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STABILITY OF A FINITE ELEMENT METHOD FOR 3D EXTERIOR STATIONARY NAVIER-STOKES FLOWS

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Abstract. We consider numerical approximations of stationary incompressible Navier-Stokes flows in 3D exterior domains, with nonzero velocity at infinity. It is shown that a P1-P1 stabilized finite element method proposed by C. Rebollo: A term by term stabilization algorithm for finite element solution of incompressible flow problems, Numer. Math. 79 (1998), 283–319, is stable when applied to a Navier-Stokes flow in a truncated exterior domain with a pointwise boundary condition on the artificial boundary.

Keywords: stationary incompressible Navier-Stokes flows, exterior domains, stabilized finite element methods, stability estimates

MSC 2000: 35Q30, 65N30, 76D05

1. Introduction

Consider a rigid body moving steadily and without rotation in an incompressible viscous fluid, with the external boundaries of the flow so far away that the fluid field near the body is not influenced by them. Assume that the fluid particles close to the surface of the body adhere to this surface. Further suppose that the flow is described with respect to a coordinate system attached to the moving object. Then a mathematical model of this physical situation is given by the incompressible stationary Navier-Stokes system in an exterior domain $\mathbb{R}^3 \setminus \overline{U}$, with a homogeneous Dirichlet boundary condition on ∂U and a nonzero boundary condition at infinity,

$$(1.1.)\quad -\nu \cdot \Delta w + (w \cdot \nabla)w + (1/\varrho) \cdot \nabla p = (1/\varrho) \cdot g, \quad \operatorname{div} w = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{U},$$
$$w|_{\partial U} = 0, \quad |w(x) - b^* \cdot (1,0,0)| \to 0 \quad \text{for } |x| \to \infty,$$

where $U \subset \mathbb{R}^3$ is the bounded domain which models the moving object. The quantities ν (kinematic viscosity), ϱ (constant density of the fluid) and b^* (speed of the object with respect to the far field of the fluid) are positive reals. The functions g (volume force) and w (velocity field of the fluid) are vector fields on $\mathbb{R}^3 \setminus \overline{U}$, and p (pressure in the fluid) is a scalar function with the same domain. The functions w and p are unknown, all the other quantities are given.

Since from the mathematical point of view it is more convenient to deal with homogeneous boundary conditions at infinity, we transform the velocity in problem (1.1) by a translation. In addition, we introduce a change of scale in order to convert the Navier-Stokes system into a dimensionless form. In this way, we arrive at a modified Navier-Stokes system in an exterior domain $\mathbb{R}^3 \setminus \overline{\Omega}$, with a homogeneous boundary condition at infinity and a nonhomogeneous boundary condition on $\partial\Omega$,

$$(1.2) \qquad \begin{gathered} -\Delta u + \tau \cdot \partial_1 u + \tau \cdot (u \cdot \nabla) u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ u|_{\partial\Omega} = (-1, 0, 0), \quad |u(x)| \to 0 \quad \text{for } |x| \to \infty, \end{gathered}$$

where $\tau \in (0, \infty)$ is the Reynolds number of the fluid. The set Ω is supposed to be open, bounded and polyhedral, with Lipschitz boundary. Note that an additional term, namely $\tau \cdot \partial_1 u$ ("Oseen term"), arises in these differential equations. This is why we call them "modified Navier-Stokes equations".

We would like to apply our theory also to the linear Oseen system which is obtained from (1.2) by removing the nonlinear term $\tau \cdot (u \cdot \nabla)u$. Therefore we consider a problem slightly more general than (1.2), which is obtained from (1.2) by replacing the factor τ in the nonlinearity by a constant $\widetilde{\tau} \in [0, \tau]$. Thus we arrive at the following boundary value problem:

$$(1.3) \qquad \begin{gathered} -\Delta u + \tau \cdot \partial_1 u + \widetilde{\tau} \cdot (u \cdot \nabla) u + \nabla \pi = f, \quad \operatorname{div} u = 0 \quad \text{in } \mathbb{R}^3 \setminus \overline{\Omega}, \\ u|_{\partial\Omega} = (-1, 0, 0), \quad |u(x)| \to 0 \quad \text{for } |x| \to \infty. \end{gathered}$$

For $\widetilde{\tau} = 0$, the preceding differential equations reduce to the Oseen system, whereas in the case $\widetilde{\tau} = \tau$, they coincide with the modified Navier-Stokes system from (1.2). If we suppose $f \in L^{6/5}(\mathbb{R}^3)^3$, it is well known that problem (1.2) admits a solution (u, π) with

$$(1.4) \qquad u \in W^{2,6/5}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{\Omega})^3, \quad \nabla u \in L^2(\mathbb{R}^3 \setminus \overline{\Omega})^9, \quad \pi \in W^{1,6/5}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{\Omega});$$

see [24, Theorem IX.4.1, IX.1.1]. The arguments in the proof of these theorems yield an analogous existence result for problem (1.3). Let us fix a solution (u, π) of (1.3) with properties as in (4). We call this solution "exterior flow". If the function f decays sufficiently fast for $|x| \to \infty$, then this exterior flow decays as well; more details may be found in [22], [3], [17], [18], [16], [13]. In the work at hand, we study the problem of how this exterior flow may be computed numerically by finite element methods.

It is not obvious how to perform such computations since the exterior domain $\mathbb{R}^3 \setminus \overline{\Omega}$ is unbounded and thus does not lend itself to standard finite element methods. Of course, we may choose a bounded computational domain like $B_R \setminus \overline{\Omega}$ (a "truncated exterior domain"), where B_R denotes the open ball with radius $R > 0$ and center at the origin. But then the question arises as to which boundary condition should be imposed on the outer ("artificial") boundary ∂B_R. The answer to this question characterizes the different methods for computing exterior flows.

The approach we choose here follows the one proposed by Guirguis, Gunzburger [30] for the Stokes system in exterior domains; also see [31, Chapters 16.3, 16.4], [10]. The crucial step of this approach consists in prescribing a local (that is, pointwise) artificial boundary condition on ∂B_R. In our case, we choose the nonlinear condition proposed in [16, Lemma 6.5]. Then we consider the boundary value problem in $B_R \setminus \overline{\Omega}$ consisting of this artificial boundary condition, a modified Navier-Stokes system as in (1.3), and a Dirichlet boundary condition on $\partial\Omega$ also as in (1.3). This boundary value problem, stated in [16, Lemma 6.5], is written in variational form in [16, (1.7)]. It is this latter problem which is actually studied in [16]. A solution of [16, (1.7)] will not in general coincide with the restriction of the exterior flow to $B_R \setminus \overline{\Omega}$. This discrepancy (the "truncation error") need not constitute a major flaw, however, because any subsequent discretization of the problem in $B_R \setminus \overline{\Omega}$ introduces an error in any case. But the truncation error should decay as fast as possible when R is increased. In the case of problem [16, (1.7)], this decay rate measured in a suitable norm was shown to be $O(R^{-1})$ for $R \to \infty$, at least if the Reynolds number is small ([16, Theorem 7.1]). No further results seem to be known in this respect, neither for the artificial boundary condition from [16, Lemma 6.5] nor for any other.

In the present work, we discretize the variational problem in [16, (1.7)] by the P1-P1-finite element method proposed by Rebollo [46]. This discretization is given by the finite element variational problem stated in equations (2.5)–(2.7) below.

Some remarks are in order with respect to this discretization. First it should be noted that actually we do not choose $B_R \setminus \overline{\Omega}$ as our truncated exterior domain. Instead we approximate $B_R \setminus \overline{\Omega}$ by a polyhedron $P_{h,R}$, which is better suited for finite element methods than a domain with a spherical exterior boundary. The role of the truncating sphere ∂B_R will be played by the surface $\partial P_{h,R} \setminus \partial\Omega$ (a set which will be abbreviated by $\partial_{h,R}$ in the following; see Section 2).

We further remark that the index h is a discretization parameter which indicates that $P_{h,R}$ may be considered to be the interior of the union of certain closed tetrahedrons. These tetrahedrons constitute a graded mesh with the property that the tetrahedrons near Ω have a diameter of about h. Following an idea due to Goldstein [26], [27], we increase the mesh size with increasing distance from Ω in such a way that the tetrahedrons near ∂B_R have a diameter of about $h \cdot R$. Of course, such a graded mesh reduces the number of unknowns of the algebraic system which corresponds to (2.5)–(2.7); see a remark in this respect in Section 2, following assumption (A3). For the case of the Poisson equation, Goldstein could show that the usual finite element error estimates remain valid for this type of mesh, provided the solution of the exterior problem decays fast enough near infinity. We refer to [30], [31, Chapters 16.3, 16.4], [10] for similar results pertaining to the Stokes system.

We finally remark with respect to (2.5)–(2.7) that there seems to be a discrepancy between the almost rotationally symmetric form of the truncating surface $\partial_{h,R}$, which should be considered an approximation of the sphere ∂B_R, and the asymmetric character of the exterior flow (u, π), which exhibits a wake ([23, p. 357–358], [24, p. 60–61]). However, the bilinear form a in variational problem (2.5) takes account of this type of flow field, via an asymmetric surface integral on ∂B_R figuring in the definition of a.

In the present work, we are going to show that a solution of (2.5)–(2.7) is stable in the sense that its velocity part may be bounded in a suitable norm independently of h and R, under the assumption that $\widetilde{\tau} = \tau$ and h satisfies a smallness condition with respect to τ (Theorem 2.2), or that $\widetilde{\tau}$ is smaller than a constant which is given a priori and does not depend on the data (Theorem 2.3). Simultaneously, we prove existence of solutions. In a companion paper [14], we exploit some of these results in order to estimate the difference between a solution of (2.5)–(2.7) and the exterior flow, under the assumption that $\widetilde{\tau} = 0$ and $\kappa = 0$. This means that the paper [14] corresponds to an application of the Guirguis-Gunzburger approach to the Oseen system. The same error estimates hold for stationary Navier-Stokes flows ($\tau = \widetilde{\tau}$) with small Reynolds number τ, with and without convection stabilization ($\kappa = 0$ or $\kappa = 1$; paper in preparation). Preconditioners adapted to the systems of equations arising by our method are discussed in [9]. An article reporting on some numerical tests we performed with our method is also in preparation.

It might be asked why we have chosen Rebollo's finite element method for our discretization of [16, (1.7)]. There are essentially three reasons for this choice. Firstly, due to the article [46], optimal error estimates are available when this method is applied to the stationary Navier-Stokes system in a fixed bounded domain. Thus a theory is available which we could build on. Secondly, Rebollo proposed a stabilized P1-P1 method which circumvents the LBB-condition by means of a stabilization term ("pressure stabilization") that does not involve any parameter. Thus, implementing Rebollo's method is relatively simple (although it is never a really easy affair to program a 3D Navier-Stokes solver). Finally, Rebollo's theory also takes into account stabilization with respect to the convective term ("velocity stabilization"). Although this point is not at the center of our interest here, we wanted to include it in our theory.

It should be pointed out that Rebollo's theory does not carry over easily to problem (2.5)–(2.7). This is because certain features of these equations are not present in the problem considered in [46]. These features include the Oseen term, the graded meshes, the inhomogeneous boundary conditions on $\partial\Omega$, and the parameter R which has to be controlled in stability and error estimates, in addition to the parameter h. As a consequence, stability turned out to be a major problem in our context, whereas it was not an issue in [46]. Another aspect of problem (2.5)–(2.7) is that it cannot be separated into a "good" linear and a "bad" nonlinear part. Actually it is the linear part which gives rise to the most serious difficulties, due to the interaction between the Oseen term, the pressure, the graded mesh and the parameter R.

