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MIXED FINITE ELEMENT ANALYSIS OF SEMI-COERCIVE UNILATERAL CONTACT PROBLEMS WITH GIVEN FRICTION

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Abstract. A unilateral contact 2D-problem is considered provided one of two elastic bodies can shift in a given direction as a rigid body. Using Lagrange multipliers for both normal and tangential constraints on the contact interface, we introduce a saddle point problem and prove its unique solvability. We discretize the problem by a standard finite element method and prove a convergence of approximations. We propose a numerical realization on the basis of an auxiliary "bolted" problem and the algorithm of Uzawa.

Keywords: unilateral contact, Tresca's model of friction, mixed variational formulation, Uzawa algorithm

MSC 2000: 49J40, 65N30, 74M10, 74M15

Introduction

Unilateral contact of elastic bodies has been modelled in terms of displacements by many authors (see e.g. [9], [14], [16], [12] and the literature therein). The weak variational formulation leads to elliptic variational inequalities. If a given friction (Tresca's model) has to be considered, a non-differentiable term appears in the inequality, making the approximate solution more complicated. A remedy can be a mixed variational formulation of the problem which employs Lagrange multipliers ([8], [9], [10], [11], [12], [17], [2]).

Piecewise linear finite elements are used for the approximation of displacements, as a rule. Lagrange multipliers, which can be interpreted as components of the contact

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stress vector, are approximated either by piecewise constant functions ([8], [9], [10], [11]), or by piecewise linear functions as in [2].

The present paper seems to be the first attempt to analyse the mixed variational formulation of semi-coercive unilateral contact problems with given friction. Indeed, any of the papers mentioned above deals with either a primal formulation (i.e., in terms of displacements) for semi-coercive problems ([9], [14], [12]) or mixed formulation for a coercive problem ([8], [10], [11], [12], [17], [2]). The only analysis of the mixed finite element method of semi-coercive problems was presented in [15] for a contact with zero friction. In [3] numerical experiments were displayed for a related semi-coercive problem, though the theory was restricted to coercive frictionless unilateral contact problems.

Section 1 of the present paper is devoted to a primal variational formulation of a semi-coercive unilateral contact problem with given friction, following [14]. We deduce sufficient conditions for the existence and uniqueness of a weak solution. A mixed variational formulation of the same contact problem is introduced in Section 2 in the form of a saddle point problem.

In Section 3 we propose and analyse a discretized saddle point problem. Standard spaces of linear shape functions on regular triangulations are used for displacements. Traces of these spaces on the contact interface are employed for the first Lagrange multiplier (normal contact force), following [2], [3], whereas piecewise constant approximations are used for the second multiplier (tangential contact force), as in [8], [9], [10], [11], [17].

Proofs of existence and uniqueness of a saddle point are given on both the continuous and the discretized level. Section 4 contains a convergence analysis for the mesh sizes tending to zero. We prove that the approximate displacements converge in the H^1-norm, the approximate tangential Lagrange multipliers converge in L^∞ weakly star, whereas the approximate normal multipliers converge in a functional sense. We do not use any additional assumption about a regularity of the exact solution.

To establish an effective numerical realization of the discretized saddle point problem, we introduce an auxiliary contact problem in Section 5, assuming that the two bodies under consideration are "bolted" together at a suitable nodal point of the interface (see [7], [15]). Then we propose and analyse an algorithm of Uzawa for the "bolted" saddle point problem. We show in Section 6 that the iterations of the algorithm can be constructed for every single elastic body separately, as in [15].

1. Primal variational formulation

Let us consider two elastic bodies occupying bounded domains Ω^1 and Ω^2 of $\mathbb{R}^2$ with Lipschitz boundaries $\partial\Omega^M$, $M = 1, 2$. We assume the stress-strain law

$$\tau_{ij}^M = c_{ijkm}^M \varepsilon_{km}^M, \quad i, j, k, m = 1, 2,$$

where

$$\varepsilon_{km}^M = \frac{1}{2}(\partial v_k^M / \partial x_m + \partial v_m^M / \partial x_k),$$

v_k^M are components of the displacement vector,

$$c_{ijkm}^M \in L^\infty(\Omega^M), \quad c_{ijkm}^M = c_{kmij}^M = c_{jikm}^M,$$
$$c_{ijkm}^M \xi_{ij}\xi_{km} \geqslant c_0 \xi_{ij}\xi_{ij}$$

holds for a.a. $x \in \Omega^M$ and for all symmetric matrices (ξ_{ij}). Henceforth we use the summation convention (any repeated subscript implies summation within $\{1, 2\}$).

Let

$$T_n(v) = \tau_{ij}^M(v) n_i^M n_j^M, \quad T_t(v) = \tau_{ij}^M(v) n_i^M t_j^M$$

denote the normal and tangential components of the stress vector $T_i^M(v) = \tau_{ij}^M(v) n_j^M$.

Let the body Ω^1 be fixed on a closed part $\Gamma_u \subset \partial\Omega^1$, where $\operatorname{meas}\Gamma_u > 0$. The body Ω^2 has bilateral contact conditions $v_i n_i^2 \equiv v \cdot n^2 = 0$, $T_t(v) = 0$ on a (closed) part $\Gamma_0 \subset \partial\Omega^2$ with $\operatorname{meas}\Gamma_0 > 0$. Let Γ_0 be contained in straight lines parallel with the x_2-axis. We denote by

$$\Gamma_c = \partial\Omega^1 \cap \partial\Omega^2$$

the common part of the boundaries and assume that $\operatorname{meas}\Gamma_c > 0$ and $\Gamma_c \cap \Gamma_u = \emptyset$.

Let surface loads P^1, P^2 be prescribed on the remaining parts of $\partial\Omega^1$ and $\partial\Omega^2$, respectively.

On Γ_c we consider a unilateral contact with given friction (i.e. Tresca's model of friction) as follows:

$$[v_n] \leqslant 0, \quad T_n(v) \leqslant 0, \quad [v_n] T_n(v) = 0,$$
$$|T_t(v)| \leqslant g, \quad |T_t(v)| < g \Rightarrow [v_t] = 0,$$
$$|T_t(v)| = g \Rightarrow \text{there exists } \Theta \geqslant 0 \text{ such that } [v_t] = -\Theta T_t(v),$$

where

$$[v_n] = v^1 \cdot n^1 + v^2 \cdot n^2, \quad [v_t] = v^1 \cdot t^1 + v^2 \cdot t^2,$$

n^M and t^M, $M = 1, 2$, are unit outward normal and tangential unit vectors, respectively; g is a given slip limit.

Next, we introduce a variational formulation of the equilibrium problem, i.e.,

$$\partial\tau_{ij}^M(v)/\partial x_j + F_i^M = 0 \quad \text{in } \Omega^M, \quad M = 1, 2,$$
$$T_i^M(v) = P_i^M \quad \text{on } \Gamma_p^M, \quad M = 1, 2,$$

where $\Gamma_p^1 = \partial\Omega^1 \setminus (\Gamma_u \cup \Gamma_c)$, $\Gamma_p^2 = \partial\Omega^2 \setminus (\Gamma_0 \cup \Gamma_c)$ and $i = 1, 2$.

We define the spaces of virtual displacements:

$$V^1 = \{v \in [H^1(\Omega^1)]^2 \colon\ v = 0 \text{ on } \Gamma_u\},$$
$$V^2 = \{v \in [H^1(\Omega^2)]^2 \colon\ v \cdot n^2 = 0 \text{ on } \Gamma_0\},$$
$$\mathbb{V} = V^1 \times V^2,$$

the bilinear forms

$$a^M(u, v) = \int_\Omega c_{ijkm}^M \varepsilon_{ij}^M(u)\varepsilon_{km}^M(v)\,\mathrm{d}x, \quad M = 1, 2,$$
$$a(u, v) = a^1(u, v) + a^2(u, v)$$

and the functionals

$$S(v) = \sum_{M=1,2} S^M(v) = \sum_{M=1,2}\Big(\int_{\Omega^M} F_i^M v_i^M\,\mathrm{d}x + \int_{\Gamma_p^M} P_i^M v_i^M\,\mathrm{d}s\Big),$$
$$j(v) = \int_{\Gamma_c} g|[v_t]|\,\mathrm{d}s.$$

Let us define the following set of admissible displacements:

$$\mathbb{K} = \{v \in \mathbb{V} \colon\ [v_n] \leqslant 0 \text{ on } \Gamma_c\}.$$

We say that $u \in \mathbb{K}$ is a *weak solution of the primal problem*, if

$$a(u, v - u) + j(v) - j(u) \geqslant S(v - u) \quad \forall\, v \in \mathbb{K}. \tag{1.1}$$

Let us introduce the subspace of rigid bodies displacements

$$\mathcal{R} = \{v \in [H^1(\Omega^1)]^2 \times [H^1(\Omega^2)]^2 \colon\ |v|' = 0\},$$

where

$$|v|' = \Big(\sum_{M=1,2}\int_{\Omega^M} \varepsilon_{ij}^M(v)\varepsilon_{ij}^M(v)\,\mathrm{d}x\Big)^{1/2}. \tag{1.2}$$

By assumptions on Γ_u and Γ_0, we infer

$$\mathcal{R} \cap \mathbb{V} = \{\varrho = (\varrho^1, \varrho^2)\colon\ \varrho^1 = (0,0),\ \varrho^2 = (0,a),\ a \in \mathbb{R}^1\}. \tag{1.3}$$

Obviously, we have

$$\mathcal{R} \cap \mathbb{K} = \{\varrho = (\varrho^1, \varrho^2)\colon\ \varrho^1 = (0,0),\ \varrho^2 = (0,a),\ a \geqslant 0\} \tag{1.4}$$

provided

$$n_2^1 > 0 \quad \text{holds on } \Gamma_c. \tag{1.5}$$

Throughout the paper, we assume that

$$F_i^M \in L^2(\Omega^M), \quad P_i^M \in L^2(\Gamma_p^M), \quad g \in L^\infty(\Gamma_c), \quad g \geqslant 0.$$

Theorem 1.1. *Assume that*

$$S(y) < j(y) \quad \forall\, y \in \mathcal{R} \cap \mathbb{K} \setminus \{0\}. \tag{1.6}$$

Then there exists a weak solution of the primal problem. If

$$|S(w)| > j(w) \quad \forall\, w \in \mathcal{R} \cap \mathbb{V} \setminus \{0\}, \tag{1.7}$$

there exists at most one solution.

P r o o f is based on a slight generalization of Theorem 1.4 of [16, Chapter 13]. (See also [14, Theorem 4.1].) □

Corollary 1.1. *By the above assumptions on* Γ_u, Γ_0 *and by* (1.5) *we infer that* (1.6) *is satisfied if and only if*

$$\mathcal{V}_2^2 \equiv \int_{\Omega^2} F_2^2 \,\mathrm{d}x + \int_{\Gamma_p^2} P_2^2 \,\mathrm{d}s < \int_{\Gamma_c} g|t_2^2| \,\mathrm{d}s.$$

Condition (1.7) *is satisfied if and only if*

$$|\mathcal{V}_2^2| > \int_{\Gamma_c} g|t_2^2| \,\mathrm{d}s.$$

As a consequence, there exists a unique solution, if

$$\mathcal{V}_2^2 < -\int_{\Gamma_c} g|t_2^2| \,\mathrm{d}s.$$

R e m a r k 1.1. Domain decomposition method and piecewise linear triangular finite elements have been applied to the solution of the primal problem in [4].

