \geqslant \tilde{g}^L_\alpha \geqslant (\tilde{f}^+_n)^L_\alpha$ and $\tilde{g}^L_1 \geqslant \tilde{g}^L_\alpha \geqslant |(\tilde{f}^-_n)^U_\alpha|$ for all $\alpha \in [0, 1]$, and $\tilde{g}^U_0 \geqslant \tilde{g}^U_\alpha \geqslant (\tilde{f}^+_n)^U_\alpha$ and $\tilde{g}^U_0 \geqslant \tilde{g}^U_\alpha \geqslant |(\tilde{f}^-_n)^L_\alpha|$ for all $\alpha \in [0, 1]$ (i.e., independent of α). Now we consider the following inequality

$$\int_E (\tilde{f}^+_n)^L_\alpha \, \mathrm{d}\tilde{\mu}^L_1 \leqslant \int_E (\tilde{f}^+_n)^L_1 \, \mathrm{d}\tilde{\mu}^L_1. \tag{9}$$

Since $\tilde{f}^+_n$ is integrable, i.e., $(\tilde{f}^+_n)^L_\alpha \in L^1(\tilde{\mu}^L_\alpha)$ for all $\alpha \in [0, 1]$, it follows that $(\tilde{f}^+_n)^L_\alpha \in L^1(\tilde{\mu}^L_1)$ from (9). Similarly, since $(\tilde{f}^-_n)^U_\alpha \in L^1(\tilde{\mu}^L_\alpha)$, $(\tilde{f}^-_n)^L_\alpha \in L^1(\tilde{\mu}^U_\alpha)$, $(\tilde{f}^+_n)^U_0 \in L^1(\tilde{\mu}^U_0)$ (note that $(\tilde{f}^-_n)^L_\alpha$ and $(\tilde{f}^-_n)^U_\alpha$ are nonpositive) and $\int_E (\tilde{f}^-_n)^U_\alpha \, \mathrm{d}\tilde{\mu}^L_1 \leqslant \int_E (\tilde{f}^-_n)^U_\alpha \, \mathrm{d}\tilde{\mu}^L_\alpha$, $\int_E (\tilde{f}^-_n)^L_\alpha \, \mathrm{d}\tilde{\mu}^U_0 \leqslant \int_E (\tilde{f}^-_n)^L_\alpha \, \mathrm{d}\tilde{\mu}^U_\alpha$, $\int_E (\tilde{f}^+_n)^U_\alpha \, \mathrm{d}\tilde{\mu}^U_0 \leqslant \int_E (\tilde{f}^+_n)^U_0 \, \mathrm{d}\tilde{\mu}^U_0$, we have $(\tilde{f}^-_n)^U_\alpha \in L^1(\tilde{\mu}^L_1)$ and $(\tilde{f}^-_n)^L_\alpha$, $(\tilde{f}^+_n)^U_\alpha \in L^1(\tilde{\mu}^U_0)$ for each $n = 1, 2, \ldots$ and all $\alpha \in [0, 1]$. Since $(\tilde{f}^+_n)^L_\alpha \in L^1(\tilde{\mu}^L_1)$ and $(\tilde{f}^-_n)^L_\alpha \in L^1(\tilde{\mu}^U_0)$ for each $n = 1, 2, \ldots$ and all $\alpha \in [0, 1]$, using Propositions 4.2, 6.2 and the Lebesgue's Dominated Convergence Theorem, we have

$$\left| \int_E (\tilde{f}^+_n)^L_\alpha \, \mathrm{d}\tilde{\mu}^L_1 - \int_E (\tilde{f}^+)^L_\alpha \, \mathrm{d}\tilde{\mu}^L_1 \right| < \frac{\varepsilon}{2} \quad \text{and} \quad \left| \int_E (\tilde{f}^-_n)^L_\alpha \, \mathrm{d}\tilde{\mu}^U_0 - \int_E (\tilde{f}^-)^L_\alpha \, \mathrm{d}\tilde{\mu}^U_0 \right| < \frac{\varepsilon}{2}$$

for n sufficiently large and all $\alpha \in [0, 1]$ (i.e., independent of α). From Theorems 5.2 and 5.3, we can show that

$$\left| \left(\int_E \tilde{f}_n \, \mathrm{d}\tilde{\mu} \right)^L_\alpha - \left(\int_E \tilde{f} \, \mathrm{d}\tilde{\mu} \right)^L_\alpha \right| < \varepsilon$$

for n sufficiently large and all $\alpha \in [0, 1]$. Similarly, we also have

$$\left| \left(\int_E \tilde{f}_n \, \mathrm{d}\tilde{\mu} \right)^U_\alpha - \left(\int_E \tilde{f} \, \mathrm{d}\tilde{\mu} \right)^U_\alpha \right| < \varepsilon$$

for n sufficiently large and all $\alpha \in [0, 1]$. The result follows from Proposition 6.1 immediately. □

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