In addition to the articles previously mentioned, we want to point out some further references related to the computation of exterior flows. We begin by indicating some theoretical results concerning pointwise artificial boundary conditions for elliptic equations. Such results may be found in [40] (Laplace equations), [41] (Stokes system), [15] (Oseen system) and [42] (Navier-Stokes system with zero velocity at infinity).

A large number of articles deal with computation of exterior flows by coupling boundary integral equations with partial differential equations. Typically this approach has been used in order to approximate linear exterior flows. As examples, we mention [29], [47], [48], where the Stokes system is considered, and reference [32], where theoretical aspects of the approximation of solutions of the Oseen system in half-space are discussed. But there are also some articles where the same kind of approach is applied to nonlinear fluid flow models. In this respect, we mention Feistauer, Schwab [19]–[21], who approximate exterior Navier-Stokes flows by an Oseen flow outside a fixed ball B_R, and by a Navier-Stokes flow inside that ball. The Oseen flow is then reduced to an integral equation on ∂B_R, which may be viewed as a nonlocal artificial boundary condition on ∂B_R for a boundary value problem in $B_R \setminus \overline{\Omega}$. It is shown in [19]–[21] that the coupled problem arising in this way admits a solution. In [33], the same general idea was applied to 2D exterior time-dependent Navier-Stokes flows. Error estimates are obtained under the assumption that the exterior Navier-Stokes flow and the solution of a certain Oseen-like problem are bounded pointwise by some ε outside B_R. It is not clear, however, how these conditions are linked to any known results on the asymptotic behaviour of time-dependent 2D exterior Navier-Stokes flows.

Nishida [44] approximates exterior Oseen flows by a collocation method applied to the Oseen fundamental solution. He presents numerical tests, but no theoretical results. Stationary 2D exterior Navier-Stokes flows are studied in [4]. That reference offers a numerical method with a free parameter which is numerically determined as a function of the total drag exerted on the rigid body. A survey of papers treating the computation of exterior flows, with the emphasis on compressible flows, may be found in [49]. Numerical and theoretical studies on artificial boundary conditions for flows in pipe-like domain are presented in [7], [8], [34], [35], [36], [37]–[39]. Exterior magnetic fields are treated in [2], where pointwise artificial boundary conditions and graded meshes are considered.

Let us indicate the structure of this article. In Section 2, we specify the assumptions on our grids, introduce our finite element variational problem (2.5)–(2.7), and state our results on existence and stability of solutions for this problem (Theorems 2.2 and 2.3). Section 3 contains some further notation, as well as some auxiliary results. Section 4 is devoted to a linearization of (2.5)–(2.7) given by variational problem (4.27), (4.28). We establish certain stability estimates for solutions of this problem (Corollaries 4.2, 4.3). The main difficulty in the corresponding proofs consists in controlling the pressure (Theorems 4.1 and 4.2). Another, less serious obstacle concerns the extension of the boundary value $(-1, 0, 0)$ to a piecewise linear function $\widetilde{e}$ on $P_{h,R}$. We will construct such an extension $\widetilde{e}$ which in addition has compact support, small divergence (of order h), and gives rise to a factor ε when inserted as a test function into the Navier-Stokes nonlinearity (Theorem 4.4). Once our linear theory is available, we may prove Theorems 2.2 and 2.3 by applying Brower's point theorem as in [46]. This is done in Section 5. In the Appendix, we sketch proofs of certain results from analysis stated in Section 3.

2. Triangulations and function spaces. Formulation of a finite element variational problem. Statement of main results

Let Ω be an open bounded polyhedron in $\mathbb{R}^3$, which will be kept fixed throughout. We suppose that Ω is Lipschitz bounded in the sense of [1, 4.5], and that Ω and $\overline{\Omega}^c$ are connected. Here and in the sequel, we use the notation $V^c := \mathbb{R}^3 \setminus V$ for $V \subset \mathbb{R}^3$. If V is Lebesgue measurable, we denote its measure by $|V|$. We further put $B_r(x) := \{y \in \mathbb{R}^3 : \ |y - x| < r\}$ for $r > 0$, $x \in \mathbb{R}^3$, and $B_R := B_R(0)$, $\Omega_R := B_R \setminus \overline{\Omega}$ for $R > 0$.

We fix some $S > 0$ with $\overline{\Omega} \subset B_S$. The set B_S is to contain the region where we would like to compute the flow field described by (1.3). Therefore our computational domain should at least contain the set Ω_S (but quite possibly must be larger in order to yield sufficiently precise results). We introduce annular domains around Ω_S, which will be used to characterize our grids: we put $U_0 := \Omega_S$, $U_j := B_{2^j \cdot S} \setminus B_{2^{j-1} \cdot S}$ for $j \in \mathbb{N}$. It will be convenient to use the notation $U_{-1} := \emptyset$.

For $h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$, we choose an open polyhedron $P_{h,R}$, an integer $k(h,R) \in \mathbb{N}$ and a family $\mathcal{T}_{h,R} = (K_l^{(h,R)})_{1 \leqslant l \leqslant k(h,R)}$ of closed tetrahedrons $K_l^{(h,R)}$ which decompose $P_{h,R}$. We will always write k and $K_1, \dots, K_k$ instead of $k(h,R)$, $K_1^{(h,R)}, \dots, K_{k(h,R)}^{(h,R)}$, respectively. In conditions (A1)–(A8) below, we state the assumptions which are to be verified by $P_{h,R}$ and $\mathcal{T}_{h,R}$. Conditions (A1) and (A2) specify that $P_{h,R}$ is a subset of Ω_R and $\mathcal{T}_{h,R}$ is a triangulation of $P_{h,R}$:

(A1) $\overline{P_{h,R}} = \bigcup_{l=1}^{k} K_l \; \forall h \in (0, \frac{1}{2}S) \; \forall R \in (S, \infty)$.

(A2) $K_l \subset \Omega_R$; $K_l \cap K_m$ is either empty or a common vertex or a common side or a common face of K_l and K_m ($h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$, $l, m \in \{1, \dots, k\}$ with $l \neq m$).

Next we require our triangulations to be regular in the sense of [45, Definition 3.4.1], and we specify the mesh-grading process from [26], [27]:

(A3) There is $\sigma_1 > 0$ such that $\sigma_1 \cdot 2^j \cdot h \leqslant \sup\{r \in (0,\infty) \colon B_r(x) \subset K_l$ for some $x \in K_l\}$ for $h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$, $l \in \{1, \dots, k\}$, $j \in \mathbb{N}_0$ with $K_l \cap U_j \neq \emptyset$; $\operatorname{diam} K_l \leqslant 2^j \cdot h$ for h, R, l, j as before.

Due to (A3), the number of unknowns arising in the finite element method associated to the mesh $\mathcal{T}_{h,R}$ is proportional to $h^{-3} \cdot \ln(R/S)$; see [26], [27], [10, Lemma 6.3]. The remaining assumptions read as follows:

(A4) $B_S \cap P_{h,R} = \Omega_S$; if $l \in \{1, \dots, k\}$ and if $x \in \partial P_{h,R} \setminus \partial\Omega$ is a vertex of K_l, then $x \in \partial B_R$ ($h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$).

(A5) The domain $P_{h,R} \cup \overline{\Omega}$ is convex ($h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$).

(A6) There is $\sigma_2 \in (0, \infty)$ such that for $h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$, $l \in \{1, \dots, k\}$, the macroelement $(K_l)_\Delta := \bigcup\{K_m \colon m \in \{1, \dots, k\}$ with $K_m \cap K_l \neq \emptyset\}$ is star-shaped with respect to the ball $B_{\sigma_2 \cdot \operatorname{diam} K_l}(x)$ for some $x \in (K_l)_\Delta$. (This means that for all $y \in (K_l)_\Delta$, the closed convex hull of $\{y\} \cup B_{\sigma_2 \cdot \operatorname{diam} K_l}(x)$ is a subset of $(K_l)_\Delta$; compare [5, (4.2.2)].)

(A7) For any $l \in \{1, \dots, k\}$, at least one vertex of K_l is located in the (open) set $P_{h,R}$ ($h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$).

(A8) There is $\varphi_1 \in (0, \frac{1}{2}\pi)$ such that for $h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$, the relation

$$x + \{z \in \mathbb{R}^3 \setminus \{0\} \colon |x|^{-1} \cdot |z|^{-1} \cdot (x \cdot z) > \cos\varphi_1\} \subset \mathbb{R}^3 \setminus \overline{P'_{h,R} \cup \Omega}$$

holds for any $x \in \partial P'_{h,R} \setminus \partial\Omega$, where $P'_{h,R}$ denotes the interior of the union of the tetrahedrons K_l with $K_l \subset \Omega_{2 \cdot S}$.

Assumption (A4) may be interpreted in the sense that the polyhedral domain $P_{h,R}$ coincides with Ω_R, except for a remainder set near ∂B_R. As will become evident in the following, the convexity of $P_{h,R} \cup \overline{\Omega}$ (assumtption (A5)) will be useful in many respects. Condition (A6) will be needed for introducing interpolation operators of Clément type (Theorem 3.2); compare [5, p. 120–123], where (A6) is used implicitly.

As concerns assumptions (A7) and (A8), they are imposed so that we may use [11, Theorem 4.1], which pertains to the LBB condition for the Mini element on domains similar to $P_{h,R}$, and which is stated in a slightly modified version in Theorem 3.1 below. Condition (A8) means that the tetrahedrons located in $\Omega_{2\cdot S}$ form a polyhedron $P'_{h,R}$ whose "outer boundary" $\partial P'_{h,R} \setminus \partial\Omega$ verifies an exterior cone condition uniformly in h and R.

The convexity of $P_{h,R} \cup \overline{\Omega}$ and our assumptions on $\partial\Omega$ imply that $P_{h,R}$ is Lipschitz bounded ([28, Corollary 1.2.2.3]). This means in particular that the outward unit normal to $P_{h,R}$ is well defined ([43, p. 88–89]). We denote this outward unit normal by $n^{(h,R)}$. Due to assumption (A4), the boundary of $P_{h,R}$ consists of the "inner part" $\partial\Omega$ and the "outer part" $\partial P_{h,R} \setminus \partial\Omega$. It will be convenient to use the abbreviation $\partial_{h,R} := \partial P_{h,R} \setminus \partial\Omega$ for this "outer part", which may be viewed as the surface which cuts off the exterior domain $\overline{\Omega}^c$.

Next we introduce some notation related to Sobolev spaces. Let $V \subset \mathbb{R}^3$ be open. For a function $v\colon V \mapsto \mathbb{R}$ with appropriate regularity, the terms $\partial_l v$ $(1 \leqslant l \leqslant 3)$, ∇v, $\partial^a v$ (with $a \in \mathbb{N}_0^3$) are used in an obvious way in order to denote partial derivatives of v. For functions $w, \widetilde{w}\colon V \mapsto \mathbb{R}^3$, the terms $\operatorname{div} w$ and $(w \cdot \nabla)\widetilde{w}$ are defined in the usual way. If $p \in [1, \infty]$ and $m \in \mathbb{N}$, the Sobolev space of order m and exponent p is denoted by $W^{m,p}(V)$. In the case $p = 2$, we write $H^m(V)$ instead of $W^{m,p}(V)$. The standard norm in $W^{m,p}(V)$ is denoted by $\|\cdot\|_{m,p}$. The subspace $H_0^1(V)$ of $H^1(V)$ is defined in the usual way. By $H_0^{-1}(V)$ we denote the dual space of $H_0^1(V)$.

Let $\sigma \in \mathbb{N}$, $\sigma \geqslant 2$, let $\mathcal{H}$ be a vector space, and let $\|\cdot\|$ be a norm or seminorm on $\mathcal{H}$. Then we use the same symbol $\|\cdot\|$ for the norm or seminorm on $\mathcal{H}^\sigma$ which associates the value $\Big(\sum\limits_{j=1}^{\sigma} \|F_j\|^2\Big)^{1/2}$ to each element $(F_1, \dots, F_\sigma) \in \mathcal{H}^\sigma$. For example, if $v \in H^1(V)^3$, the term $\|\nabla v\|_2$ denotes the L^2-norm of ∇v in $L^2(V)^9$.