2. Mixed variational formulation

The main goal of computational mechanics in contact problems is the displacement field and the stress field. In particular, the normal and tangential components of the stress vector on the contact are of great importance. To this end, a mixed variational formulation of the contact problem may serve, which has both the displacements and the stress vector components on the contact involved as the pivot variables. At the same time, the mixed formulation transforms the non-differentiable term $j(v)$ into a differentiable functional by means of Lagrange multipliers. A mixed finite element analysis of a coercive Signorini problem with given friction was presented in [8], [9], [17], [10], [11], [2].

In the present paper, we analyse a suitable variant of the mixed finite element method for a semi-coercive unilateral contact problem of two elastic bodies, considered in Section 1.

First, we have to introduce some auxiliary definitions and lemmas.

Definition 2.1. Let us define a mapping $\delta\colon \mathbb{V} \to L^2(\Gamma_c) \times L^2(\Gamma_c)$ by

$$\delta v = ([v_n], [v_t]).$$

We denote $\mathbb{W} = W \times W = \delta(\mathbb{V})$ so that $[v_n] \in W$, $[v_t] \in W$, $\delta v \in \mathbb{W}$. To define the norm in $\mathbb{W}$, we introduce

$$\|\!|v|\!\| = \Big[(|v|')^2 + \sum_{M=1}^{2} \|v^M\|^2_{0,\Omega^M}\Big]^{1/2} \tag{2.1}$$

(see (1.2) for the seminorm $|v|'$) and

$$\|\varphi\|_{\mathbb{W}} = \inf_{v\in\mathbb{V},\, \delta v=\varphi} \|\!|v|\!\|. \tag{2.2}$$

Remark 2.1. If the intersection $\Gamma_c = \partial\Omega^1 \cap \partial\Omega^2$ is sufficiently smooth, then $\mathbb{W} = H^{1/2}(\Gamma_c) \times H^{1/2}(\Gamma_c)$. If the arcs of Γ_c are polygonal, these spaces are not equal, but only isomorphic.

Definition 2.2. Let us define the space

$$H(\operatorname{div}, Q) = \{(\sigma_{ij})^2_{i,j=1} \in [L^2(\Omega^1)]^4 \times [L^2(\Omega^2)]^4\colon\ \sigma_{ij} = \sigma_{ji},\\ \operatorname{div}\sigma^M = (\sigma^M_{1j,j}, \sigma^M_{2j,j}) \in [L^2(\Omega^M)]^2,\ M = 1, 2\}$$

with the scalar product

$$(\sigma, \tau)_{H(\operatorname{div},Q)} = \sum_{M=1}^{2} [(\sigma^M_{ij}, \tau^M_{ij})_{0,\Omega^M} + (\sigma^M_{ij,j}, \tau^M_{ik,k})_{0,\Omega^M}] \tag{2.3}$$

where $\sigma^M_{ij,j} = \partial\sigma^M_{ij}/\partial x_j$ (and $Q = \Omega^1 \cup \Omega^2$).

R e m a r k 2.2. The space $H(\mathrm{div}, Q)$ with the scalar product (2.3) is a Hilbert space.

Lemma 2.1 (Green's formula). *Let $\mathbb{W}'$ denote the dual space with respect to $\mathbb{W}$. There exists a unique mapping $\mathbb{T} = (T_n, T_t) \in \mathcal{L}(H(\mathrm{div}, Q), \mathbb{W}')$ such that*

$$\begin{aligned}(2.4)\qquad \sum_{M=1}^{2} [(\tau_{ij}^M, \varepsilon_{ij}(v^M))_{0,\Omega^M} + (\tau_{ij,j}^M, v_i^M)_{0,\Omega^M}] &= \langle \mathbb{T}(\tau), \delta v\rangle \\ &\equiv \langle T_n, [v_n]\rangle + \langle T_t, [v_t]\rangle \quad \forall\, \tau \in H(\mathrm{div}, Q),\ \ \forall\, v \in \mathbb{V}.\end{aligned}$$

Moreover, $\mathbb{T}$ maps $H(\mathrm{div}, Q)$ onto $\mathbb{W}'$.

P r o o f is a slight generalization of those in [1] or [8]. □

Lemma 2.2. *Given any $\mu \in \mathbb{W}'$, let $u(\mu)$ denote the solution of the following auxiliary problem: find $u \equiv u(\mu) \in \mathbb{V}$ such that*

$$(2.5)\qquad \sum_{M=1}^{2} [(\varepsilon_{ij}(u^M), \varepsilon_{ij}(v^M))_{0,\Omega^M} + (u_i^M, v_i^M)_{0,\Omega^M}] = \langle \mu, \delta v\rangle \quad \forall\, v \in \mathbb{V}.$$

Then

$$(2.6)\qquad \|\mu\|_{\mathbb{W}'} = |\!|\!| u(\mu) |\!|\!|.$$

P r o o f. Let $\tau \in H(\mathrm{div}, Q)$ be such that $\mu = \mathbb{T}(\tau)$. By Lemma 2.1 we have

$$\sum_M [(\tau_{ij}^M, \varepsilon_{ij}(v^M))_{0,\Omega^M} + (\tau_{ij,j}^M, v_i^M)_{0,\Omega^M}] = \langle \mu, \delta v\rangle \quad \forall\, v \in \mathbb{V}$$

so that

$$\langle \mu, \varphi\rangle \leqslant \|\tau\|_{H(\mathrm{div},Q)} |\!|\!| v |\!|\!|$$

holds for all $v \in \mathbb{V}$ such that $\delta v = \varphi \in \mathbb{W}$.

Making use of the definition (2.2), we obtain

$$(2.7)\qquad \|\mu\|_{\mathbb{W}'} \leqslant \|\tau\|_{H(\mathrm{div},Q)} \quad \forall\, \tau \in H(\mathrm{div}, Q),\ \ \mathbb{T}(\tau) = \mu.$$

Let $u \equiv u(\mu)$ be the solution of problem (2.5). Then

$$\sigma \equiv \varepsilon(u(\mu)) \in H(\mathrm{div}, Q),$$

$$(2.8)\qquad \|\sigma\|_{H(\mathrm{div},Q)} = \Bigl(\sum_{M=1}^{2} [(\varepsilon_{ij}^M(u), \varepsilon_{ij}^M(u))_{0,\Omega^M} + (u_i^M, u_i^M)_{0,\Omega^M}]\Bigr)^{1/2} = |\!|\!| u(\mu) |\!|\!|$$

follows from Green's formula (2.4) for $v \in [C_0^\infty(\Omega^1)]^2 \times [C_0^\infty(\Omega^2)]^2$ and (2.5). Inserting $v := u(\mu)$ into (2.5), we infer

$$\langle \mu, \delta u(\mu)\rangle = |||u(\mu)|||^2 = \|\sigma\|_{H(\mathrm{div},Q)} |||u(\mu)||| \geqslant \|\sigma\|_{H(\mathrm{div},Q)} \|\delta u(\mu)\|_{\mathbb{W}},$$

using also (2.2). As a consequence,

$$\|\mu\|_{\mathbb{W}'} \geqslant \|\sigma\|_{H(\mathrm{div},Q)}. \tag{2.9}$$

The assertion (2.6) follows from (2.7), (2.8) and (2.9), since $\mu = \mathbb{T}(\sigma)$. □

For a suitable mixed variational formulation of the contact problem under consideration we shall need the set of Lagrange multipliers $\mathcal{M} = \mathcal{M}_n \times \mathcal{M}_t$, where

$$\begin{aligned} \mathcal{M}_n &= \{\mu_n \in W' \colon\ \mu_n \geqslant 0 \text{ on } \Gamma_c\}, \\ \mathcal{M}_t &= \{\mu_t \in L^2(\Gamma_c) \colon\ |\mu_t| \leqslant 1 \text{ a.e. on } \operatorname{supp} g,\ \mu_t = 0 \text{ on } \Gamma_c \setminus \operatorname{supp} g\}. \end{aligned}$$

Let us define

$$b(\mu, v) = \langle \mu_n, [v_n]\rangle + (g\mu_t, [v_t])_{0,\Gamma_c}$$

and the Lagrangian

$$\mathcal{L}(\nu, \mu) = \frac{1}{2} a(v, v) - S(v) + b(\mu, v).$$

Instead of the primal problem (1.1) we will solve the following *saddle point problem*: find a couple (w, λ) such that $w \in \mathbb{V}$, $\lambda \in \mathcal{M}$ and

$$\mathcal{L}(w, \mu) \leqslant \mathcal{L}(w, \lambda) \leqslant \mathcal{L}(v, \lambda) \tag{2.10}$$

holds for all $\mu \in \mathcal{M}$ and $v \in \mathbb{V}$.

Theorem 2.1. *Let assumptions* (1.6), (1.7) *be fulfilled. Moreover, let* $-T_n(u) \in \mathcal{M}_n$ *hold for the solution* u *of the primal problem,* $\Gamma_c \cap \Gamma_u = \emptyset$ *and* $\Gamma_c \cap \Gamma_0 = \emptyset$.

Then there exists a unique saddle point $(w, \lambda) \in \mathbb{V} \times \mathcal{M}$, w *coincides with* u *and*

$$\lambda_n = -T_n(u), \quad g\lambda_t = -T_t(u).$$

Proof. An equivalent formulation of (2.10) is represented by the following problem: find $(w, \lambda) \in \mathbb{V} \times \mathcal{M}$ such that

$$a(w, v) + b(\lambda, v) = S(v) \quad \forall\, v \in \mathbb{V}, \tag{2.11}$$

$$b(\mu - \lambda, w) \leqslant 0 \quad \forall\, \mu \in \mathcal{M}. \tag{2.12}$$

Let us observe that Green's formula from Lemma 2.1 can be extended to cover the whole boundaries $\partial\Omega^1 \cup \partial\Omega^2$, by replacing $\mathbb{V}$ by $[H^1(\Omega^1)]^2 \times [H^1(\Omega^2)]^2$, as follows:

$$
\begin{aligned}
(2.13) \qquad & \sum_{M=1}^{2} [(\tau_{ij}^M, \varepsilon_{ij}(v^M))_{0,\Omega^M} + (\tau_{ij,j}^M, v_i^M)_{0,\Omega^M}] \\
&= \sum_{M=1}^{2} [\langle T_n^M(\tau), v_n^M \rangle_{\partial\Omega^M} + \langle T_t^M(\tau), v_t^M \rangle_{\partial\Omega^M}] \\
&\quad \forall\, \tau = H(\mathrm{div}, Q), \ \ \forall\, v \in [H^1(\Omega^1)]^2 \times [H^1(\Omega^2)]^2,
\end{aligned}
$$

where

$$T_n^M(\tau) \in W'(\partial\Omega^M), \quad T_t^M(\tau) \in W'(\partial\Omega^M), \quad v_n^M, v_t^M \in W(\partial\Omega^M).$$

Equation (2.11) can be rewritten as

$$
\begin{aligned}
& \sum_{M=1}^{2} (\tau_{ij}^M(w), \varepsilon_{ij}(v^M))_{0,\Omega^M} + \langle \lambda_n, [v_n] \rangle + (g\lambda_t, [v_t])_{0,\Gamma_c} \\
&= \sum_{M=1}^{2} [(F_i^M, v_i^M)_{0,\Omega^M} + (P_i^M, v_i^M)_{0,\Gamma_p^M}].
\end{aligned}
$$

Setting $\tau = \tau(w)$ in (2.13), we obtain

$$
\begin{aligned}
(2.14) \qquad a(w, v) = & - \sum_{M=1}^{2} (\tau_{ij,j}^M(w), v_i^M)_{0,\Omega^M} \\
& + \sum_{M=1}^{2} [\langle T_n^M(w), v_n^M \rangle_{\partial\Omega^M} + \langle T_t^M(w), v_t^M \rangle_{\partial\Omega^M}] \quad \forall\, v \in \mathbb{V}.
\end{aligned}
$$

Let us insert (2.14) into (2.11). Choosing $v^M \in [C_0^\infty(\Omega^M)]^2$, we obtain

$$-\tau_{ij,j}^M(w) = F_i^M \quad \text{in } \Omega^M, \ \ M = 1, 2.$$

If we choose $v \in \mathbb{V}$ such that $v_n^1 = -v_n^2$, $v_t^1 = v_t^2 = 0$ on Γ_c and $\operatorname{supp} v_n^1 \subset \Gamma_c$, we obtain $\langle T_n^1(w) - T_n^2(w), v_n^1 \rangle = 0$, so that $T_n^1(w) = T_n^2(w)$ follows. We infer $T_t^1(w) = T_t^2(w)$ likewise.

If we choose $v \in \mathbb{V}$ such that $v^M = 0$ on Γ_p^M and $v_t^2 = 0$ on Γ_0, we obtain

$$(2.15) \qquad \lambda_n = -T_n(w), \quad g\lambda_t = -T_t(w).$$

Choosing $\mu_t = \lambda_t$ and $\mu_n = 0$, $\mu_n = 2\lambda_n$ in the inequality (2.12), we arrive at

$$\langle \lambda_n, [w_n] \rangle = 0, \quad \langle \mu_n, [w_n] \rangle \leqslant 0 \quad \forall \, \mu_n \in \mathcal{M}_n. \tag{2.16}$$

As a consequence, $[w_n] \leqslant 0$ on Γ_c, so that $w \in \mathbb{K}$. The choice $\mu_n = \lambda_n$ in (2.12) yields

$$(g\mu_t, [w_t])_{0,\Gamma_c} \leqslant (g\lambda_t, [w_t])_{0,\Gamma_c} \quad \forall \, \mu_t \in \mathcal{M}_t.$$

Choosing $\mu_t = \operatorname{sign}[w_t]$, we obtain

$$j(w) = (g, |[w_t]|)_{0,\Gamma_c} \leqslant (g\lambda_t, [w_t])_{0,\Gamma_c}. \tag{2.17}$$

Using (2.16), (2.17), (2.11) and the definitions of $\mathcal{M}_n$, $\mathcal{M}_t$, we arrive at

$$a(w, v - w) + j(v) - j(w) \geqslant S(v - w) \quad \forall \, v \in \mathbb{K}.$$

As a consequence, w is a solution of the primal problem and since this solution is unique, $w = u$. Using also (2.15), we conclude that there exists at most one saddle point (w, λ).

Conversely, let $u \in \mathbb{K}$ be the solution of the primal problem and $-T_n(u) \in \mathcal{M}_n$. Then $u \in \mathbb{V}$, $(-T_n(u), -T_t(u)) \in \mathcal{M}_n \times \mathcal{M}_t$. Indeed, following the arguments of [8], we infer

$$|T_t(u)| \leqslant g \quad \text{a.e. on } \Gamma_c.$$

Let us verify conditions (2.11), (2.12). By the formula (2.14), where w is replaced by u, we obtain

$$-\tau^M_{ij,j}(u) = F^M_i \quad \text{in } \Omega^M, \quad M = 1, 2, \tag{2.18}$$

setting $v = u \pm \varphi$, $\varphi^M \in [C_0^\infty(\Omega^M)]^2$. Choosing $v = u \pm z$, where $z^1 = 0$ on $\Gamma_u \cup \Gamma_c$ and $z^2 = 0$ on $\Gamma_0 \cup \Gamma_c$ and using (2.18), we arrive at

$$T^M(u) = P^M \quad \text{on } \Gamma^M_p, \quad M = 1, 2. \tag{2.19}$$

The choice $v = u \pm z$ such that $z^1 = 0$, $z^2_n = 0$ on Γ_0, $z^2 = 0$ on Γ_c, yields

$$T^2_t(u) = 0 \quad \text{on } \Gamma_0. \tag{2.20}$$

Finally, inserting (2.18), (2.19) and (2.20) into (2.13), we infer that condition (2.11) is satisfied.

Next, let us verify condition (2.12) for $\lambda_n = -T_n(u)$ and $g\lambda_t = -T_t(u)$. Making use of the variational inequality (1.1), formula (2.13) (with u instead of w), (2.18), (2.19) and (2.20), we arrive at

$$\langle T_n(u), [v_n] - [u_n]\rangle + \langle T_t(u), [v_t] - [u_t]\rangle + j(v) - j(u) \geqslant 0 \quad \text{for all } v \in \mathbb{K}. \tag{2.21}$$

Let $\{\psi^k\}$, $k \to \infty$, be a sequence of functions such that $\psi^k \in \mathbb{V}$, $[\psi_n^k] = 0$ on Γ_c and

$$[\psi_t^k] \to -[u_t] \quad \text{in } L^1(\Gamma_c) \quad \text{as } k \to \infty.$$

Then $v^k = u + \psi^k \in \mathbb{K}$ and

$$\langle T_t(u), [\psi_t^k]\rangle + (g, |[u_t + \psi_t^k]| - |[u_t]|)_{0,\Gamma_c} \geqslant 0.$$

Passing to the limit with $k \to \infty$, we are led to

$$-\langle T_t(u), [u_t]\rangle - (g, |[u_t]|)_{0,\Gamma_c} \geqslant 0.$$

Since

$$T_t(u)[u_t] + g|[u_t]| \geqslant 0 \tag{2.22}$$

follows from the bound $|T_t(u)| \leqslant g$ on Γ_c, we infer

$$T_t(u)[u_t] + g|[u_t]| = 0 \quad \text{a.e. on } \Gamma_c. \tag{2.23}$$

Next, we may write

$$\begin{aligned} b(\mu - \lambda, u) &= \langle \mu_n - \lambda_n, [u_n]\rangle + \langle g(\mu_t - \lambda_t), [u_t])_{0,\Gamma_c} \\ &= \langle \mu_n, [u_n]\rangle + (g\mu_t + T_t(u), [u_t])_{0,\Gamma_c}, \quad \forall\, \mu \in \mathcal{M}, \end{aligned}$$

since

$$\langle T_n(u), [u_n]\rangle = 0$$

follows from (2.21) and (2.23) by setting $v = 0$ and $v = 2u$.

The first term is nonpositive by the definitions of $\mathcal{M}_n$ and $\mathbb{K}$, and

$$(g\mu_t, [u_t])_{0,\Gamma_c} \leqslant (g, |[u_t]|)_{0,\Gamma_c} = (-T_t(u), [u_t])_{0,\Gamma_c} \quad \forall\, \mu_t \in \mathcal{M}_t$$

follows from (2.23). As a consequence, condition (2.12) is satisfied and $(u, (\lambda_n, \lambda_t))$, where $\lambda_n = -T_n(u)$, $g\lambda_t = -T_t(u)$, is a saddle point. $\square$

3. Finite element approximation

In the present section we propose and analyse a variant of mixed finite element approximation, based on the saddle point formulation (2.10). We use the standard spaces of linear elements on regular triangulations $\mathcal{T}_h = \mathcal{T}_h^1 \cup \mathcal{T}_h^2$ of polygonal domains Ω^1 and Ω^2.

We assume that Γ_c is a straight line segment. If a frictionless problem is considered, we need not restrict ourselves to straight segments (see [3] and [15]). We define spaces

$$\mathbb{V}_h^M = \{v_h \in [C(\overline{\Omega}^M)]^2 \cap V^M\colon\ v_{h|\kappa} \in [P_1(\kappa)]^2\ \forall\, \kappa \in \mathcal{T}_h^M\},$$

where κ denotes any triangle of $\mathcal{T}_h^M$, and

$$\mathbb{V}_h = \mathbb{V}_h^1 \times \mathbb{V}_h^2.$$

Assume that the triangulation $\mathcal{T}_h$ is compatible with the end-points of Γ_u, Γ_0 and Γ_c. Moreover, let the nodes s_i of $\mathcal{T}_h^1$ and $\mathcal{T}_h^2$ coincide on Γ_c and form a *uniform* partition

$$N_h = (s_0, s_1, \ldots, s_m), \quad m = m(h)$$

of Γ_c. We define

$$W_{hn} = \{\psi_h\colon \text{ there exists } v_h \in \mathbb{V}_h \text{ such that } \psi_h = [v_{hn}]\}$$

and assume that $g_h \in W_{hn}$ is a non-negative approximation of the slip limit g.

Instead of the set $\mathcal{M}_n$ we define

$$M_{hn} = \{\mu_{hn}\colon\ \mu_{hn} \text{ is a real function defined on } N_h \text{ such that } \mu_{hn}(s_i) \geqslant 0,\ \ i = 0, 1, \ldots, m\}. \tag{3.1}$$

Let $\{\mathcal{T}_H\}$ be a partition of Γ_c, whose nodes will be denoted correspondingly by z_i (cf. [2], [17]):

$$z_0 = s_0, \quad z_{m+1} = s_m, \quad z_{i+1} = \frac{1}{2}(s_i + s_{i+1}), \quad i = 0, 1, \ldots, m-1.$$

We introduce

$$L_H = \{\mu_H \in L^\infty(\Gamma_c)\colon\ \mu_H|_{(z_i, z_{i+1})} \in P_0(z_i, z_{i+1}),\ 0 \leqslant i \leqslant m\}$$

and

$$M_{Ht} = \{\mu_H \in L_H\colon\ |\mu_H| \leqslant 1,\ \mu_H = 0 \text{ on } \Gamma_c \setminus \operatorname{supp} g_h\}. \tag{3.2}$$

Finally, we define

$$\mathbb{M}_{hH} = M_{hn} \times M_{Ht}.$$

Instead of problem (2.10) we will solve the following saddle point problem: find a pair $(u_h, \lambda_{hH}) \in \mathbb{V}_h \times \mathbb{M}_{hH}$ such that

$$\mathcal{L}_{hH}(u_h, \mu_{hH}) \leqslant \mathcal{L}_{hH}(u_h, \lambda_{hH}) \leqslant \mathcal{L}_{hH}(v_h, \lambda_{hH}) \ \forall (v_h, \mu_{hH}) \in \mathbb{V}_h \times \mathbb{M}_{hH}, \tag{3.3}$$

where

$$\begin{aligned} \mathcal{L}_{hH}(v_h, \mu_{hH}) &= \frac{1}{2} a(v_h, v_h) - S(v_h) + b_{hH}(\mu_{hH}, v_h), \\ b_{hH}(\mu_{hH}, v_h) &= \{\overline{\mu}_{hn}, [v_{hn}]\}_h + (g_h \mu_{Ht}, [v_{ht}])_{0,\Gamma_c} \end{aligned} \tag{3.4}$$

and

$$\{\overline{\mu}_{hn}, [v_{hn}]\}_h = \sum_{i=0}^{m} \mathcal{M}_i \mathcal{H}_i \mathcal{V}_i, \tag{3.5}$$

where

$$\overline{\mu}_{hn} = \sum_{i=0}^{m} \mathcal{M}_i \psi_i, \quad [v_{hn}] = \sum_{i=0}^{m} \mathcal{V}_i \psi_i,$$

$\mathcal{M}_i = \mu_{hn}(s_i)$ and $\{\psi_0, \psi_1, \ldots, \psi_m\}$ denotes the standard 1D-basis of W_{hn}; $\mathcal{H}_0 = \mathcal{H}_m = \frac{1}{2} h_0$, $\mathcal{H}_i = h_0$ for $i = 1, 2, \ldots, m-1$, $h_0 = s_{i+1} - s_i$. That is, $\{\cdot, \cdot\}_h$ denotes the numerical integration by the trapezoidal rule on the partition $\mathcal{T}_h \cap \Gamma_c$.