Let $h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$. We introduce a special notation for the space of all H^1-functions on $P_{h,R}$ vanishing on $\partial\Omega$. In fact, we put $W_{h,R} := \{v \in H^1(P_{h,R})^3\colon v|_{\partial\Omega} = 0\}$. Moreover, for $v \in W_{h,R}$, we set $\|v\|^{(h,R)} := (\|\nabla v\|_2^2 + R^{-1} \cdot \|v|_{\partial_{h,R}}\|_2^2)^{1/2}$. The mapping $\|\cdot\|^{(h,R)}$ is a norm on $W_{h,R}$ which is equivalent to the norm $\|\cdot\|_{1,2}|_{W_{h,R}}$. This is an immediate consequence of Theorem 3.4 and a standard trace theorem. For $l \in \{1, \dots, k\}$, let $P_1(K_l)$ denote the space of all polynomials over K_l of degree at most 1. We put

$$V_{h,R} := \{v \in C^0(\overline{P_{h,R}})^3\colon\ v|_{K_l} \in P_1(K_l)^3 \quad \text{for } 1 \leqslant l \leqslant k\},$$
$$M_{h,R} := \{\varrho \in C^0(\overline{P_{h,R}})\colon\ \varrho|_{K_l} \in P_1(K_l) \quad \text{for } 1 \leqslant l \leqslant k\},$$

and $Y_{h,R} := \{v \in V_{h,R}\colon\ v|_{\partial\Omega} = 0\}$. It will be convenient to use the notation

$$\|F\|_{-1} := \sup\{|F(w)|/\|w\|^{(h,R)}\colon\ w \in Y_{h,R},\ \ w \neq 0\} \quad \text{for } F \in Y'_{h,R}. \tag{2.1}$$

We denote by $\mathcal{B}_{h,R}$ the space of all functions $v\colon \overline{P_{h,R}} \mapsto \mathbb{R}$ such that for $l \in \{1, \dots, k\}$ we have $v|_{K_l} = \alpha_l \cdot b_{K_l}$ for some $\alpha_l \in \mathbb{R}$, where b_{K_l} is the standard bubble function on K_l (that is, the polynomial of order 4 on K_l vanishing on ∂K_l as defined in [46, p. 287]). Note that $\mathcal{B}_{h,R} \subset C^0(\overline{P_{h,R}}) \cap H_0^1(P_{h,R})$. Next we introduce bi- and trilinear forms which enter into our finite element variational problem (2.5)–(2.7) below. To this end, take $\tau \in (0, \infty)$ and $\widetilde{\tau} \in [0, \tau]$. These parameters will be kept fixed throughout. Then, for $v, w, z \in H^1(P_{h,R})^3$, $q \in L^2(P_{h,R})$, set

$$
\begin{aligned}
a(v, w) := a_{h,R,\tau}(v, w) := &\int_{P_{h,R}} \left(\sum_{k=1}^{3} \partial_k v \cdot \partial_k w + \tau \cdot \partial_1 v \cdot w \right) \mathrm{d}x \\
&+ \int_{\partial_{h,R}} \left(R^{-1} + \frac{1}{2} \tau \cdot (1 - n_1^{(h,R)}) \right) \cdot (v \cdot w) \, \mathrm{d}o_x, \\
b(z, v, w) := b_{h,R,\widetilde{\tau}}(z, v, w) := \widetilde{\tau} \cdot &\int_{P_{h,R}} \left((z \cdot \nabla) v \cdot w + \operatorname{div} z \cdot \tfrac{1}{2} (v \cdot w) \right) \mathrm{d}x \\
&- \tfrac{1}{2} \widetilde{\tau} \cdot \int_{\partial_{h,R}} (z \cdot n^{(h,R)}) \cdot (v \cdot w) \, \mathrm{d}o_x, \\
c(v, q) := c_{h,R}(v, q) := - &\int_{P_{h,R}} \operatorname{div} v \cdot q \, \mathrm{d}x.
\end{aligned}
$$

Thus the linear differential operator $-\Delta + \tau \cdot \partial_1$ in (1.3) corresponds to the bilinear form a, and the nonlinearity $\widetilde{\tau} \cdot (u \cdot \nabla) u$ is represented by the trilinear form b (which is defined in such a way that $b(z, \cdot, \cdot)$ is skew-symmetric on $W_{h,R}$). In addition, the linear part of the artificial boundary condition from [16, Lemma 6.5] is included in the form a, and the nonlinear part in b, each time in the form of a boundary integral. (Of course, the terms in the artificial boundary condition which correspond to a "natural" boundary condition in the sense of variational calculus do not appear in the above definitions.) We still have to introduce Rebollo's stabilization terms. To this end, we consider mappings

$$
\begin{aligned}
A_1 &:= A_{1,h,R,\tau,\widetilde{\tau}}\colon\ V_{h,R} \times H_0^1(P_{h,R})^3 \times H_0^1(P_{h,R})^3 \mapsto \mathbb{R}, \\
A_2 &:= A_{2,h,R}\colon\ H_0^1(P_{h,R})^3 \times H_0^1(P_{h,R})^3 \mapsto \mathbb{R}
\end{aligned}
$$

for which the following properties hold:

(S1) $A_1(z, \cdot, \cdot)$ and A_2 are bilinear and symmetric forms ($z \in V_{h,R}$).

(S2) There are constants $\alpha_1, \alpha_2, \alpha_3 \in (0, \infty)$, independent of h and R, such that

$$
A_1(z, V, V) \geqslant \alpha_1 \cdot \|\nabla V\|_2^2, \quad A_2(V, V) \geqslant \alpha_2 \cdot \|\nabla V\|_2^2,
$$
$$
|A_2(V, W)| \leqslant \alpha_3 \cdot \|\nabla V\|_2 \cdot \|\nabla W\|_2
$$

for $V, W \in \mathcal{B}^3_{h,R}$, $z \in V_{h,R}$. Moreover,

$$\sup\{|A_1(z, V, W)|/(\|\nabla V\|_2 \cdot \|\nabla W\|_2) \colon\ V, W \in \mathcal{B}^3_{h,R} \setminus \{0\}\} < \infty \ \ \forall\, z \in V_{h,R}.$$

(S3) For $z, \widetilde{z} \in V_{h,R}$, set

$$\mathcal{A}(z, \widetilde{z}) := \sup\{|A_1(z, V, W) - A_1(\widetilde{z}, V, W)| \cdot (\|\nabla V\|_2 \cdot \|\nabla W\|_2)^{-1} \colon \\ V, W \in \mathcal{B}^3_{h,R} \setminus \{0\}\}.$$

Then, for $z \in V_{h,R}$ and for sequences (z_n) in $V_{h,R}$ with $\|z_n - z\|^{(h,R)} \to 0$, the term $\mathcal{A}(z, z_n)$ tends to zero for $n \to \infty$.

The next step in Rebollo's construction of stabilization terms consists in introducing solution operators with respect to variational problems associated to the forms A_1 and A_2, respectively:

Theorem 2.1 ([46, p. 288–289])**.** *Let $z \in V_{h,R}$. Then there are mappings*

$$\mathcal{R}_1(z, \cdot) := \mathcal{R}_{1,A_1}(z, \cdot) \colon\ H_0^{-1}(P_{h,R})^3 \mapsto \mathcal{B}^3_{h,R}, \quad \mathcal{R}_2 := \mathcal{R}_{2,A_2} \colon\ H_0^{-1}(P_{h,R})^3 \mapsto \mathcal{B}^3_{h,R}$$

such that for $F \in H_0^{-1}(P_{h,R})^3$, $W \in \mathcal{B}^3_{h,R}$,

$$A_1(z, \mathcal{R}_1(z, F), W) = F(W), \quad A_2(\mathcal{R}_2(F), W) = F(W).$$

For $v, w \in H^1(P_{h,R})^3$ we further define

$$\mathcal{S}(v, w) := \mathcal{S}_{h,R,\tau,\widetilde{\tau}}(v, w) \colon\ H_0^1(P_{h,R})^3 \ni z \\ \mapsto \int_{P_{h,R}} (\tau \cdot \partial_1 w + \widetilde{\tau} \cdot (v \cdot \nabla) w) \cdot z \, \mathrm{d}x \in \mathbb{R}. \tag{2.2}$$

By the Hölder and Poincaré inequalities and by the imbedding of $H^1(P_{h,R})$ into $L^6(P_{h,R})$, the linear form $\mathcal{S}(v, w)$ belongs to $H_0^{-1}(P_{h,R})^3$. Thus we have introduced an operator

$$\mathcal{S} := \mathcal{S}_{h,R,\tau,\widetilde{\tau}} \colon\ H^1(P_{h,R})^3 \times H^1(P_{h,R})^3 \mapsto H_0^{-1}(P_{h,R})^3.$$

For brevity, we will use the abbreviations

$$\mathcal{R}_1(\mathcal{S}(y, w)) := \mathcal{R}_1(y, \mathcal{S}(y, w)), \\ A_1(\mathcal{R}_1(\mathcal{S}(y, w)), W) := A_1(y, \mathcal{R}_1(y, \mathcal{S}(y, w)), W), \tag{2.3}$$

where $y \in V_{h,R}$, $w \in H^1(P_{h,R})^3$, $W \in \mathcal{B}^3_{h,R}$. For $q \in M_{h,R}$ we consider the mapping $\mathcal{Z}_q\colon H^1_0(P_{h,R})^3 \ni w \mapsto -\int_{P_{h,R}} q \cdot \operatorname{div} w \, \mathrm{d}x \in \mathbb{R}$. This mapping belongs to $H^{-1}_0(P_{h,R})^3$. Following [46], we will denote it by ∇q. This convention, Theorem 2.1 and (2.2) yield

$$A_1(\mathcal{R}_1(\mathcal{S}(z,v)), W) = \int_{P_{h,R}} (\tau \cdot \partial_1 v + \widetilde{\tau} \cdot (z \cdot \nabla)v) \cdot W \, \mathrm{d}x, \tag{2.4}$$

$$A_2(\mathcal{R}_2(\nabla q), W) = -\int_{P_{h,R}} q \cdot \operatorname{div} W \, \mathrm{d}x \quad \forall\, z, v \in V_{h,R} \ \forall\, q \in M_{h,R}, \ \forall\, W \in \mathcal{B}^3_{h,R}.$$

Now we are in a position to state our finite element variational problem. Take $\kappa \in \{0, 1\}$. Then this problem reads as follows: for $F \in Y'_{h,R}$, find $u_{h,R} = u_{h,R,\tau,\widetilde{\tau},F,\kappa,A_1,A_2} \in V_{h,R}$, $\pi_{h,R} = \pi_{h,R,\tau,\widetilde{\tau},F,\kappa,A_1,A_2} \in M_{h,R}$ such that

$$a(u_{h,R}, w) + b(u_{h,R}, u_{h,R}, w) + \kappa \cdot A_1\big(\mathcal{R}_1(\mathcal{S}(u_{h,R}, u_{h,R})), \mathcal{R}_1(\mathcal{S}(u_{h,R}, w))\big) + c(w, \pi_{h,R}) = F(w) \quad \forall\, w \in Y_{h,R}, \tag{2.5}$$

$$c(u_{h,R}, q) = A_2(\mathcal{R}_2(\nabla \pi_{h,R}), \mathcal{R}_2(\nabla q)) \quad \forall\, q \in M_{h,R}, \tag{2.6}$$

$$u_{h,R}|_{\partial\Omega} = (-1, 0, 0). \tag{2.7}$$

As explained in [46], the term $A_1\big(\mathcal{R}_1(\mathcal{S}(u_{h,R}, u_{h,R})), \mathcal{R}_1(\mathcal{S}(u_{h,R}, w))\big)$ in (2.5) is introduced in order to avoid oscillations due to convection ("velocity stabilization"). Moreover, the LBB-condition, not satisfied by the P1-P1-space $(V_{h,R}, M_{h,R})$, is circumvented by adding the term $A_2(\mathcal{R}_2(\nabla \pi_{h,R}), \mathcal{R}_2(\nabla q))$ in (2.6) ("pressure stabilization"). Due to the parameter κ, we actually consider two problems, one with a velocity stabilization term (if $\kappa = 1$), and another one without such a term (if $\kappa = 0$).