The last term in (3.4) is piecewise quadratic and can be evaluated by Simpson's rule exactly.

To verify the existence of a saddle point (3.3), we can employ the following abstract theorem.

Proposition 3.1. *Let V and Y be two real Hilbert spaces, $A \subset V$ and $B \subset Y$ nonempty, closed and convex subsets. Assume that*

$$\begin{aligned} &\forall \mu \in B, \ v \to \mathcal{L}(v, \mu) \text{ is convex, weakly lower semicontinuous,} \\ &\forall v \in A, \ \mu \to \mathcal{L}(v, \mu) \text{ is concave, weakly upper semicontinuous,} \\ &\sup_{\mu \in B} \mathcal{L}(v, \mu) \to +\infty \quad \text{as } v \in A, \ \|v\|_V \to \infty; \end{aligned} \tag{3.6}$$

there exists $v_0 \in A$ such that

$$\mathcal{L}(v_0, \mu) \to -\infty \quad \text{as } \mu \in B, \ \|\mu\|_Y \to \infty. \tag{3.7}$$

Then there exists a saddle point of $\mathcal{L}(v, \mu)$ on $A \times B$.

Proof. See [5, Proposition VI.2.4 and Remark 2.4], or [9, Theorem 3.9] and its counterpart. □

Next, let us define

$$K_h = \{v_h \in \mathbb{V}_h \colon \{\overline{\mu}_{hn}, [v_{hn}]\}_h \leqslant 0 \ \forall\, \mu_{hn} \in M_{hn}\}. \tag{3.8}$$

From definitions (3.1), (3.5) and (3.8), we infer that

$$K_h = \mathbb{V}_h \cap \mathbb{K}. \tag{3.9}$$

Lemma 3.1. *Let* $\mu_H \in L_H$, $\mu_H = 0$ *on* $\Gamma_c \setminus \operatorname{supp} g_h$ *and*

$$(\mu_H g_h, [v_{ht}])_{0,\Gamma_c} = 0 \quad \forall\, v_h \in \mathbb{V}_h. \tag{3.10}$$

Then

$$\mu_H = 0 \quad \text{on } \Gamma_c. \tag{3.11}$$

Proof. Let $\mathcal{G}_i$ and $\mathcal{M}_i$, $i = 0, 1, \ldots, m$, denote the nodal values of $g_h \in W_{hn}$ and $[\mu_H]$, respectively, on N_h. Then the proof of (3.11) leads to a linear system for unknowns $\mathcal{M}_i$, with a tridiagonal matrix $A \equiv \{a_{ij}\}$, such that

$$\begin{aligned}
a_{ii} &= 14\mathcal{G}_i + 2\mathcal{G}_{i-1} + 2\mathcal{G}_{i+1}, \quad i = 1, \ldots, m-1,\\
a_{i-1,i} &= \mathcal{G}_{i-1} + 2\mathcal{G}_i, \quad i = 1, \ldots, m,\\
a_{i,i+1} &= 2\mathcal{G}_i + \mathcal{G}_{i+1}, \quad i = 1, \ldots, m-1,\\
a_{00} &= 7\mathcal{G}_0 + 2\mathcal{G}_1,\\
a_{mm} &= 7\mathcal{G}_m + 2\mathcal{G}_{m-1}.
\end{aligned}$$

As a consequence, the matrix A is diagonally dominant. By [6, Theorem 5.17] the matrix A is regular and every of its principal submatrices is regular, as well. This implies that (3.11) holds. □

Theorem 3.1. *Let assumptions* (1.6), (1.7) *be fulfilled. Then there exists a unique solution of the saddle point problem* (3.3).

Proof. (i) *Existence.* Using Proposition 3.1, we set

$$A = \mathbb{V}_h, \quad V = \mathbb{V}_h, \quad B = \mathbb{M}_{hH}, \quad Y = \mathbb{R}^{m+1} \times L^2(\Gamma_c).$$

It is readily seen that it suffices to verify conditions (3.6) and (3.7).

To verify (3.6), we first consider $v_h \in K_h$. Then

$$\begin{aligned} \sup_{\mu_{hH} \in \mathbb{M}_{hH}} b_{hH}(\mu_{hH}, v_h) &= \sup_{\mu_{Ht} \in M_{Ht}} (g_h \mu_{Ht}, [v_{ht}])_{0,\Gamma_c} \\ &= \sum_{i=0}^{m} \left| \int_{z_i}^{z_{i+1}} g_h [v_{ht}] \, \mathrm{d}s \right| \equiv j_H(v_h). \end{aligned} \tag{3.12}$$

Second, we consider $v_h \notin K_h$. Then there exists $\mu^0_{hn} \in M_{hn}$ such that

$$\{\overline{\mu}^0_{hn}, [v_{hn}]\}_h > 0.$$

Setting $\mu_{hn} = t\mu^0_{hn}$, $t \to +\infty$, we infer that

$$\sup_{\mu_{hn} \in M_{hn}} \{\overline{\mu}_{hn}, [v_{hn}]\}_h = +\infty.$$

On the other hand,

$$|(g_h \mu_{Ht}, [v_{ht}])_{0,\Gamma_c}| \leqslant \int_{\Gamma_c} g_h |[v_{ht}]| \, \mathrm{d}s < +\infty \tag{3.13}$$

holds for any $\mu_{Ht} \in M_{Ht}$, so that

$$\sup_{\mu_{hH} \in \mathbb{M}_{hH}} b_{hH}(\mu_{hH}, v_h) = +\infty \tag{3.14}$$

follows for $v_h \notin K_h$.

As a consequence of (3.12) and (3.13), we have

$$\sup_{\mu_{hH} \in \mathbb{M}_{hH}} \mathcal{L}_{hH}(v_h, \mu_{hH}) = \begin{cases} J_H(v_h) & \text{if } v_h \in K_h, \\ +\infty & \text{if } v_h \in \mathbb{V}_h \setminus K_h, \end{cases} \tag{3.15}$$

where

$$J_H(v_h) = \frac{1}{2} a(v_h, v_h) - S(v_h) + j_H(v_h).$$

By assumption (1.6) the functional $J_H(\cdot)$ is coercive on K_h. Indeed,

$$J_H(v_h) \to +\infty \quad \text{as } v_h \in K_h \text{ and } \|v_h\| \to \infty \tag{3.16}$$

can be deduced by a slight modification of the proof of Theorem 4.1 in [14], where we replace $j(v)$ by $j_H(v_h)$ and use the fact that

$$j_H(y_h) = j(y_h) \quad \text{for } y_h \in \mathcal{R} \cap \mathbb{V}_h. \tag{3.17}$$

Then (3.6) follows from (3.14) and (3.15). To verify condition (3.7), we can find $v_0 \in K_h$ such that

$$\max_{s \in \Gamma_c} [v_{0n}](s) = \omega < 0.$$

Then we may write

$$-\{\overline{\mu}_{hn}, [v_{0n}]\}_h \geqslant -h_0 \omega \sum_{i=0}^{m} |\mathcal{M}_i|$$

so that

$$\{\overline{\mu}_{hn}, [v_{0n}]\}_h \to -\infty \quad \text{as } \|\mathcal{M}\|_{m+1} = \|\mu_{hn}\|_{m+1} \to \infty. \tag{3.18}$$

As a consequence of (3.18) and (3.13),

$$\mathcal{L}_{hH}(v_0, \mu_{hH}) \to -\infty \quad \text{as } \|\mu_{hH}\|_Y \to \infty, \ \ \mu_{hH} \in \mathbb{M}_{hH},$$

so that condition (3.7) is satisfied. By Proposition 3.1, there exists at least one saddle point (3.3).

(ii) *Uniqueness.* First, we will show that the component u_h of the saddle point coincides with the solution $u_h \in K_h$ of the variational inequality

$$a(u_h, v_h - u_h) + j_H(v_h) - j_H(u_h) \geqslant S(v_h - u_h) \quad \forall\, v_h \in K_h. \tag{3.19}$$

Indeed, by virtue of (3.15) we have

$$\mathcal{L}_{hH}(u_h, \lambda_{hH}) = \min_{v_h \in \mathbb{V}_h} \sup_{\mu_{hH} \in \mathbb{M}_{hH}} \mathcal{L}_{hH}(v_h, \mu_{hH}) = \min_{v_h \in K_h} J_H(v_h).$$

The latter minimization problem is equivalent with the variational inequality (3.19).

Let $\overline{u}_h$ and u_h be two solutions of (3.19). Then

$$a(u_h, \overline{u}_h - u_h) + j_H(\overline{u}_h) - j_H(u_h) \geqslant S(\overline{u}_h - u_h),$$
$$a(\overline{u}_h, u_h - \overline{u}_h) + j_H(u_h) - j_H(\overline{u}_h) \geqslant S(u_h - \overline{u}_h),$$

so that

$$c_0(|\overline{u}_h - u_h|')^2 \leqslant a(\overline{u}_h - u_h, \overline{u}_h - u_h) \leqslant 0$$

and $w\colon\ \overline{u}_h - u_h \in \mathcal{R} \cap \mathbb{V}_h$ follows. Since inequality (3.19) implies

$$u_h = \arg \min_{v_h \in K_h} J_H(v_h),$$

we have

$$J_H(\overline{u}_h) = J_H(u_h) \Rightarrow |S(w)| = |j_H(u_h + w) - j_H(u_h)| \leqslant j_H(w) = j(w) \tag{3.20}$$

using also (3.17).

On the other hand, assumption (1.7) implies

$$|S(w)| > j(w) \quad \forall\, w \in \mathcal{R} \cap \mathbb{V}_h \setminus \{0\}, \tag{3.21}$$

since $\mathcal{R} \cap \mathbb{V}_h = \mathcal{R} \cap \mathbb{V}$. As a consequence, (3.20) and (3.21) imply $w = 0$, so that the first component of the saddle point is unique.

Next, let $\overline{\lambda}_{hH}$ and λ_{hH} be two second components of the saddle point. Since

$$a(u_h, v_h) + b_{hH}(\lambda_{hH}, v_h) = S(v_h) \quad \forall\, v_h \in \mathbb{V}_h$$

and a parallel condition for $\overline{\lambda}_{hH}$ holds, we obtain

$$b_{hH}(\lambda_{hH} - \overline{\lambda}_{hH}, v_h) = 0 \quad \forall\, v_h \in \mathbb{V}_h.$$

Denoting $\mu_{hH} = \lambda_{hH} - \overline{\lambda}_{hH}$, we have $\mu_{hH} \equiv (\mu_{hn}, \mu_{Ht})$ and

$$\{\overline{\mu}_{hn}, [v_{hn}]\}_h + (g_h \mu_{Ht}, [v_{ht}])_{0,\Gamma_c} = 0 \quad \forall\, v_h \in \mathbb{V}_h.$$

Choosing $[v_{ht}] = 0$ and $[v_{hn}] = \overline{\mu}_{hn}$, we obtain $\mu_{hn} = 0$. Let $[v_{hn}] = 0$ and using Lemma 3.1 we conclude $\mu_{Ht} = 0$. □

4. Convergence analysis

We are going to prove convergence of saddle-point components u_h, λ_{hn} and λ_{Ht} as the mesh sizes h and H ($H = h_0 \leqslant h$) tend to zero. To this end, we shall need the following lemma.

Lemma 4.1. *Let assumptions* (1.6), (1.7) *be fulfilled. Then the sequence* $\{u_h\}$, $h \to 0_+$, *is bounded in* $\mathbb{V}$.

P r o o f. From the proof of uniqueness in Theorem 3.1 we know that $u_h \in K_h$ is a solution of inequality (3.19). Setting $v_h = 0$ and $v_h = 2u_h$, we obtain

$$J_H(u_h) + \frac{1}{2} a(u_h, u_h) = a(u_h, u_h) + j_H(u_h) - S(u_h) = 0. \tag{4.1}$$

Assume that $\|u_h\| \to \infty$ as $h \to 0_+$. Using (3.16), we infer $J_H(u_h) \to +\infty$. We arrive at a contradiction with (4.1) so that $\{u_h\}$ must be bounded in $\mathbb{V}$. □

Theorem 4.1. *Let the assumptions of Theorem* 2.1 *be fulfilled. In addition to that, let there be only a finite number of points* $\Gamma_p \cap \Gamma_u$, $\Gamma_p \cap \Gamma_0$, *and let* $\operatorname{supp} g$ *consist of a finite number of segments* G_p, $p \leqslant \overline{p}$, *such that the endpoints of* G_p *coincide with some nodes of* $\mathcal{T}_h \cap \Gamma_c$ *for all* $\mathcal{T}_h$ *under consideration. Let* $g \in H^1(G_p)$, $p \leqslant \overline{p}$. *Assume that* g_h *is the Lagrange linear interpolate of* g *on* $\mathcal{T}_h \cap \Gamma_c$.

Then

$$u_h \to u \quad \text{in } \mathbb{V}, \tag{4.2}$$

$$\lambda_{Ht} \rightharpoonup \lambda_t \quad \text{weakly* in } L^\infty(\Gamma_c), \tag{4.3}$$

$$\overline{\lambda}_{hn} \rightharpoonup \lambda_n \quad \text{weakly* in } H^{-1/2}(\Gamma_c) \tag{4.4}$$

as $h \to 0_+$, *where* (u, λ), $\lambda \equiv (\lambda_n, \lambda_t)$ *is the saddle point of* $\mathcal{L}(v, \mu)$ *on* $\mathbb{V} \times \mathcal{M}$, *i.e., the solution of problem* (2.10).

P r o o f. By virtue of Lemma 4.1 and the definition of M_{Ht}, one can find subsequences of $\{u_h\}$ and $\{\lambda_{Ht}\}$ (we will denote them by the same symbols) such that

$$u_h \rightharpoonup u^* \quad \text{weakly in } \mathbb{V}, \tag{4.5}$$

$$\lambda_{Ht} \rightharpoonup \lambda_t^* \quad \text{weakly* in } L^\infty(\Gamma_c), \tag{4.6}$$

where u^* and λ_t^* are some elements of $\mathbb{V}$ and $L^\infty(\Gamma_c)$, respectively.

Let us show that $\lambda_t^* \in \mathcal{M}_t$. Since

$$\|\lambda_{Ht}\|_{0,\infty} \leqslant 1 \quad \text{for all } H \to 0_+, \tag{4.7}$$

$$\|\lambda_t^*\|_{0,\infty} \leqslant \liminf_{H \to 0} \|\lambda_{Ht}\|_{0,\infty} \leqslant 1$$

follows from (4.6).

It is readily seen that $\operatorname{supp} g_h = \operatorname{supp} g$. We observe that

$$\lambda_{Ht} = 0 \quad \text{on } \Gamma_c \setminus \mathcal{O}_H(\operatorname{supp} g),$$

where

$$\mathcal{O}_H(\operatorname{supp} g) = \Big\{ s \in \Gamma_c \colon \operatorname{dist}(s, \operatorname{supp} g) < \frac{1}{2} H \Big\}.$$

Assume that $|\lambda_t^*| > 0$ on a set T_0 such that $T_0 \subset \Gamma_c \setminus \operatorname{supp} g$ and $\operatorname{meas} T_0 > 0$. Then (4.6) implies

$$\int_{T_0} [(\lambda_t^*)^2 - \lambda_{Ht} \lambda_t^*] \, \mathrm{d}s \to 0 \quad \text{as } H \to 0_+.$$

On the other hand,

$$\int_{T_0} \lambda_{Ht}\lambda_t^* \,\mathrm{d}s = \int_{T_0\cap\mathcal{O}_H(\operatorname{supp} g)} \lambda_{Ht}\lambda_t^* \,\mathrm{d}s \to 0$$

and we arrive at a contradiction. As a consequence, $\lambda_t^* = 0$ a.e. on $\Gamma_c \setminus \operatorname{supp} g$. Combining this result with (4.7), we conclude that $\lambda_t^* \in \mathcal{M}_t$.

Next, we show that u^* is a solution of the primal problem (1.1). Since $u_h \in K_h \subset \mathbb{K}$ by (3.9) and $\mathbb{K}$ is weakly closed, $u^* \in \mathbb{K}$ follows.

From (3.3) we infer

$$b_{hH}(\mu_{hH} - \lambda_{hH}, u_h) \leqslant 0 \quad \forall\, \mu_{hH} \in \mathbb{M}_{hH}. \tag{4.8}$$

Setting $\mu_{Ht} = \lambda_{Ht}$, $\mu_{hn} = 0$ and $\mu_{hn} = 2\lambda_{hn}$, we obtain

$$\{\overline{\lambda}_{hn}, [u_{hn}]\}_h = 0, \tag{4.9}$$

$$\{\overline{\mu}_{hn}, [u_{hn}]\}_h \leqslant 0 \quad \forall\, \mu_{hH} \in M_{hn}. \tag{4.10}$$

Since $K_h \subset \mathbb{V}_h$, (3.3) yields

$$\mathcal{L}_{hH}(u_h, \lambda_{hH}) \leqslant \mathcal{L}_{hH}(v_h, \lambda_{hH}) \quad \forall\, v_h \in K_h. \tag{4.11}$$

If $v_h \in K_h$, then (4.9) and the definition of M_{hn} imply

$$\begin{aligned} b_{hH}(\lambda_{hH}, v_h - u_h) &= \{\overline{\lambda}_{hn}, [v_{hn}]\}_h + (g_h\lambda_{Ht}, [v_{ht}] - [u_{ht}])_{0,\Gamma_c} \\ &\leqslant (g_h\lambda_{Ht}, [v_{ht}] - [u_{ht}])_{0,\Gamma_c}. \end{aligned} \tag{4.12}$$

From (4.11) and (4.12) we infer

$$a(u_h, v_h - u_h) + (g_h\lambda_{Ht}, [v_{ht}] - [u_{ht}])_{0,\Gamma_c} - S(v_h - u_h) \geqslant 0 \quad \forall\, v_h \in K_h. \tag{4.13}$$

The set

$$\mathbb{K}_0 = \mathbb{K} \cap ([C^\infty(\overline{\Omega}^1)]^2 \times [C^\infty(\overline{\Omega}^2)]^2)$$

is dense in $\mathbb{K}$ by virtue of the assumptions of Theorem 4.1. (For a proof we refer to [12, § 2.3.3, Lemma 3.2] and [8, Remark 3.2].) For any $v \in \mathbb{K}$ we may therefore find a sequence $\{v_h\}$, $h \to 0_+$, such that

$$v_h \in K_h, \quad v_h \to v \quad \text{in } \mathbb{V}. \tag{4.14}$$

Passing to the limit in (4.13) and using (4.5), (4.6), compactness of the trace mapping and (4.14), we obtain

$$a(u^*, v) + (g\lambda_t^*, [v_t] - [u_t^*])_{0,\Gamma_c} - S(v - u^*) \geqslant a(u^*, u^*). \tag{4.15}$$

Making use of (4.8) with $\mu_{hn} = \lambda_{hn}$, we infer

$$(g_h\mu_{Ht}, [u_{ht}])_{0,\Gamma_c} \leqslant (g_h\lambda_{Ht}, [u_{ht}])_{0,\Gamma_c} \quad \forall\, \mu_{Ht} \in M_{Ht}. \tag{4.16}$$

Since M_{Ht} is dense in $\mathcal{M}_t$ with respect to the $L^2(\Gamma_c)$-norm, we can construct a sequence $\{\mu_{Ht}\}$ such that $\mu_{Ht} \in M_{Ht}$ and

$$\mu_{Ht} \to \operatorname{sign}[u_t^*] \quad \text{in } L^2(\operatorname{supp} g) \quad \text{as } H \to 0_+.$$

Then using (4.5), (4.6), we infer from (4.16) that

$$\begin{aligned} j(u^*) = (g, |[u_t^*]|)_{0,\Gamma_c} &= (g \operatorname{sign}[u_t^*], [u_t^*])_{0,\Gamma_c} \\ &\leqslant (g\lambda_t^*, [u_t^*])_{0,\Gamma_c}. \end{aligned} \tag{4.17}$$

Here we have employed also the well-known estimate for the Lagrange linear interpolate of g:

$$\|g_h - g\|_{0,\Gamma_c} \leqslant Ch_0 \sum_{p \leqslant \overline{p}} \|g\|_{1,G_p}.$$

On the other hand,

$$(g\lambda_t^*, [v_t])_{0,\Gamma_c} \leqslant (g, |[v_t]|)_{0,\Gamma_c} = j(v). \tag{4.18}$$

Combining (4.15), (4.17) and (4.18), we arrive at

$$a(u^*, v - u^*) + j(v) - S(v - u^*) \geqslant j(u^*) \quad \forall\, v \in \mathbb{K}.$$

As a consequence, u^* is a solution of the primal problem. By Theorem 1.1 the solution is unique, so that $u^* = u$ follows. Moreover, the whole sequence $\{u_h\}$ tends weakly to u in $\mathbb{V}$.

Next, we prove the *strong convergence* in $\mathbb{V}$. By virtue of inequality (3.19), we have

$$J_H(u_h) \leqslant J_H(v_h) \quad \forall\, v_h \in K_h,$$

where

$$J_H(v) = J_0(v) + j_H(v), \quad J_0(v) = \frac{1}{2}a(v, v) - S(v).$$

Employing Taylor's formula, we obtain

$$J_0(v_h) + j_H(v_h) \geqslant J_0(u) + J_0'(u, u_h - u) + \frac{1}{2}a(u_h - u, u_h - u) + j_H(u_h),$$

so that

$$\frac{1}{2}c_0(|u_h - u|')^2 \leqslant J_0(v_h) - J_0(u) - J_0'(u, u_h - u) + j_H(v_h) - j_H(u_h). \tag{4.19}$$

We also observe that

$$|j_H(v_h) - j_H(u_h)| \leqslant C\|[v_{ht}] - [u_{ht}]\|_{0,\Gamma_c}. \tag{4.20}$$

Let us choose $v_h \in K_h$ such that $v_h \to u$ in $\mathbb{V}$ and use the continuity of J_0, weak convergence of $\{u_h\}$, (4.20) and compactness of the trace mapping to deduce that

$$|u_h - u|' \to 0 \quad \text{as } h \to 0_+ \tag{4.21}$$

follows from (4.19). The weak convergence $u_h \rightharpoonup u$ in $\mathbb{V}$ yields that

$$\|u_h^M - u^M\|_{0,\Omega^M} \to 0 \quad \text{as } h \to 0_+, \ \ M = 1, 2, \tag{4.22}$$

by the Rellich theorem. From coerciveness of strains (i.e., 1st Korn's inequality) we obtain

$$(|u_h - u|')^2 + \sum_{M=1}^{2} \|u_h^M - u^M\|_{0,\Omega^M}^2 \geqslant C\|u_h - u\|^2 \tag{4.23}$$

(see [16, Chapter 6, Theorem 3.4]).