In the next two theorems, we present our results on existence and stability of solutions to (2.5)–(2.7). We recall that the parameters σ_1, σ_2 and φ mentioned in these theorems were introduced in (A3), (A6) and (A8), respectively, and that S is a fixed positive real with $\overline{\Omega} \subset B_S$ (see the beginning of this section).

Theorem 2.2. *Suppose $\widetilde{\tau} = \tau$. Then there are constants $R_0 \in [8S, \infty[$, $C_1(\tau), C_2(\tau) > 0$ depending on Ω, S, σ_1, σ_2, φ_1, and in the case of $C_1(\tau)$ and $C_2(\tau)$ also on α_1, α_2, α_3 and τ, such that for $h \in (0, C_1(\tau))$, $R \in (R_0, \infty)$, $F \in Y'_{h,R}$, there exists a pair of functions $(u_{h,R}, \pi_{h,R}) \in V_{h,R} \times M_{h,R}$ which satisfies (2.5)–(2.7) and the inequalities*

$$\big((\|u\|^{(h,R)})^2 + \tau \|u_{h,R}|_{\partial_{h,R}}\|_2^2 + \kappa \|\nabla \mathcal{R}_1(\mathcal{S}(u_{h,R}, u_{h,R}))\|_2^2 + \|\nabla \mathcal{R}_2(\nabla \pi_{h,R})\|_2^2\big)^{1/2} \leqslant C_2(\tau)(1 + \|F\|_{-1}), \tag{2.8}$$

$$R^{-1} \|\pi_{h,R}\|_2 + \|\pi_{h,R}|_{\Omega_{2S}}\|_2 \leqslant C_2(\tau)(1 + \widetilde{\tau}\|F\|_{-1})^2 (1 + \|F\|_{-1}). \tag{2.9}$$

Theorem 2.3. *There are constants $h_0 \in (0, \frac{1}{4}S)$, $R_0 \in [8S, \infty[$, $C_3, C_4(\tau) > 0$ depending on Ω, S, σ_1, σ_2, φ_1, and also on α_1, α_2, α_3 in the case of C_3 and $C_4(\tau)$, and in addition on τ as concerns $C_4(\tau)$, such that the following holds true:*

Suppose that $\widetilde{\tau} = 0$, or $\widetilde{\tau} = \tau$ with $\tau \leqslant C_3$. Let $h \in (0, h_0)$, $R \in (R_0, \infty)$, $F \in Y_{h,R}$. Then there is a pair of functions $(u_{h,R}, \pi_{h,R}) \in V_{h,R} \times M_{h,R}$ such that equations (2.5)–(2.7) are satisfied, and such that inequalities (2.8) and (2.9) are valid with $C_2(\tau)$ replaced by $C_4(\tau)$.

The proof of Theorem 2.2 shows that the smallness condition on h arises due to the boundary constraint (2.7). Therefore we might interpret this smallness condition in the sense that existence and stability hold only if the flow is resolved in a sufficiently precise way near $\partial\Omega$. We further point out that the smallness condition on τ in Theorem 2.3 in the case $\widetilde{\tau} = \tau$ refers to a quantity (namely C_3) which does not depend on the data F. This means that our condition is less restrictive than the usual uniqueness condition for the stationary Navier-Stokes system (see [25, (IV.2.12)] or [46, (20)] for example). Thus, under the assumptions of Theorem 2.3 (and the more so under those of Theorem 2.2, of course), we cannot expect that solutions of (2.5)–(2.7) are unique.

Rebollo [46] gives examples for mappings A_1 and A_2 satisfying (S1)–(S3), with the choice of A_1 motivated by the SUPG-method, and that of A_2 by static condensation of bubble functions. Condition (S3) is not formulated explicitly in [46], but it is also required in that reference; see a remark in [46, p. 305]. For the convenience of the reader, we state Rebollo's examples in Theorem 2.4 below. The example for the mapping A_1 is slightly modified with respect to [46] because we have to take into account the Oseen term, and since we consider the normalized version of the Navier-Stokes system characterized by the Reynolds number, whereas Rebollo refers to the "physical" form of this system, with the viscosity as key parameter. We further note that our graded meshes, being regular in the sense of [45, Definition 3.4.1] (see (A3)), do not give rise to additional problems in these examples. We finally remark that the theorem below only treats the case $\widetilde{\tau} = \tau$. Of course, if $\widetilde{\tau} = 0$, an analogous result holds.

Theorem 2.4. *Assume $\widetilde{\tau} = \tau$. Let $A, P \in (0, \infty)$, $h \in (0, \frac{1}{2}S)$, $R \in (S, \infty)$, $p \in (3, \infty)$. For $z \in V_{h,R}$, $l \in \{1, \dots, k\}$, put $\mathcal{P}e_l := \mathcal{P}e(z)_l := \tau \| z|_{K_l} \|_p (\operatorname{diam} K_l)$ ("local Péclet number"),*

$$\sigma_l := A(\operatorname{diam} K_l)^2 \quad \text{if } \mathcal{P}e_l \leqslant P,$$
$$\sigma_l := (1/\tau) AP (\operatorname{diam} K_l) \| z|_{K_l} \|_p^{-1} \quad \text{if } \mathcal{P}e_l > P;$$
$$C_l := |K_l| (840^2 \sigma_l \| \nabla b_{K_l} \|_2^2)^{-1},$$

where the bubble function b_{K_l} was introduced following (2.1). *For $z \in V_{h,R}$, $v, w \in H^1_0(P_{h,R})^3$, we set*

$$A_1(z,v,w) := \sum_{l=1}^{k} C_l \int_{K_l} \nabla v \cdot \nabla w \,\mathrm{d}x, \quad A_2(v,w) := \int_{P_{h,R}} \nabla v \cdot \nabla w \,\mathrm{d}x.$$

The mappings A_1 and A_2 satisfy (S1)–(S3).

For $v \in V_{h,R}$, define $\overline{v}\colon\ P_{h,R} \mapsto \mathbb{R}^3$ by

$$\overline{v}|_{K_l} := \left[\left(\int_{K_l} v_j \cdot b_{K_l} \,\mathrm{d}x\right)\left(\int_{K_l} b_{K_l} \,\mathrm{d}x\right)^{-1}\right]_{1 \leqslant j \leqslant 3}$$

for $l \in \{1,\dots,k\}$. Then, for $z, v^{(1)}, w^{(1)}, v^{(2)}, w^{(2)} \in V_{h,R}$, the equation

$$\begin{aligned} &A_1\big(z, \mathcal{R}_1(z, \mathcal{S}(v^{(1)}, w^{(1)})), \mathcal{R}_1(z, \mathcal{S}(v^{(2)}, w^{(2)}))\big) \\ &\qquad = \tau^2 \sum_{l=1}^{k} \sigma_l |K_l| \sum_{t=1}^{3} \prod_{r=1}^{2} (\overline{v^{(r)} + (1,0,0)}|_{K_l}) \cdot \nabla(w_t^{(r)}|_{K_l}) \end{aligned}$$

holds. Moreover, for $p, q \in M_{h,R}$,

$$A_2(\mathcal{R}_2(\nabla p), \mathcal{R}_2(\nabla q)) = \sum_{l=1}^{k} |K_l|^2 (840\|\nabla b_{K_l}\|_2)^{-2} (\nabla(p|_{K_l}) \cdot \nabla(q|_{K_l})).$$

P r o o f. For a proof of this theorem, we refer to [46, p. 288–289 and p. 299–304], in particular [46, Lemma 5.2] and the proof of [46, Lemma 5.1]. $\square$

3. Further notation. Some auxiliary results

We first remark that by (A3), (A4),

$$\Omega_{R(1-h^2/S^2)^{1/2}} \subset P_{h,R} \subset \Omega_R \quad \forall\, h \in (0, \tfrac{1}{4}S)\ \forall\, R \in (8S, \infty); \tag{3.1}$$

see [11, Lemma 4.2] and its proof, and note that the parameter h here corresponds to $\frac{1}{4}h$ in [11]. We further remark that the set $P_{h,R} \cup \overline{\Omega}$ satisfies an interior cone condition, uniform in h and R. In fact, by (3.1) and the convexity of $P_{h,R} \cup \overline{\Omega}$ (see (A5)), there is $\varphi_0 \in (0, \frac{1}{2}\pi)$ (for example, $\varphi_0 = \frac{1}{12}\pi$) such that the relation

$$(x - \{z \in \mathbb{R}^3 \setminus \{0\}\colon\ |z|^{-1}|x|^{-1}(xz) > \cos\varphi_0\}) \cap B_R(x) \subset P_{h,R} \cup \overline{\Omega} \tag{3.2}$$

holds for $h \in (0, \frac{1}{4}S)$, $R \in (8S, \infty)$, $x \in \partial_{h,R}$. Moreover, again by the convexity of $P_{h,R} \cup \overline{\Omega}$, the ray $\{r(\cos\varrho\cos\vartheta, \sin\varrho\cos\vartheta, \sin\vartheta)\colon\ r \in (0,\infty)\}$ has precisely one point of intersection with $\partial_{h,R}$ for any $\varrho, \vartheta \in \mathbb{R}$. Due to this observation and to conditions (A1)–(A4) and (A7), our triangulations $\mathcal{T}_{h,R}$ verify all the assumptions stated in [12], so that we may apply the results from that reference in what follows. As a first such application, we note there is a constant $S_0 \in (0, \frac{1}{8}S]$ such that

$$\int_{\partial_{h,R}} \mathrm{d}o \leqslant 2\int_{\partial B_R} \mathrm{d}o \quad \forall\, h \in (0, S_0)\ \ \forall\, R \in (8S, \infty); \tag{3.3}$$

see [12, (2.3), (2.7), Corollary 2.2]. The ensuing theorem states that the pair $(Y_{h,R} \oplus \mathcal{B}_{h,R}, M_{h,R})$ of finite element spaces based on the Mini element satisfies the LBB condition with a constant which is not only independent of h, but also of R.

Theorem 3.1. *There are constants $C_0 \in (0,\infty)$, $\gamma_0 \in (0, \frac{1}{4}S]$, $R_0 \in [8S, \infty)$ such that for $h \in (0, \gamma_0)$, $R \in (R_0, \infty)$, $\varrho \in M_{h,R}$, we have*

$$\|\varrho\|_2 \leqslant C_0 \sup\Big\{\int_{P_{h,R}} \frac{\varrho \cdot \operatorname{div}(w+W)}{\|w+W\|^{(h,R)}}\,\mathrm{d}x\colon\ w \in Y_{h,R},\ W \in \mathcal{B}^3_{h,R},\ w+W \neq 0\Big\}.$$

The constants C_0, γ_0, R_0 depend on the parameters σ_1, σ_2 and φ_1 (see (A3), (A6), (A8)*, respectively), and on Ω and S.*

P r o o f. A result analogous to Theorem 3.1 is given by [11, Theorem 4.1], where triangulations of Ω_R (instead of $P_{h,R}$) are considered. These triangulations of Ω_R involve curved elements near ∂B_R and thus are more difficult to handle than our decompositions $\mathcal{T}_{h,R}$ of $P_{h,R}$. This means that Theorem 3.1 may be established by somewhat simplifying the proof of [11, Theorem 4.1]. (Note that the interior cone condition [11, (A8)] is verified according to (3.2).) □

We put $h_0 := \min\{\gamma_0, S_0\}$, with γ_0 from Theorem 3.1 and S_0 from (3.3). This means in particular that $h_0 \leqslant \frac{1}{8}S$. We write $\mathcal{C}$ for constants depending only on Ω, S, α_1, α_2, α_3 and the parameters σ_1, σ_2, φ_1 from (A3), (A6) and (A8), respectively. Recall that τ and $\widetilde{\tau}$ were already fixed in Section 2, following (2.1). In our estimates, there will frequently arise factors of the form $\mathcal{C}(1 \vee \tau)^{\mu}$ with a numerical constant μ. For brevity, we will write $\mathcal{C}(\tau)$ for such factors. When the quantities h and R arise in the following, they will always be supposed to belong to $(0, h_0)$ and (R_0, ∞) respectively, except when indicated otherwise. We set

$$J := J_R := \min\{j \in \mathbb{N}\colon\ 2^j S \geqslant R\},$$
$$I(h,R) := \{l \in \{1,\ldots,k\}\colon\ K_l \cap \partial B_R \neq \emptyset\},$$
$$A_{h,R} := \bigcup\{K_l\colon\ l \in I(h,R)\}.$$

Next we consider functions from $M_{h,R}$ which vanish at the vertices of the tetrahedrons K_l, except at those vertices which are located on ∂B_R (or equivalently, on $\partial_{h,R}$; see (A4)).

Lemma 3.1. *Let $m \in M_{h,R}$. Denote by $\overline{m}$ the uniquely determined element from $M_{h,R}$ such that for any $l \in \{1,\dots,k\}$ and for any vertex x of K_l, the relation $\overline{m}(x) = m(x)$ holds if $x \in \partial B_R$, and $\overline{m}(x) = 0$ elsewhere. Then*

$$m|_{\partial_{h,R}} = \overline{m}|_{\partial_{h,R}}, \quad \operatorname{supp}(\overline{m}) \subset A_{h,R}, \quad \|\overline{m}\|_2 \leqslant \mathcal{C}\|m|_{A_{h,R}}\|_2.$$

Proof. The relation $m|_{\partial_{h,R}} = \overline{m}|_{\partial_{h,R}}$ holds by (A4). The definitions of $\overline{m}$ and $A_{h,R}$ yield $\operatorname{supp}(\overline{m}) \subset A_{h,R}$. The inequality at the end of the lemma may be reduced to the fact that the shape functions of the P1 finite element on a reference tetrahedron are linearly independent. □

If $V \subset \mathbb{R}^3$ is open and bounded, we write $H^m_{\text{loc}}(V)$ for the space of all functions $v\colon V \mapsto \mathbb{R}$ such that $v|_K \in H^m(K)$ for any $K \subset \mathbb{R}^3$ open with $\overline{K} \subset V$. For $r \in \{0,1,2\}$, $p \in (1,\infty)$, we define the semi-norm $|\cdot|_{r,p}$ by

$$|v|_{r,p} := \Bigl(\sum_{a \in \mathbb{N}_0^3,\, |a|_1 = r} \|\partial^a v\|_p^p\Bigr)^{1/p} \quad \forall\, v \in W^{r,p}(V),$$

where $|a|_1 := a_1 + a_2 + a_3$ for $a \in \mathbb{N}_0^3$. Most of the time we write $\|\nabla v\|_2$ instead of $|v|_{1,2}$ for $v \in H^1(V)$. Of course, in the case $r = 0$, the mapping $|\cdot|_{r,p}$ coincides with the norm $\|\cdot\|_p$.

Let us present some observations related to the preceding notation. By the definition of J, we have $2^J S \geqslant R > 2^{J-1} S$, hence by (A3),

$$\operatorname{diam} K_l \leqslant 2^J h \leqslant 2h \cdot R/S \quad \forall\, l \in \{1,\dots,k\}. \tag{3.4}$$

Assumption (A3) further implies, either directly or via (3.4),

$$A_{h,R} \subset \overline{P_{h,R}} \setminus B_{R(1-2h/S)}, \quad \bigcup\{(K_l)_\Delta\colon\ l \in I(h,R)\} \subset \overline{P_{h,R}} \setminus B_{R(1-4h/S)}; \tag{3.5}$$

$$\sum_{l=1}^{k} \int_{(K_l)_\Delta} v \,\mathrm{d}x \leqslant \mathcal{C} \int_{P_{h,R}} v \,\mathrm{d}x \quad \forall\, v \in L^1(P_{h,R}),\ v \geqslant 0; \tag{3.6}$$

$$\sum_{l=1,\, K_l \cap U_j \neq \emptyset}^{k} \int_{(K_l)_\Delta} v \,\mathrm{d}x \leqslant \mathcal{C} \int_{(U_{j-1} \cup U_j \cup U_{j+1}) \cap P_{h,R}} v \,\mathrm{d}x \tag{3.7}$$

$$\text{for } v \text{ as in (3.6)},\quad 0 \leqslant j \leqslant J;$$

$$\sum_{l \in I(h,R)} \int_{(K_l)_\Delta} v \,\mathrm{d}x \leqslant \mathcal{C} \int_{P_{h,R} \setminus B_{R(1-4h/S)}} v \,\mathrm{d}x \quad \text{for } v \text{ as in (3.6)}. \tag{3.8}$$

Obviously,

$$(3.9)\qquad \int_{P_{h,R}} v\,\mathrm{d}x = \sum_{j=0}^{J} \int_{U_j\cap P_{h,R}} v\,\mathrm{d}x \quad \forall\, v \in L^1(P_{h,R}),\ v \geqslant 0.$$

In the next theorem, we introduce an interpolation operator.

Theorem 3.2. *There is a linear operator* $\Pi_{h,R}\colon\ W_{h,R} \mapsto Y_{h,R}$ *such that*

$$(3.10)\qquad |(\Pi_{h,R}(w) - w)|_{K_l}|_{r,2} \leqslant \mathcal{C}(\operatorname{diam} K_l)^{\nu - r}|w|_{(K_l)_\Delta}|_{\nu,2}$$

for $r \in \{0,1\}$, $\nu \in \{1,\, 2\}$, $w \in W_{h,R}$, $l \in \{1,\dots,k\}$ *with* $w|_{(K_l)_\Delta} \in H^\nu((K_l)_\Delta)^3$*;*

$$(3.11)\qquad \|(\Pi_{h,R}(w) - w)|_{K_l}\|_\infty \leqslant \mathcal{C}(\operatorname{diam} K_l)\|\nabla w|_{(K_l)_\Delta}\|_\infty$$

for $w \in W_{h,R}$, $l \in \{1,\dots,k\}$ *with* $w|_{(K_l)_\Delta} \in W^{1,\infty}((K_l)_\Delta)^3$.
(Strictly speaking, the sets K_l *and* $(K_l)_\Delta$ *should be replaced by the interiors of* K_l *and* $(K_l)_\Delta$*, respectively, in the preceding inequalities. But we did not take this fact into account, nor will we do so in the following in similar situations.)*

Proof. Theorem 3.2 follows from (A3), (A6) and [5, Section 4.8], in particular [5, (4.8.10)]. □

Corollary 3.1. *For* $h \in W_{h,R}$ *we have*

$$\|(\Pi_{h,R}(w) - w)|_{\partial_{h,R}}\|_2 \leqslant \mathcal{C}(hR)^{1/2}\|\nabla w|_{P_{h,R}\setminus B_{R(1-4h/S)}}\|_2.$$

Proof. By proceeding as in the proof of [12, Theorem 3.1], we may deduce from Theorem 3.2 that

$$(3.12)\qquad \|(\Pi_{h,R}(w) - w)|_{\partial_{h,R}}\|_2^2 \leqslant \mathcal{C}\sum_{l\in I(h,R)} (\operatorname{diam} K_l)\|\nabla w|_{(K_l)_\Delta}\|_2^2 \quad \forall\, w \in W_{h,R}.$$

Corollary 3.1 follows from (3.12), (A3) and (3.8). □

Next we state some estimates involving bubble functions.

Lemma 3.2. *For $w \in \mathcal{B}^3_{h,R}$, $l \in \{1, \dots, k\}$, we have*

$$\|w|_{K_l}\|_2 \leqslant \mathcal{C}(\operatorname{diam} K_l)\|\nabla w|_{K_l}\|_2, \quad \|w\|_2 \leqslant \mathcal{C}hR\|\nabla w\|_2,$$
$$\|w\|_6 \leqslant \mathcal{C}\|\nabla w\|_2, \quad \|w\|_3 \leqslant \mathcal{C}(hR)^{1/2}\|\nabla w\|_2,$$
$$\|\nabla b_{K_l}\|_2 \leqslant \mathcal{C}(\operatorname{diam} K_l)^{-5/2} \int_{K_l} b_{K_l} \, \mathrm{d}x,$$

where the bubble function b_{K_l} was introduced in the paragraph following (2.1).

P r o o f. The first two inequalities follow from [46, Lemma 4.1 b)] and (3.4). The third one is a standard Sobolev inequality; see [23, (II.2.7)], for example. As for the forth estimate, it follows from the second and third one by interpolation. The fifth one may be established via a transformation to a standard tetrahedron. □

Lemma 3.3. $\|v\|^{(h,R)} + \|\nabla w\|_2 \leqslant \mathcal{C}\|v + w\|^{(h,R)}$ *for $v \in Y_{h,R}$, $w \in \mathcal{B}^3_{h,R}$.*

P r o o f. Lemma 3.3 is shown by the argument in [46, p. 291]. □

Theorem 3.3. $\|v|_{\Omega_{2S}}\|_2 \leqslant \mathcal{C}\|v\|^{(h,R)}$ *for $v \in H^1(P_{h,R})^3$ with $v|_{\partial\Omega} = 0$ or $v|_{\partial\Omega} = (-1, 0, 0)$.*

P r o o f. In the case $v|_{\partial\Omega} = 0$, we refer to the proof of [12, Theorem 3.4]; also see the proof of [13, Lemma 7.1]. If $v|_{\partial\Omega} = (-1, 0, 0)$, then the constant extension of v to $P_{h,R} \cup \overline{\Omega}$ belongs to $H^1(P_{h,R} \cup \overline{\Omega})^3$, and we may again refer to the proof of [12, Theorem 3.4]. □

Theorem 3.4. $\|v\|_6 \leqslant \mathcal{C}\|v\|^{(h,R)}$ *for $v \in H^1(P_{h,R})^3$ with $v|_{\partial\Omega} = 0$ or $v|_{\partial\Omega} = (-1, 0, 0)$.*

Theorem 3.5. *For any $g \in L^2(\Omega_{2S})$, there is a function $\mathcal{F}(g) \in H^1(P_{h,R})^3$ with $\mathcal{F}(g)|_{\partial\Omega} = 0$, $\operatorname{div} \mathcal{F}(g) = \widetilde{g}$, where $\widetilde{g}$ denotes the zero extension of g to $P_{h,R}$,*

$$\|\mathcal{F}(g)|_{\Omega_{4S}}\|_{1,2} \leqslant \mathcal{C}\|g\|_2, \quad |\mathcal{F}(x)||x|^2 + |\nabla\mathcal{F}(x)||x|^3 \leqslant \mathcal{C}\|g\|_2 \quad \forall\, x \in B^c_{4S}. \tag{3.13}$$