Combining (4.21), (4.22) and (4.23), we infer the strong convergence $u_h \to u$ in $\mathbb{V}$ as $h \to 0_+$.

Let us recall (4.15) with $u^* = u$, i.e.,

$$a(u, v - u) + (g\lambda_t^*, [v_t] - [u_t])_{0,\Gamma_c} \geqslant S(v - u) \quad \forall\, v \in \mathbb{K}.$$

Using Green's formula, (2.13) and (2.14), we deduce that $g\lambda_t^* = -T_t(u)$ by an argument similar to that in the proof of Theorem 2.1. As a consequence, $\lambda_t^* = \lambda_t$ and the whole sequence $\{\lambda_{Ht}\}$ tends to λ_t weakly* in $L^\infty(\Gamma_c)$.

To verify (4.4), we first show that the sequence $\{\overline{\lambda}_{hn}\}$ is bounded in $H^{-1/2}(\Gamma_c)$.

By Lemma 2.2 the norm of $\mu \equiv (\overline{\lambda}_{hn}, 0) \in [H^{-1/2}(\Gamma_c)]^2$ is equal to the norm $|\!|\!| z(\lambda_h) |\!|\!|$, where $z(\lambda_h) \in \mathbb{V}$ is the solution of the problem

$$[z(\lambda_h), v] = \langle \overline{\lambda}_{hn}, [v_n] \rangle \quad \forall\, v \in \mathbb{V}, \tag{4.24}$$

where

$$[z,v] = \sum_{M=1,2} ((\varepsilon_{ij}(z^M), \varepsilon_{ij}(v^M))_{0,\Omega^M} + (z_i^M, v_i^M)_{0,\Omega^M})$$

and $|||z||| = [z,z]^{1/2}$.

Instead of problem (4.24) we will solve the following approximate problem: find $z_h(\lambda_h) \in \mathbb{V}_h$ such that

$$[z_h(\lambda_h), v_h] = \{\overline{\lambda}_{hn}, [v_{hn}]\}_h \quad \forall\, v_h \in \mathbb{V}_h. \tag{4.25}$$

Making use of definition (3.3) and of the boundedness of $\{u_h\}$ and $\{\lambda_{Ht}\}$, we may write

$$\{\overline{\lambda}_{hn}, [v_{hn}]\}_h = S(v_h) - a(u_h, v_h) - (g_h \lambda_{Ht}, [v_{ht}])_{0,\Gamma_c} \leqslant C\|v_h\|. \tag{4.26}$$

Inserting $v_h := z_h(\lambda_h)$ into (4.25), (4.26) and using the coerciveness of strains, we obtain

$$|||z_h(\lambda_h)|||^2 \leqslant C\|z_h(\lambda_h)\| \leqslant \tilde{C}|||z_h(\lambda_h)|||,$$

so that

$$|||z_h(\lambda_h)||| \leqslant \tilde{C} \quad \forall\, h \to 0_+. \tag{4.27}$$

From (4.24) and (4.25) we infer

$$[z(\lambda_h) - z_h(\lambda_h), v_h] = 0 \quad \forall\, v_h \in \mathbb{V}_h, \tag{4.28}$$

provided we set

$$\langle \overline{\lambda}_{hn}, [v_{hn}] \rangle := \{\overline{\lambda}_{hn}, [v_{hn}]\}_h. \tag{4.29}$$

Then (4.28) implies that

$$|||z(\lambda_h) - z_h(\lambda_h)||| = \inf_{v_h \in \mathbb{V}_h} |||z(\lambda_h) - v_h|||.$$

Obviously, the infimum tends to zero as $h \to 0_+$. Using this and (4.27), we are led to the estimate

$$|||z(\lambda_h)||| \leqslant \tilde{C} + 1 \quad \text{for } h < h_1,$$

so that

$$\|\overline{\lambda}_{hn}\|_{-1/2,\Gamma_c} = |||z(\lambda_h)||| \leqslant \tilde{C} + 1 \quad \forall\, h < h_1.$$

Since the space $H^{1/2}(\Gamma_c)$ is reflexive and separable, its dual $H^{-1/2}(\Gamma_c)$ is separable as well.

There exist a subsequence of $\{\overline{\lambda}_{hn}\}$ (we will denote it by the same symbol) and $\lambda^* \in H^{-1/2}(\Gamma_c)$ such that

$$\overline{\lambda}_{hn} \rightharpoonup \lambda^* \quad \text{weakly* in } H^{-1/2}(\Gamma_c) \text{ as } h \to 0_+. \tag{4.30}$$

Let $v \in \mathbb{V}$ be an arbitrary element. There exists a sequence $\{v_h\}$, $h \to 0_+$, such that $v_h \in \mathbb{V}_h$ and $v_h \to v$ in $\mathbb{V}$. Let us consider the equation in (4.26) and pass to the limit with $h \to 0_+$. Using (4.2), (4.3), (4.29), the convergence

$$[v_{hn}] \to [v_n] \quad \text{in } H^{1/2}(\Gamma_c)$$

and (4.30), we arrive at

$$\langle \lambda^*, [v_n] \rangle = S(v) - a(u, v) - (g\lambda_t, [v_t])_{0,\Gamma_c}.$$

Comparing this equation with (2.11), we infer

$$\langle \lambda^* - \lambda_n, [v_n] \rangle = 0 \quad \forall\, v \in \mathbb{V},$$

so that $\lambda^* = \lambda_n$ follows. Since the saddle point is unique by Theorem 2.1, the whole sequence $\{\overline{\lambda}_{hn}\}$ tends to λ_n weakly* in $H^{-1/2}(\Gamma_c)$. $\square$

R e m a r k 4.1. If Γ_c consists of more than one straight segment S_i $\left(\Gamma_c = \bigcup_{i \leqslant I} S_i\right)$, we have to replace $H^{-1/2}(\Gamma_c)$ by $\prod_{i \leqslant I} H^{-1/2}(S_i)$.

5. An algorithm of Uzawa type

We will propose an iterative algorithm for the search of the saddle point (u_h, λ_{hH}), i.e., for the solution of problem (3.3). To establish an effective realization of such an algorithm, we will use a method of Uzawa, combined with the idea of an artificial bolt (see [7] and [15]). The latter idea is based on the following observation.

Lemma 5.1. *Let assumptions* (1.5), (1.6) *and* (1.7) *be fulfilled. Then there exists at least one node* $s_\alpha \in N_h$ *such that*

$$[u_{hn}](s_\alpha) = 0.$$

P r o o f. By (3.9) we have $u_h \in K_h \subset \mathbb{K}$, so that $[u_{hn}] \leqslant 0$ holds on Γ_c. Let us assume that

$$\overline{u} := \max_{s \in \Gamma_c} [u_{hn}](s) < 0. \tag{5.1}$$

Setting

$$\tilde{u}_h = u_h + y, \quad y = (y^1, y^2), \quad y^1 = (0,0), \quad y^2 = (0, \overline{u}),$$

we observe that

$$y \in \mathcal{R} \cap \mathbb{V}_h.$$

Using (1.5), we obtain

$$[\tilde{u}_{hn}] = [u_{hn}] + \overline{u} n_2^2 \leqslant \overline{u}(1 + n_2^2) \leqslant 0 \quad \text{on } \Gamma_c,$$

so that $\tilde{u}_h \in K_h$.

By (3.19) we may write

$$a(u_h, v_h - u_h) + j_H(v_h) - j_H(u_h) \geqslant S(v_h - u_h) \quad \forall\, v_h \in K_h. \tag{5.2}$$

We also have the estimate (3.20), i.e.,

$$|j_H(u_h + y) - j_H(u_h)| \leqslant j(y). \tag{5.3}$$

Since $-y \in \mathcal{R} \cap \mathbb{K} \setminus \{0\}$, assumptions (1.6) and (1.7) imply

$$S(-y) < -j(-y) = -j(y). \tag{5.4}$$

Combining (5.2) with (5.3) and (5.4), we obtain

$$\begin{aligned} a(\tilde{u}_h, v_h - \tilde{u}_h) &+ j_H(v_h) - j_H(\tilde{u}_h) \\ &\geqslant a(u_h, v_h - u_h) + j_H(v_h) - j_H(u_h) - j(y) \\ &\geqslant S(v_h - u_h) + S(-y) = S(v_h - \tilde{u}_h). \end{aligned}$$

Consequently, $\tilde{u}_h$ is another solution of inequality (3.19). In the proof of Theorem 3.1 we concluded that (3.19) has a unique solution, so that we are led to a contradiction with (5.1). □

Next, we introduce an auxiliary "bolted" saddle point problem. We choose a suitable nodal point $s_\alpha \in N_h$ and assume that the bodies Ω^1 and Ω^2 have a "bolted joint" at s_α. This "bolted" problem is coercive so that we overcome the difficulties connected with semi-definiteness of the stiffness matrix corresponding to problem (3.3). The "bolted" problem will be solved by a method of Uzawa. If the resulting normal contact force at the "bolt" is positive (i.e., tension), we replace s_α by a neighbouring nodal point and repeat the algorithm. If the normal contact force is non-positive, we stop the procedure.

Definition 5.1. Let us define the following spaces and forms:

$$\begin{aligned} \mathbb{V}_h^\alpha &= \{v_h \in \mathbb{V}_h \colon\ [v_{hn}](s_\alpha) = 0\}, \\ M_{hn}^\alpha &= \{\mu_{hn} \colon\ \mu_{hn} \text{ is defined on } N_h \setminus \{s_\alpha\},\ \ \mu_{hn}(s_i) \geqslant 0\ \ \forall\, i \neq \alpha\} \\ \mathbb{M}_{hH}^\alpha &= M_{hn}^\alpha \times M_{Ht}, \\ b_{hH}^\alpha(\mu_{hH}, v_h) &= \{\mu_{hn}, [v_{hn}]\}_h^\alpha + (g_h \mu_{Ht}, [v_{ht}])_{0,\Gamma_c} \end{aligned}$$

where

$$\{\mu_{hn}, [v_{hn}]\}_h^\alpha = \sum_{i \neq \alpha}^m \mathcal{M}_i \mathcal{H}_i \mathcal{V}_i.$$

Let $(u_h^\alpha, \lambda_{hH}^\alpha) \in \mathbb{V}_h^\alpha \times \mathbb{M}_{hH}^\alpha$ be such that

$$a(u_h^\alpha, v_h) + b_{hH}^\alpha(\lambda_{hH}^\alpha, v_h) = S(v_h) \quad \forall\, v_h \in \mathbb{V}_h^\alpha, \tag{5.5}$$

$$b_{hH}^\alpha(\mu_{hH} - \lambda_{hH}^\alpha, u_h^\alpha) \leqslant 0 \quad \forall\, \mu_{hH} \in \mathbb{M}_{hH}^\alpha. \tag{5.6}$$

Then we say that $(u_h^\alpha, \lambda_{hH}^\alpha)$ is a solution of the "bolted" saddle point problem.

Theorem 5.1. *Let* (1.5) *be fulfilled. Then there exists a unique solution of problem* (5.5)–(5.6).

P r o o f. We use [9, Theorem 3.8] to prove the *existence* of a saddle point $(u_h^\alpha, \lambda_{hH}^\alpha) \in \mathbb{V}_h^\alpha \times \mathbb{M}_{hH}^\alpha$ of the functional $\mathcal{L}_{hH}^\alpha(v_h, \mu_{hH})$. In the theorem mentioned above we set

$$V = \mathbb{V}_h, \quad A = \mathbb{V}_h^\alpha, \quad Y = \mathbb{R}^m \times L^2(\Gamma_c), \quad B = \mathbb{M}_{hH}^\alpha.$$

Choosing $\mu_{hH} = 0 \in \mathbb{M}_{hH}^\alpha$ and $v_h \in \mathbb{V}_h^\alpha$ such that $\|v_h\| \to \infty$, we first infer that

$$\mathcal{L}_{hH}^\alpha(v_h, 0) = \frac{1}{2} a(v_h, v_h) - S(v_h) \to +\infty, \tag{5.7}$$

since

$$a(v_h, v_h) \geqslant c\|v_h\|^2 \quad \forall\, v_h \in \mathbb{V}_h^\alpha. \tag{5.8}$$

Indeed, (5.8) follows from assumption (1.5) since the latter implies that

$$\mathbb{V}_h^\alpha \cap \mathcal{R} = \{0\}.$$

Second, we can find $v_0 \in \mathbb{V}_h^\alpha$ such that $[v_{0t}] = 0$ on Γ_c and

$$\mathcal{V}_i^0 \equiv [v_{0n}](s_i) \leqslant \omega < 0 \quad \forall\, s_i \in N_h \setminus \{s_\alpha\}.$$

Then

$$\mathcal{L}_{hH}^\alpha(v_0, \mu_{hH}) = \frac{1}{2}a(v_0, v_0) - S(v_0) + \{\overline{\mu}_{hn}, [v_{0n}]\}_h^\alpha \to -\infty \tag{5.9}$$

as $\mu_{hn} \in M_{hn}^\alpha$, $\|\mu_{hn}\| \to \infty$, since

$$\{\mu_{hn}, [v_{0n}]\}_h^\alpha = \sum_{i \neq \alpha} \mathcal{M}_i \mathcal{H}_i \mathcal{V}_i^0 \leqslant \sum_{i \neq \alpha} |\mathcal{M}_i| \mathcal{H}_0 \omega,$$

where $\mathcal{H}_0 = \min\limits_{i \neq \alpha} \mathcal{H}_i > 0$ and $\mathcal{M}_i = \mu_{hn}(s_i)$.

To prove *uniqueness*, let us assume that (u, λ) and $(\overline{u}, \overline{\lambda})$ are two solutions of problem (5.5)–(5.6). Let us denote

$$w = u - \overline{u}, \quad \mu = \lambda - \overline{\lambda}, \quad \mu \equiv (\mu_n, \mu_t).$$

From (5.5) we deduce (dropping the subscripts and superscripts)

$$a(w, w) + b(\mu, w) = 0 \tag{5.10}$$

and (5.6) yields

$$b(\mu, w) \geqslant 0. \tag{5.11}$$

Using (5.8), (5.10) and (5.11), we arrive at

$$c\|w\|^2 \leqslant a(w, w) \leqslant 0,$$

so that $u = \overline{u}$.

From (5.5) we infer that

$$b(\mu, v) = 0 \quad \forall\, v \in \mathbb{V}_h^\alpha.$$

Choosing $v \in \mathbb{V}_h^\alpha$ such that $[v_t] = 0$ and $[v_n] = \overline{\mu}_n$, we obtain

$$\{\mu_n, \mu_n\}_h^\alpha = 0$$

so that $\mu_n = 0$ follows.

Now, let us choose $v \in \mathbb{V}_h^\alpha$ such that $[v_n] = 0$ and use Lemma 3.1 to conclude that $\mu_t = 0$. □

In what follows, we define iterations for $k = 0, 1, \ldots$, of the *algorithm of Uzawa*. We will identify the functions $\lambda_n^k \in M_{hn}^\alpha$ with vectors $\mathcal{M}^k \in \mathbb{R}^m$ (with components $\mathcal{M}_i^k$, $i \neq \alpha$, $0 \leqslant i \leqslant m$); for $\lambda_t^k \in M_{Ht}$ we introduce vectors $\Lambda^k \in \mathbb{R}^{m+1}$ such that

$$\lambda_t^k = \sum_{i=0}^{m} \Lambda_i^k \chi_i,$$

where χ_i denotes the characteristic function of the interval (z_i, z_{i+1}).

We also write

$$[u_n^k] = \sum_{i\neq\alpha} \mathcal{U}_i^k \psi_i, \quad \mathcal{T}_i^k = \int_{\Gamma_c} \chi_i g_h [u_t^k]\,\mathrm{d}s, \quad 0 \leqslant i \leqslant m.$$

In every iteration step $k = 0, 1, \ldots$ we solve the following problem: find $u^k \in \mathbb{V}_h^\alpha$ such that

$$a(u^k, v) + b_{hH}^\alpha(\lambda^k, v) = S(v) \quad \forall\, v \in \mathbb{V}_h^\alpha. \tag{5.12}$$

We set $\mathcal{M}_i^0 = 0$ for all $i \neq \alpha$ and $\mathcal{M}_\alpha^0 = \mathcal{V}_2^2/(\mathcal{H}_\alpha n_2^2)$,

$$\Lambda_i^0 = 0, \ \ 0 \leqslant i \leqslant m, \quad \varrho_k \in \mathbb{R}^1, \ \ \varrho_k > 0,$$

$$\mathcal{M}_i^{k+1} = (\mathcal{M}_i^k + \varrho_k \mathcal{U}_i^k)^+ \quad \forall\, i \neq \alpha, \tag{5.13}$$

$$\Lambda_i^{k+1} = \pi(\Lambda_i^k + \varrho_k \mathcal{T}_i^k) \quad \forall\, i = 0, 1, \ldots, m. \tag{5.14}$$

In addition to that we define $\mathcal{M}_\alpha^{k+1}$ by the following equilibrium condition:

$$\Big(\mathcal{M}_\alpha^{k+1}\mathcal{H}_\alpha + \sum_{i\neq\alpha} \mathcal{M}_i^{k+1}\mathcal{H}_i\Big) n_2^2 + t_2^2 \sum_{i=0}^{m} \int_{\Gamma_c} \Lambda_i^{k+1} \chi_i g_h \,\mathrm{d}s = \mathcal{V}_2^2. \tag{5.15}$$

Here $\pi(\cdot)$ denotes the one-dimensional projection of $\mathbb{R}^1$ onto the interval $[-1,1]$, i.e.,

$$\pi(x) = \begin{cases} 1 & \text{if } x \geqslant 1, \\ x & \text{if } |x| < 1, \\ -1 & \text{if } x \leqslant -1 \end{cases}$$

and $\mathcal{V}_2^2$ has been defined in Corollary 1.1.

Theorem 5.2. *Assume that* ϱ_k *belong to a suitable interval* $[\varrho_{\min}, \varrho_{\max}]$, $\varrho_{\min} > 0$. *Then algorithm* (5.12)–(5.14) *converges in the first component, i.e.*

$$u^k \to u_h^\alpha \quad \text{as } k \to \infty.$$

Proof. Let us denote

$$\lambda_{hn}^\alpha = \{\mathcal{M}_i^\alpha\}_{i\neq\alpha}, \quad [u_{hn}^\alpha] = \sum_{i\neq\alpha} \mathcal{U}_i \psi_i.$$

Condition (5.6) implies that the vector $\mathcal{U}$ is a normal to the closed convex set M_{hn}^α. Indeed,

$$\{\mathcal{M} - \mathcal{M}^\alpha, \varrho\mathcal{U}\}_h^\alpha \leqslant 0 \quad \forall\, \mathcal{M} \in M_{hn}^\alpha, \ \forall\, \varrho \in \mathbb{R}^1, \ \varrho > 0, \tag{5.16}$$

so that $\mathcal{M}^\alpha$ is a projection of $(\mathcal{M}^\alpha + \varrho\mathcal{U})$ in $\mathbb{R}^m$ onto the set M_{hn}^α with respect to the scalar product $\{X, Y\}_h^\alpha$. It is easy to see that the projection is determined componentwise by the non-negative parts , i.e.

$$\mathcal{M}_i^\alpha = (\mathcal{M}_i^\alpha + \varrho\mathcal{U}_i)^+ \quad \forall\, i \neq \alpha. \tag{5.17}$$

Second, (5.6) implies that

$$(g_h(\mu_t - \lambda_{Ht}^\alpha), [u_{ht}^\alpha])_{0,\Gamma_c} \leqslant 0 \quad \forall\, \mu_t \in M_{Ht}. \tag{5.18}$$

Inserting

$$\mu_t = \sum_{i=0}^m \mathcal{M}_i^t \chi_i, \quad \lambda_{Ht}^\alpha = \sum_{i=0}^m \Lambda_i^\alpha \chi_i$$

into (5.18), we obtain

$$0 \geqslant \left(\sum_{i=0}^m (\mathcal{M}_i^t - \Lambda_i^\alpha)\chi_i, g_h[u_{ht}^\alpha] \right)_{0,\Gamma_c} = \sum_{i=0}^m (\mathcal{M}_i^t - \Lambda_i^\alpha)\mathcal{T}_i, \tag{5.19}$$

where

$$\mathcal{T}_i = \int_{\Gamma_c} \chi_i g_h[u^{\alpha}_{ht}]\,\mathrm{d}s, \quad 0 \leqslant i \leqslant m. \tag{5.20}$$

From (5.19) we infer that the vector $\mathcal{T}$ is a normal to the closed convex set $\hat{M}_{Ht}$ of "nodal parameters" of M_{Ht} with respect to the standard scalar product in $\mathbb{R}^{m+1}$. As a consequence, we may write

$$\Lambda^{\alpha}_i = \pi(\Lambda^{\alpha}_i + \varrho\,\mathcal{T}_i), \quad \forall\,\varrho \in \mathbb{R}^1,\ \ \varrho > 0,\ \ i = 0, 1, \ldots, m. \tag{5.21}$$

Using the norm

$$\|X\|_h \equiv (\{X, X\}^{\alpha}_h)^{1/2},$$

from (5.13) and (5.17) we infer

$$\|\mathcal{M}^{k+1} - \mathcal{M}^{\alpha}\|_h \leqslant \|\mathcal{M}^k - \mathcal{M}^{\alpha} + \varrho_k(\mathcal{U}^k - \mathcal{U})\|_h. \tag{5.22}$$

In what follows, we denote

$$r^k_n = \mathcal{M}^k - \mathcal{M}^{\alpha} \quad \text{and} \quad r^k_t = \Lambda^k - \Lambda^{\alpha}.$$

Then

$$\|r^{k+1}_n\|_h \leqslant \|r^k_n + \varrho_k\,(\mathcal{U}^k - \mathcal{U})\|_h. \tag{5.23}$$

From (5.14) and (5.21) we infer

$$\|r^{k+1}_t\|_{m+1} \leqslant \|r^k_t + \varrho_k(\mathcal{T}^k - \mathcal{T})\|_{m+1}. \tag{5.24}$$