Theorem 3.6. *Let $\varepsilon \in (0, \infty)$. Then there is $\varphi_\varepsilon \in C^\infty(\overline{\Omega_S})^3$ with $\operatorname{supp}(\varphi_\varepsilon) \subset B_S$, $\operatorname{div} \varphi_\varepsilon = 0$,*

$$\varphi_\varepsilon|_{\partial\Omega} = (-1, 0, 0), \quad \int_{\Omega_S} \sum_{j,k,l,m=1}^{3} |v_j \partial_m v_k \varphi_{\varepsilon,l}| \, \mathrm{d}x \leqslant \varepsilon\|\nabla v\|_2^2 \quad \text{for } v \in W_{h,R}.$$

P r o o f s of Theorems 3.4–3.6 are indicated in Appendix. □

When we apply the interpolation operator $\Pi_{h,R}$ to the function $\mathcal{F}(g)$ introduced in Theorem 3.5, we obtain the following estimates:

Lemma 3.4. *Let $g \in L^2(\Omega_{2S})$, and put $w := \mathcal{F}(g)$ with $\mathcal{F}(g)$ from Theorem* 3.5. *Then*

$$\begin{aligned}\|\Pi_{h,R}(w)\|_{1,2} + R\|\Pi_{h,R}(w)|_{\partial_{h,R}}\|_2 + \|\Pi_{h,R}(w)\|_3 + \|\Pi_{h,R}(w)\|_6 \\ + R^2\|\Pi_{h,R}(w)|_{\partial_{h,R}}\|_\infty \leqslant \mathcal{C}\|g\|_2.\end{aligned}$$

Proof. Let $a \in \mathbb{N}_0^3$ with $|a|_1 := a_1 + a_2 + a_3 \leqslant 1$. Then by (3.9), (3.10), (A3) and (3.7) we get

$$\begin{aligned}(3.14)\qquad &\|\partial^a(\Pi_{h,R}(w) - w)\|_2 \\ &\leqslant \Bigl(\sum_{j=0}^{J} \sum_{l=1, K_l \cap U_j \neq \emptyset}^{k} \|\partial^a(\Pi_{h,R}(w) - w)|_{K_l}\|_2^2\Bigr)^{1/2} \\ &\leqslant \mathcal{C}\Bigl(\sum_{j=0}^{J}(2^j h)^{2(1-|a|_1)}\|\nabla w|(U_{j-1} \cup U_j \cup U_{j+1}) \cap P_{h,R}\|_2^2\Bigr)^{1/2} \\ &\leqslant \mathcal{C}h^{1-|a|_1}\Bigl(\sum_{j=0}^{J} 2^{2j(1-|a|_1)}\|\nabla w|U_j \cap P_{h,R}\|_2^2\Bigr)^{1/2}.\end{aligned}$$

By (3.13), for $j \in \{3, \ldots, J\}$ we have

$$(3.15)\qquad \|\nabla w|_{U_j \cap P_{h,R}}\|_2 \leqslant \mathcal{C}2^{-3j}|U_j|^{1/2}\|g\|_2 \leqslant \mathcal{C}2^{-3j/2}\|g\|_2.$$

Inequality (3.13) also gives an upper bound for $\|\nabla w|_{\Omega_{4S}}\|_2$. From this estimate, (3.14) and (3.15), we may conclude that $\|\partial^a(\Pi_{h,R}(w) - w)\|_2 \leqslant \mathcal{C}h^{1-|a|_1}\|g\|_2$, hence by Theorem 3.5 $\|\partial^a\Pi_{h,R}(w)\|_2 \leqslant \mathcal{C}\|g\|_2$. Thus we have shown that $\|\Pi_{h,R}(w)\|_{1,2} \leqslant \mathcal{C}\|g\|_2$. By Corollary 3.1, we further get

$$\begin{aligned}(3.16)\qquad \|\Pi_{h,R}(w)|_{\partial_{h,R}}\|_2 &\leqslant \|(\Pi_{h,R}(w) - w)|_{\partial_{h,R}}\|_2 + \|w|_{\partial_{h,R}}\|_2 \\ &\leqslant \mathcal{C}(hR)^{1/2}\|\nabla w|_{P_{h,R} \setminus B_{R(1-4h/S)}}\|_2 \\ &\quad + \|w|_{\partial_{h,R}}\|_\infty\Bigl(\int_{\partial_{h,R}} \mathrm{d}o\Bigr)^{1/2}.\end{aligned}$$

Observing that $\partial_{h,R} \subset B_R \setminus B_{R(1-h^2/S^2)^{1/2}}$ (see (3.1)) and using (3.13) and (3.3), we may deduce from (3.16) that $\|\Pi_{h,R}(w)|_{\partial_{h,R}}\|_2 \leqslant \mathcal{C}R^{-1}\|g\|_2$. Since we already showed that $\|\Pi_{h,R}(w)\|_{1,2} \leqslant \mathcal{C}\|g\|_2$, we may conclude $\|\Pi_{h,R}(w)\|^{(h,R)} \leqslant \mathcal{C}\|g\|_2$. Now Theorem 3.4 implies $\|\Pi_{h,R}(w)\|_6 \leqslant \mathcal{C}\|g\|_2$, and interpolation yields $\|\Pi_{h,R}(w)\|_3 \leqslant \mathcal{C}\|g\|_2$.

Let $x \in \partial_{h,R}$, and let $l \in \{1, \dots, k\}$ with $x \in K_l$. The function $\Pi_{h,R}(w) \in Y_{h,R}$ is continuous, so $|\Pi_{h,R}(w)(x)| \leqslant \|\Pi_{h,R}(w)\|_\infty$. Now we find with (3.11), (3.4), (3.5) and (3.13),

$$|\Pi_{h,R}(w)(x)| \leqslant \mathcal{C}[(\operatorname{diam} K_l)\|\nabla w|_{(K_l)_\Delta}\|_\infty + \|w|_{K_l}\|_\infty] \leqslant \mathcal{C}R^{-2}\|g\|_2.$$

$\square$

We end this section by indicating some properties of the trilinear form b.

Lemma 3.5. *Let $v, w, z \in H^1(P_{h,R})^3$ with $\{v, w, z\} \cap W_{h,R} \neq \emptyset$. Then*

$$b(v, w, z) = \tfrac{1}{2}\widetilde{\tau} \int_{P_{h,R}} [(v \cdot \nabla)w \cdot z - (v \cdot \nabla)z \cdot w] \, \mathrm{d}x. \tag{3.17}$$

In particular, $b(v, w, z) = -b(v, z, w)$ and $b(v, w, w) = 0$. Moreover, the estimates

$$|b(v, w, z)| \leqslant \mathcal{C}\widetilde{\tau}\|v\|^{(h,R)}\|w\|^{(h,R)}(\|z\|_3 + R\|z|_{\partial_{h,R}}\|_\infty), \tag{3.18}$$

$$|b(v, w, z)| \leqslant \mathcal{C}\widetilde{\tau}\|v\|_3\|w\|^{(h,R)}\|z\|^{(h,R)} \tag{3.19}$$

hold for $v, w, z \in H^1(P_{h,R})^3$ with $\{v, w, z\} \cap W_{h,R} \neq \emptyset$ and $y|_{\partial\Omega} = 0$ or $y|_{\partial\Omega} = (-1, 0, 0)$ for all $y \in \{v, w, z\}$.

P r o o f. Partial integration yields (3.17). Inequality (3.18) follows from the definition of b, Hölder's inequality and Theorem 3.4. As concerns inequality (3.19), it is a consequence of (3.17), Hölder's inequality and Theorem 3.4. $\square$

4. Study of a linear problem

In this section we consider a linearization of problem (2.5)–(2.7), that is, equations (4.27), (4.28). Our aim is to find suitable a priori bounds for the solution of this linearized problem.

Let $z \in H^1(P_{h,R})^3$ with $z|_{\partial\Omega} = (-1, 0, 0)$. This function is to be fixed throughout this section. We begin our study by some estimates related to the form A_1.

Lemma 4.1. *Let $v \in H^1(P_{h,R})^3$ with $v|_{\partial\Omega} = 0$ or $v|_{\partial\Omega} = (-1,0,0)$. Let $W \in \mathcal{B}^3_{h,R}$. Then*

$$
\begin{aligned}
&|A_1(\mathcal{R}_1(\mathcal{S}(z,v)),W)| \\
&\qquad \leqslant \mathcal{C}(\tau)(1+hR)(\widetilde{\tau}\|z\|^{(h,R)}+1)\|v\|^{(h,R)}\|\nabla W\|_2, \qquad (4.1)\\
&|A_1(\mathcal{R}_1(\mathcal{S}(z,v)),W)| \\
&\qquad \leqslant \mathcal{C}(\widetilde{\tau}\|z\|^{(h,R)}\|v\|_3+\tau\|v\|_2)\|\nabla W\|_2, \qquad (4.2)\\
&\|\nabla\mathcal{R}_1(\mathcal{S}(z,v))\|_2 \leqslant \mathcal{C}(\tau)(1+hR)(\widetilde{\tau}\|z\|^{(h,R)}+1)\|v\|^{(h,R)}. \qquad (4.3)
\end{aligned}
$$

Proof. Inequality (4.1) follows from (2.4), Hölder's inequality, Lemma 3.2 and Theorem 3.4. Abbreviate $\mathcal{R} := \mathcal{R}_1(\mathcal{S}(z,v))$. Since $W \in \mathcal{B}^3_{h,R} \subset H^1_0(P_{h,R})^3$, we obtain from (2.4) by partial integration

$$A_1(\mathcal{R},W) = -\int_{P_{h,R}} (\widetilde{\tau}\cdot \operatorname{div} z\cdot(v\cdot W) + \widetilde{\tau}\cdot(z\cdot\nabla)W\cdot v + \tau\cdot v\cdot\partial_1 W)\,\mathrm{d}x.$$

This equation, Hölder's inequality and Lemma 3.2 yield (4.2). Inequality (4.3) follows from (4.1) with $W = \mathcal{R}$ and from (S2). □

Lemma 4.2. *Let $W \in \mathcal{B}^3_{h,R}$, $e \in V_{h,R}$ with* $\operatorname{supp}(e) \subset \overline{\Omega_{2S}}$. *Put $\mathcal{E}_e := \|\nabla e\|_3 + \|\nabla e\|_2$. Then*

$$
\begin{aligned}
|A_1(\mathcal{R}_1(\mathcal{S}(z,e)),W)| &\leqslant \mathcal{C}(\tau)\mathcal{E}_e(\widetilde{\tau}\|z\|^{(h,R)}+1)h\|\nabla W\|_2,\\
\|\nabla\mathcal{R}_1(\mathcal{S}(z,e))\|_2 &\leqslant \mathcal{C}(\tau)\mathcal{E}_e(\widetilde{\tau}\|z\|^{(h,R)}+1)h.
\end{aligned}
$$

Proof. By (2.4) and Theorem 3.4 we get

$$|A_1(\mathcal{R}_1\mathcal{S}(z,e)),W)| \leqslant \mathcal{C}\mathcal{E}_e(\widetilde{\tau}\|z\|^{(h,R)}+\tau)\|W|_{\Omega_{2S}}\|_2. \qquad (4.4)$$

Next we note that $\operatorname{diam} K_l \leqslant 2h$ for $l \in \{1,\dots,k\}$ with $K_l \cap \Omega_{2S} \neq \emptyset$ (see (A3)). Thus Lemma 3.2 yields $\|W|_{\Omega_{2S}}\|_2 \leqslant \mathcal{C}h\|\nabla W\|_2$, and the first inequality of the lemma follows. The latter one is a consequence of (S2) and the former one with $W = \mathcal{R}_1(\mathcal{S}(z,e))$. □

Now we turn to estimating the pressure. As a first step, we prove a variant of the LBB-inequality.