If we set $v_h = u^k - u^{\alpha}_h$ in (5.5) and drop the superscripts "α" and the subscripts "h, H" for the time being, we obtain

$$a(u, u^k - u) + b(\lambda, u^k - u) = S(u^k - u). \tag{5.25}$$

Setting $v = u - u^k$ in (5.12), we get

$$a(u^k, u - u^k) + b(\lambda^k, u - u^k) = S(u - u^k). \tag{5.26}$$

The sum of (5.25) and (5.26) becomes

$$a(u - u^k, u^k - u) + b(\lambda - \lambda^k, u^k - u) = 0.$$

By virtue of (5.8) we may therefore write

$$c\|u^k - u\|^2 \leqslant a(u^k - u, u^k - u) = b(\lambda - \lambda^k, u^k - u) \tag{5.27}$$
$$= -\{r_n^k, \mathcal{U}^k - \mathcal{U}\}_h^\alpha - (r_t^k)^T(\mathcal{T}^k - \mathcal{T}).$$

Inequalities (5.23) and (5.24) yield

$$\|r_n^{k+1}\|_h^2 \leqslant \|r_n^k\|_h^2 + 2\varrho_k\{r_n^k, \mathcal{U}^k - \mathcal{U}\}_h^\alpha + \varrho_k^2\|\mathcal{U}^k - \mathcal{U}\|_h^2, \tag{5.28}$$
$$\|r_t^{k+1}\|_{m+1}^2 \leqslant \|r_t^k\|_{m+1}^2 + 2\varrho_k(r_t^k)^T(\mathcal{T}^k - \mathcal{T}) + \varrho_k^2\|\mathcal{T}^k - \mathcal{T}\|_{m+1}^2. \tag{5.29}$$

Let us introduce the following norm of the pairs $r \equiv (r_n, r_t)$:

$$|\!|\!|r|\!|\!|^2 = \|r_n\|_h^2 + \|r_t\|_{m+1}^2.$$

Using (5.27), (5.28) and (5.29), we arrive at

$$|\!|\!|r^{k+1}|\!|\!|^2 \leqslant |\!|\!|r^k|\!|\!|^2 + 2\varrho_k(-c\|u^k - u\|^2) + \varrho_k^2(\|\mathcal{U}^k - \mathcal{U}\|_h^2 + \|\mathcal{T}^k - \mathcal{T}\|_{m+1}^2).$$

On the other hand,

$$\|\mathcal{U}^k - \mathcal{U}\|_h^2 + \|\mathcal{T}^k - \mathcal{T}\|_{m+1}^2 \tag{5.30}$$
$$\leqslant C(\|[u_n^k] - [u_n]\|_{0,\Gamma_c}^2 + \|[u_t^k] - [u_t]\|_{0,\Gamma_c}^2) \leqslant \tilde{C}\|u^k - u\|^2.$$

As a consequence, we have

$$|\!|\!|r^{k+1}|\!|\!|^2 + (2c\varrho_k - \tilde{C}\varrho_k^2)\|u^k - u\|^2 \leqslant |\!|\!|r^k|\!|\!|^2. \tag{5.31}$$

There exist an interval $[\varrho_{\min}, \varrho_{\max}]$ and a positive constant β such that $0 < \varrho_{\min} < \varrho_{\max}$ and

$$\varrho_k \in [\varrho_{\min}, \varrho_{\max}] \Rightarrow 2c\varrho_k - \tilde{C}\varrho_k^2 \geqslant \beta.$$

Then

$$|\!|\!|r^{k+1}|\!|\!|^2 + \beta\|u^k - u\|^2 \leqslant |\!|\!|r^k|\!|\!|^2$$

follows. The sequence $|\!|\!|r^k|\!|\!|$ is non-increasing, so that

$$|\!|\!|r^k|\!|\!| \to l, \quad 0 \leqslant l < +\infty,$$

and

$$\|u^k - u\| \to 0 \quad \text{as } k \to \infty.$$

□

Theorem 5.3. *If ϱ_k belong to a suitable interval $[\varrho_{\min}, \varrho_{\max}]$, $\varrho_{\min} > 0$, then algorithm* (5.12)–(5.14) *converges in the multipliers $(\mathcal{M}^k, \Lambda^k)$, i.e.,*

$$\mathcal{M}^k \to \mathcal{M}^\alpha \quad \textit{and} \quad \Lambda^k \to \Lambda^\alpha \quad \textit{as } k \to \infty.$$

P r o o f. Using (5.31), we observe that

$$\|\mathcal{M}^k - \mathcal{M}^\alpha\|_h \leqslant \|\!|r^k|\!\| \leqslant \|\!|r^0|\!\|$$

so that

$$\|\mathcal{M}^k\|_h \leqslant \|\mathcal{M}^\alpha\|_h + \|\!|r^0|\!\| \tag{5.32}$$

holds for all $k = 0, 1, \ldots$.

By virtue of (5.14) we have

$$|\Lambda_i^k| \leqslant 1 \quad \forall\, k.$$

Since $\{\mathcal{M}^k\}$, $\{\Lambda^k\}$ and $\{\varrho_k\}$ are bounded sequences, we can find subsequences (and denote them by the same symbols) such that

$$\mathcal{M}^k \to \overline{\mathcal{M}}, \quad \Lambda^k \to \overline{\Lambda} \quad \text{and} \quad \varrho_k \to \varrho \quad \text{as } k \to \infty. \tag{5.33}$$

From (5.13) and (5.14) we infer

$$\{\mathcal{M}^k + \varrho_k \mathcal{U}^k - \mathcal{M}^{k+1}, \mathcal{M} - \mathcal{M}^{k+1}\}_h^\alpha \leqslant 0 \quad \forall\, \mathcal{M} \in M_{hn}^\alpha \tag{5.34}$$

and

$$(\Lambda^k + \varrho_k \mathcal{T}^k - \Lambda^{k+1})^T (\Lambda - \Lambda^{k+1}) \leqslant 0 \quad \forall\, \Lambda \in \hat{M}_{Ht}, \tag{5.35}$$

respectively. We also have

$$\mathcal{U}^k \to \mathcal{U} \quad \text{and} \quad \mathcal{T}^k \to \mathcal{T} \quad \text{as } k \to \infty$$

by (5.30) and Theorem 5.2.

Passing to the limit in (5.34), (5.35), we obtain

$$\{[\overline{\mathcal{M}} + \varrho\mathcal{U}] - \overline{\mathcal{M}}, \mathcal{M} - \overline{\mathcal{M}}\}_h^\alpha \leqslant 0 \quad \forall\, \mathcal{M} \in M_{hn}^\alpha, \tag{5.36}$$

$$([\overline{\Lambda} + \varrho\,\mathcal{T}] - \overline{\Lambda})^T (\Lambda - \overline{\Lambda}) \leqslant 0 \quad \forall\, \Lambda \in \hat{M}_{Ht}. \tag{5.37}$$

In addition to that, (5.12), (5.33) and Theorem 5.2 imply

$$a(u, v) + b(\overline{\lambda}, v) = S(v) \quad \forall\, v \in \mathbb{V}_h^\alpha, \tag{5.38}$$

where $\overline{\lambda} \equiv (\overline{\lambda}_n, \overline{\lambda}_t)$ corresponds to $(\overline{\mathcal{M}}, \overline{\Lambda})$.

Summarizing (5.36), (5.37) and (5.38), we infer that $(u, \overline{\lambda})$ is a solution of problem (5.5)–(5.6). Since this problem is uniquely solvable by Theorem 5.1, $\overline{\lambda} = \lambda_{hH}^\alpha$ and the whole sequences $\{\mathcal{M}^k\}$ and $\{\Lambda^k\}$ tend to $\mathcal{M}^\alpha$ and Λ^α, respectively. □

6. Solution of problem (5.12)

The process of Uzawa requires to solve the linear problem (5.12) for any iteration step $k = 0, 1, \ldots$ on the set $\Omega^1 \cup \Omega^2$. We will propose an effective algorithm for the solution, employing some ideas of [15, Part III].

First, we decompose problem (5.12) to two separate problems $\mathcal{P}^1$ and $\mathcal{P}^2$ solved on the domains Ω^1 and Ω^2, respectively. The decomposition is made possible by the introduction of the "contact force in the bolt" $\mathcal{M}_\alpha^{k+1}$ by the equilibrium condition (5.15).

We define the following problem:

$$(\mathcal{P}^1) \qquad \text{find } u^1 \in V_h^1 \text{ such that}$$

$$a^1(u^1, v) = S^1(v) - b_{hH}(\lambda^k, v) \quad \forall\, v \in V_h^1, \tag{6.1}$$

where $\mathcal{M}_\alpha^k$ is inserted into the complete form $\{\overline{\lambda}_n^k, v_n^1\}_h$. On Ω^2 we will first solve the following auxiliary problem:

$$(\mathcal{P}^2) \qquad \text{find } \omega \in V_h^2 \text{ such that } (\omega \cdot n^2)(s_\alpha) = 0 \text{ and}$$

$$a^2(\omega, v) = S^2(v) - b_{hH}(\lambda^k, v) \quad \forall\, v \in V_h^2,\ (v \cdot n^2)(s_\alpha) = 0, \tag{6.2}$$

where $\mathcal{M}_\alpha^k$ is again inserted.

Second, we find a rigid displacement $y \in \mathcal{R} \cap \mathbb{V}_h$ such that

$$(u^1 \cdot n^1 + (\omega + y) \cdot n^2)(s_\alpha) = 0 \tag{6.3}$$

and define $u^2 = \omega + y$.

Finally, we set $u^k = (u^1, u^2)$. It is readily seen that $(u^1, \omega + y)$ is a solution of problem (5.12). Indeed, we infer

$$a(u^k, v) = S(v) - b_{hH}(\lambda^k, v) \quad \forall\, v \in \mathbb{V}_h^\alpha$$

and

$$b_{hH}(\lambda^k, v) = b_{hH}^\alpha(\lambda^k, v),$$

due to the condition $[v_n](s_\alpha) = 0$ for $v \in \mathbb{V}_h^\alpha$.

Solutions of problems $\mathcal{P}^1$ and $\mathcal{P}^2$ can be constructed as follows. We define $u^1 = u^{1s} + u^{1bk}$, where

$$u^{1s} \in V_h^1, \tag{6.4}$$
$$a^1(u^{1s}, v) = S^1(v) \quad \forall\, v \in V_h^1;$$

$$u^{1bk} \in V_h^1, \tag{6.5}$$
$$a^1(u^{1bk}, v) = -b_{hH}(\lambda^k, v) \quad \forall\, v \in V_h^1.$$

Likewise, we define

$$
\begin{aligned}
&\omega = \omega^s + \omega^{bk}, \\
(6.6) \qquad &\omega^s \in V_h^2, \ \ (\omega^s \cdot n^2)(s_\alpha) = 0, \\
&a^2(\omega^s, v) = S^2(v) \quad \forall\, v \in V_h^2, \ \ (v \cdot n^2)(s_\alpha) = 0; \\
(6.7) \qquad &\omega^{bk} \in V_h^2, \ \ (\omega^{bk} \cdot n^2)(s_\alpha) = 0, \\
&a^2(\omega^{bk}, v) = -b_{hH}(\lambda^k, v) \quad \forall\, v \in V_h^2, \ \ (v \cdot n^2)(s_\alpha) = 0.
\end{aligned}
$$

Since u^{1s} and ω^s do not depend on λ^k, they can be computed once only and remain unchanged during iterations. For problems (6.4) and (6.6) a domain decomposition (e.g. FETI) can be employed.

For an effective solution of problems (6.5) and (6.7) a pre-elimination can be recommended. To this end, we renumerate the nodal points of $\mathcal{T}_h$ in such a way that the last numbers of the numbering belong to the nodes of the contact line Γ_c. Then we can eliminate components of the solution vector which do not belong to Γ_c by partial Gauss elimination. For a detailed procedure we refer to [12, p. 203]. In this way the number of unknowns is reduced and the computing time saved is considerable.

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