Lemma 4.3. *Let $\varrho \in M_{h,R}$, and put $w := \mathcal{F}(\varrho|_{\Omega_{2S}})$, with the operator $\mathcal{F}$ introduced in Theorem* 3.5. *Then*

$$\|\varrho|_{\Omega_{2S}}\|_2^2 \leqslant |c(\Pi_{h,R}(w), \varrho)| + \mathcal{C}\big(\|\nabla \mathcal{R}_2(\nabla \varrho)\|_2 + \|\varrho\|_2 h^{1/2} R^{-3/2}\big)^2.$$

Proof. For $i \in \{1,2,3\}$, $l \in \{1,\dots,k\}$, $x \in K_l$, put

$$W_i(x) := \Big(\int_{K_l} b_{K_l}\,\mathrm{d}y\Big)^{-1}\Big(\int_{K_l} (w - \Pi_{h,R}(w))_i\,\mathrm{d}y\Big) b_{K_l}(x).$$

Then $W \in \mathcal{B}^3_{h,R}$. Let $\overline{\varrho}$ be defined as in Lemma 3.1. Abbreviate

$$\mathcal{A}_1 := \int_{P_{h,R}} \varrho \cdot \operatorname{div} \Pi_{h,R}(w)\,\mathrm{d}x, \quad \mathcal{A}_2 := \int_{P_{h,R}} \overline{\varrho} \cdot \operatorname{div}(w - \Pi_{h,R}(w))\,\mathrm{d}x,$$
$$\mathcal{A}_3 := -\int_{P_{h,R}} \overline{\varrho} \cdot \operatorname{div} W\,\mathrm{d}x, \quad \mathcal{A}_4 := \int_{P_{h,R}} \varrho \cdot \operatorname{div} W\,\mathrm{d}x.$$

Recalling that $\operatorname{div} w$ coincides with ϱ on Ω_{2S} and vanishes on Ω^c_{2S}, and using the fact that $\varrho - \overline{\varrho}$ is zero on $\partial_{h,R}$ (Lemma 3.1), whereas $w - \Pi_{h,R}(w)$ vanishes on $\partial\Omega$ (Theorem 3.5, 3.2), we obtain by partial integration

$$\|\varrho|_{\Omega_{2S}}\|_2^2 = \int_{P_{h,R}} \varrho \cdot \operatorname{div} w\,\mathrm{d}x = \mathcal{A}_1 + \mathcal{A}_2 - \sum_{l=1}^{k} (\nabla(\varrho - \overline{\varrho})|_{K_l}) \int_{K_l} (w - \Pi(w))\,\mathrm{d}x;$$

note that $\nabla(\varrho - \overline{\varrho})|_{K_l}$ is constant for $1 \leqslant l \leqslant k$. Next, applying the trick used to show that the Mini element satisfies the LBB condition ([25, p. 174–175], [6, p. 213]), we may deduce from the preceding equation and the definition of W:

$$\|\varrho|_{\Omega_{2S}}\|_2^2 = \mathcal{A}_1 + \mathcal{A}_2 + \int_{P_{h,R}} (\varrho - \overline{\varrho}) \cdot \operatorname{div} W\,\mathrm{d}x = \sum_{j=1}^{4} \mathcal{A}_j. \tag{4.5}$$

Let us now estimate $\mathcal{A}_2$, $\mathcal{A}_3$ and $\mathcal{A}_4$. Using Lemma 3.1, we find that

$$|\mathcal{A}_2| \leqslant \mathcal{C}\|\varrho\|_2 \|\nabla(w - \Pi_{h,R}(w))|_{A_{h,R}}\|_2.$$

In view of (3.10), (3.8) and (3.13), and because $R(1 - 4h/S) > \frac{1}{2}R > 4S$ (note that $h < S_0 < \frac{1}{8}S$ and $R > R_0 \geqslant 8S$), it follows that

$$|\mathcal{A}_2| \leqslant \mathcal{C}\|\varrho\|_2 \Big(\sum_{l \in I(h,R)} \|\nabla w|_{(K_l)_\Delta}\|_2^2\Big)^{1/2} \leqslant \mathcal{C}\|\varrho\|_2 \|\varrho|_{\Omega_{2S}}\|_2 h^{1/2} R^{-3/2}. \tag{4.6}$$

Turning to the term $\mathcal{A}_3$, we find by Lemma 3.1, 3.2, (3.10) that

$$
\begin{aligned}
(4.7)\qquad |\mathcal{A}_3| &\leqslant \|\varrho\|_2 \|\nabla W|_{A_{h,R}}\|_2 \\
&= \|\varrho\|_2 \Big(\sum_{l\in I(h,R)} \|\nabla W|_{K_l}\|_2^2\Big)^{1/2} \\
&\leqslant \mathcal{C}\|\varrho\|_2 \Big(\sum_{l\in I(h,R)} (\operatorname{diam} K_l)^{-5} \Big|\int_{K_l} (\Pi_{h,R}(w) - w)\,\mathrm{d}x\Big|^2\Big)^{1/2} \\
&\leqslant \mathcal{C}\|\varrho\|_2 \Big(\sum_{l\in I(h,R)} \|\nabla w|_{(K_l)_\Delta}\|_2^2\Big)^{1/2} \\
&\leqslant \mathcal{C}\|\varrho\|_2 \|\varrho|_{\Omega_{2S}}\|_2 h^{1/2} R^{-3/2}
\end{aligned}
$$

where the last inequality follows by the same estimate as that used in (4.6). Since $W \in \mathcal{B}^3_{h,R}$, we get by (2.4) that $\mathcal{A}_4 = -A_2(\mathcal{R}_2(\nabla\varrho), W)$, hence by (S2), $|\mathcal{A}_4| \leqslant \mathcal{C}\|\nabla\mathcal{R}_2(\nabla\varrho)\|_2\|\nabla W\|_2$. But the term $\|\nabla W\|_2$ may be estimated in the same way as $\|\nabla W|_{A_{h,R}}\|_2$ in (4.7). Thus

$$
|\mathcal{A}_4| \leqslant \mathcal{C}\|\nabla\mathcal{R}_2(\nabla\varrho)\|_2 \Big(\sum_{l=1}^{k} \|\nabla w|_{(K_l)_\Delta}\|_2^2\Big)^{1/2} \leqslant \mathcal{C}\|\nabla\mathcal{R}_2(\nabla\varrho)\|_2\|\nabla w\|_2
$$

where the last inequality holds due to (3.6). Now we get by Theorem 3.5 that $|\mathcal{A}_4| \leqslant \mathcal{C}\|\nabla\mathcal{R}_2(\nabla\varrho)\|_2\|\varrho|_{\Omega_{2S}}\|_2$. Lemma 4.3 follows from this inequality, (4.5), (4.6) and (4.7). □

Theorem 4.1. *Let $F \in Y'_{h,R}$, $v \in Y_{h,R}$, $\varrho \in M_{h,R}$, $e \in H^1(P_{h,R})^3$ with $e|_{\partial_{h,R}} = 0$ and $e|_{\partial\Omega} = (-1,0,0)$. Suppose that the equation*

$$
(4.8)\quad a(v,\sigma)+b(z,v,\sigma)+b(v,e,\sigma)+\kappa A_1\big(\mathcal{R}_1(\mathcal{S}(z,v)),\mathcal{R}_1(\mathcal{S}(z,\sigma))\big)+c(\sigma,\varrho) = F(\sigma)
$$

holds for any $\sigma \in Y_{h,R}$. Put

$$
(4.9)\qquad \|F\|_\star := \sup\{|F(\sigma)|/\max\{\|\sigma\|_2, \|\sigma\|_3, \|\sigma\|^{(h,R)}\}\colon\ \sigma \in Y_{h,R},\ \sigma \neq 0\}.
$$

Then

$$
\begin{aligned}
\|\varrho|_{\Omega_{2S}}\|_2 \leqslant{}& \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)} + \widetilde{\tau}\|e\|_3 + 1) \\
&\times (\|v\|^{(h,R)} + \kappa\|\nabla\mathcal{R}_1(\mathcal{S}(z,v))\|_2 + \|\nabla\mathcal{R}_2(\nabla\varrho)\|_2) \\
&+ \mathcal{C}\|F\|_\star + \mathcal{C}\|\varrho\|_2 h^{1/2} R^{-3/2}.
\end{aligned}
$$

P r o o f. In view of Lemma 4.3, we have to estimate the term $|c(\Pi_{h,R}(w),\varrho)|$ with $w:=\mathcal{F}(\varrho|_{\Omega_{2S}})$, where the operator $\mathcal{F}$ was defined in Theorem 3.5. But by (4.8) with $\sigma=\Pi_{h,R}(w)$ we get

$$
\begin{aligned}
(4.10)\quad |c(\Pi_{h,R}(w),\varrho)| \leqslant |a(v,\Pi_{h,R}(w))| + |b(z,v,\Pi_{h,R}(w))| + |b(v,e,\Pi_{h,R}(w))| \\
+ \kappa|A_1(\mathcal{R}_1,\widetilde{\mathcal{R}}_1)| + |F(\Pi_{h,R}(w))|,
\end{aligned}
$$

where we have used the abbreviations $\mathcal{R}_1:=\mathcal{R}_1(\mathcal{S}(z,v))$, $\widetilde{\mathcal{R}}_1:=\mathcal{R}(\mathcal{S}(z,\Pi_{h,R}(w)))$. Let us estimate the terms on the right-hand side of (4.10). By Hölder's inequality and Lemma 3.4 we obtain

$$
(4.11)\qquad |a(v,\Pi_{h,R}(w))| \leqslant \mathcal{C}(\tau)\|v\|^{(h,R)}\|\varrho|_{\Omega_{2S}}\|_2.
$$

Next we observe that by (3.18) and Lemma 3.4,

$$
(4.12)\qquad |b(z,v,\Pi_{h,R}(w)| \leqslant \mathcal{C}\widetilde{\tau}\|z\|^{(h,R)}\|v\|^{(h,R)}\|\varrho|_{\Omega_{2S}}\|_2.
$$

Recalling that $e|_{\partial_{h,R}}=0$, the same references yield

$$
(4.13)\qquad |b(v,e,\Pi_{h,R}(w))| = |b(v,\Pi_{h,R}(w),e)| \leqslant \mathcal{C}\widetilde{\tau}\|v\|^{(h,R)}\|\varrho|_{\Omega_{2S}}\|_2\|e\|_3.
$$

Next we note that by the symmetry of $A_1(z,\cdot,\cdot)$, (4.2) and Lemma 3.4, we have

$$
(4.14)\qquad |A_1(\mathcal{R}_1,\widetilde{\mathcal{R}_1})| \leqslant \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)}+1)\|\varrho|_{\Omega_{2S}}\|_2\|\nabla\mathcal{R}_1\|_2.
$$

Finally, we note that by the definition of $\|F\|_\star$ (see (4.9)) and Lemma 3.4, the term $|F(\Pi_{h,R}(w))|$ is bounded by $\mathcal{C}\|F\|_\star\|\varrho|_{\Omega_{2S}}\|_2$. This relation and (4.10)–(4.14) imply

$$
\begin{aligned}
&|c(\Pi_{h,R}(w),\varrho)| \\
&\quad\leqslant \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)}+\widetilde{\tau}\|e\|_3+1)(\|v\|^{(h,R)}+\kappa\|\nabla\mathcal{R}_1\|_2)\|\varrho|_{\Omega_{2S}}\|_2 \\
&\qquad+\mathcal{C}\|F\|_\star\|\varrho|_{\Omega_{2S}}\|_2.
\end{aligned}
$$

Theorem 4.1 follows from the preceding inequality and Lemma 4.3. □

Theorem 4.2. *Under the assumptions of Theorem* 4.1, *we have*

$$
\begin{aligned}
(4.15)\qquad \|\varrho\|_2 \leqslant\ & \mathcal{C}(\tau)R(\widetilde{\tau}\|z\|^{(h,R)}+\widetilde{\tau}\|e\|_3+1) \\
&\times(\|v\|^{(h,R)}+\kappa\|\nabla\mathcal{R}_1(\mathcal{S}(z,v))\|_2+\|\nabla\mathcal{R}_2(\nabla\varrho)\|_2) \\
&+\mathcal{C}\|F\|_{-1}.
\end{aligned}
$$

P r o o f. Let $w \in Y_{h,R}$. Then we have by (4.8)

$$|c(w,\varrho)| \leqslant |a(v,w)| + |b(z,v,w)| + |b(v,e,w)| + \kappa|A_1(\mathcal{R}_1,\widetilde{\mathcal{R}}_1)| + |F(w)|, \tag{4.16}$$

where $\mathcal{R}_1 := \mathcal{R}_1(\mathcal{S}(z,v))$, $\widetilde{\mathcal{R}}_1 := \mathcal{R}_1(\mathcal{S}(z,w))$. Since $P_{h,R} \subset B_R$ (see (A1), (A2)), Theorem 3.4 implies $\|w\|_2 \leqslant \mathcal{C}R\|w\|^{(h,R)}$, hence we obtain by Hölder's inequality

$$|a(v,w)| \leqslant \mathcal{C}R\|v\|^{(h,R)}\|w\|^{(h,R)}. \tag{4.17}$$

Estimating $\|z\|_2$ in the same way as $\|w\|_2$ in the proof of (4.17), and using Theorem 3.4 and an interpolation argument, we get the estimate $\|z\|_3 \leqslant \mathcal{C}R^{1/2}\|z\|^{(h,R)}$. This inequality, (3.19) and (3.18) imply

$$|b(z,v,w)| \leqslant \mathcal{C}\widetilde{\tau}R^{1/2}\|z\|^{(h,R)}\|v\|^{(h,R)}\|w\|^{(h,R)}, \tag{4.18}$$

$$|b(v,e,w)| = |b(v,w,e)| \leqslant \mathcal{C}\widetilde{\tau}\|v\|^{(h,R)}\|w\|^{(h,R)}\|e\|_3. \tag{4.19}$$

The symmetry of A_1 and inequality (4.1) yield

$$|A_1(\mathcal{R}_1,\widetilde{\mathcal{R}}_1)| \leqslant \mathcal{C}(\tau)R(\widetilde{\tau}\|z\|^{(h,R)}+1)\|w\|^{(h,R)}\|\nabla\mathcal{R}_1\|_2. \tag{4.20}$$

By the definition of $\|F\|_{-1}$ in (2.1) we have $|F(w)| \leqslant \|F\|_{-1}\|w\|^{(h,R)}$. Inserting this equation and (4.17)–(4.20) into (4.16), we obtain

$$\begin{aligned}|c(w,\varrho)| \leqslant{}& \mathcal{C}(\tau)R(\widetilde{\tau}\|z\|^{(h,R)}+\widetilde{\tau}\|e\|_3+1)\\ &\times(\|v\|^{(h,R)}+\kappa\|\nabla\mathcal{R}_1\|_2)\|w\|^{(h,R)}\\ &+\|F\|_{-1}\|w\|^{(h,R)}.\end{aligned} \tag{4.21}$$

We further find for $W \in \mathcal{B}^3_{h,R}$ (see (2.4) and (S2)):

$$\begin{aligned}|c(W,\varrho)| = \left|\int_{P_{h,R}} \varrho \operatorname{div} W \,\mathrm{d}x\right| &= |A_2(\mathcal{R}_2(\nabla\varrho),W)|\\ &\leqslant \mathcal{C}\|\nabla\mathcal{R}_2(\nabla\varrho)\|_2\|\nabla W\|_2.\end{aligned} \tag{4.22}$$

Combining Lemma 3.3, (4.21) and (4.22), we obtain that for $w \in Y_{h,R}$, $W \in \mathcal{B}^3_{h,R}$, the term $|c(w+W,\varrho)|$ is bounded by the right-hand side of (4.15) times $\|w+W\|^{(h,R)}$. Now Theorem 4.2 follows by Theorem 3.1. $\square$

The two preceding theorems yield

Corollary 4.1. *Under the assumptions of Theorem* 4.1, *we have*

$$\begin{aligned}\|\varrho|_{\Omega_{2S}}\|_2 \leqslant \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)} + \widetilde{\tau}\|e\|_3 + 1) \\ \times (\|v\|^{(h,R)} + \kappa\|\nabla\mathcal{R}_1(\mathcal{S}(z,v))\|_2 + \|\nabla\mathcal{R}_2(\nabla\varrho)\|_2) \\ + \mathcal{C}\|F\|_\star + \mathcal{C}R^{-3/2}\|F\|_{-1},\end{aligned}$$

with $\|F\|_\star$ *defined in* (4.9) *and* $\|F\|_{-1}$ *in* (2.1).

In Theorem 4.3 below we present our linearization of (2.5)–(2.7), that is, problem (4.27), (4.28). The structure of this linearization perhaps is not immediately apparent, because of the terms arising due to the transition from the inhomogeneous boundary condition in (2.7) to the homogeneous one in (4.27). But in fact, problem (4.27), (4.28) belongs to a type of system which we consider in the next lemma, and which may easily be solved.

Lemma 4.4. *Let* $e \in V_{h,R}$ *with*

$$|b(w,e,w)| \leqslant \frac{1}{2}(\|w\|^{(h,R)})^2 \quad \forall\, w \in Y_{h,R}. \tag{4.23}$$

Then, for any $F \in Y'_{h,R}$, $G \in M'_{h,R}$, *there is a unique pair of functions* $(v,\varrho) \in Y_{h,R} \times M_{h,R}$ *such that equation* (4.8) *holds, and such that*

$$c(v,p) = A_2(\mathcal{R}_2(\nabla\varrho), \mathcal{R}_2(\nabla p)) + G(p) \quad \forall\, p \in M_{h,R}. \tag{4.24}$$

This pair satisfies the estimate

$$\begin{aligned}(\|v\|^{(h,R)})^2 + \Big(\frac{1}{2}\tau\Big)\|v|_{\partial_{h,R}}\|_2^2 + \kappa\alpha_1\|\nabla\mathcal{R}_1\|_2^2 + \alpha_2\|\nabla\mathcal{R}_2\|_2^2 \\ \leqslant \mathcal{C}(|F(v)| + |G(\varrho)|),\end{aligned} \tag{4.25}$$

with α_1, α_2 *from* (S2) *and with the abbreviations* $\mathcal{R}_1 := \mathcal{R}_1(\mathcal{S}(z,v))$, $\mathcal{R}_2 := \mathcal{R}_2(\nabla\varrho)$.

P r o o f. Let $v \in Y_{h,R}$, $\varrho \in M_{h,R}$, define the abbreviations $\mathcal{R}_1$ and $\mathcal{R}_2$ as in the lemma, and denote the left-hand side of (4.25) by $\mathcal{Y}(v,\varrho)$. Via partial integration, we find $\mathcal{Y}(v,\varrho) = a(v,v) + \kappa\alpha_1\|\nabla\mathcal{R}_1\|_2^2 + \alpha_2\|\nabla\mathcal{R}_2\|_2^2$, hence by (S2) and the relation $b(z,v,v) = 0$ we have

$$\mathcal{Y}(v,\varrho) \leqslant a(v,v) + b(z,v,v) + \kappa A_1(\mathcal{R}_1,\mathcal{R}_1) + A_2(\mathcal{R}_2,\mathcal{R}_2). \tag{4.26}$$

Now suppose that the pair (v,ϱ) satisfies (4.8), (4.24) for some $F \in Y_{h,R}$, $G \in M_{h,R}$. Then we deduce from (4.26) that $\mathcal{Y}(v,\varrho) \leqslant F(v) - G(\varrho) - b(v,e,v)$, so inequality (4.25) follows with (4.23) by a simple shoestring argument. Inequality (4.25) and the estimate of $\|\varrho\|_2$ in Theorem 4.2 show that if $F = 0$, $G = 0$, then problem (4.8), (4.24) admits the zero solution only. Since this problem is linear and finite dimensional, we may conclude that it admits a unique solution. □

Theorem 4.3. *Let $e \in V_{h,R}$ with $\operatorname{supp}(e) \subset \overline{\Omega_{2S}}$, $e|_{\partial\Omega} = (-1,0,0)$. Suppose that e verifies (4.23). Then there is a unique pair of functions $(v,\varrho) \in Y_{h,R} \times M_{h,R}$ such that*

$$
\begin{aligned}
&(4.27) \qquad a(v+e,w) + b(z,v,w) + b(v+e,e,w) \\
&\qquad +\kappa A_1\big(\mathcal{R}_1(\mathcal{S}(z,v+e)), \mathcal{R}_1(\mathcal{S}(z,w))\big) + c(w,\varrho) = F(w) \quad \forall\, w \in Y_{h,R}, \\
&(4.28) \qquad c(v+e,p) = A_2(\mathcal{R}_2(\nabla\varrho), \mathcal{R}_2(\nabla p)) \quad \forall\, p \in M_{h,R}.
\end{aligned}
$$

Abbreviate $\mathcal{B}_e := \max\{\|e\|_{1,2}, \|e\|_{1,3}\}$,

$$
\begin{aligned}
\mathcal{Z}(v,\varrho,z,e) := {}& (\|v+e\|^{(h,R)})^2 + \tau\|(v+e)|_{\partial_{h,R}}\|_2^2 + \kappa\|\nabla\mathcal{R}_1(\mathcal{S}(z,v+e))\|_2^2 \\
& + \|\nabla\mathcal{R}_2(\nabla\varrho)\|_2^2.
\end{aligned}
$$

Then

$$
\begin{aligned}
&(4.29) \qquad \mathcal{Z}(v,\varrho,z,e) \\
&\qquad\qquad \leqslant \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)}(\mathcal{B}_e h + \|\operatorname{div} e\|_2) + \widetilde{\tau}\mathcal{B}_e^2 + \mathcal{B}_e + \|F\|_{-1})^2, \\
&(4.30) \qquad \|\varrho|_{\Omega_{2S}}\|_2 + R^{-1}\|\varrho\|_2 \\
&\qquad\qquad \leqslant \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)} + \widetilde{\tau}\mathcal{B}_e + 1)^2(\mathcal{B}_e + \|F\|_{-1}).
\end{aligned}
$$

P r o o f. For $w \in Y_{h,R}$ put

$$
(4.31) \qquad H(w) := F(w) - a(e,w) - b(e,e,w) - \kappa A_1\big(\mathcal{R}_1(\mathcal{S}(z,e)), \mathcal{R}_1(\mathcal{S}(z,w))\big)
$$

A pair of functions $(v,\varrho) \in Y_{h,R} \times M_{h,R}$ is a solution of (4.8), (4.24) with F, G replaced by H, $-c(e,\cdot)$, respectively, if and only if this pair is a solution of (4.27), (4.28). Therefore the existence and uniqueness result stated in Theorem 4.3 follows from the one in Lemma 4.4.

Now consider a pair (v,ϱ) which solves (4.27), (4.28). Put $\widetilde{\mathcal{R}}_1 := \mathcal{R}_1(\mathcal{S}(z,e))$, $\mathcal{R}_1 := \mathcal{R}_1(\mathcal{S}(z,v))$, $\mathcal{R}_2 := \mathcal{R}_2(\nabla\varrho)$, $\delta(v,\varrho) := (\|v\|^{(h,R)})^2 + \tau\|v|_{\partial_{h,R}}\|_2^2 + \kappa\|\nabla\mathcal{R}_1\|_2^2 + \|\nabla\mathcal{R}_2\|_2^2$. Then we have by (4.25), Corollary 4.1 and Theorem 4.2

$$
\begin{aligned}
&(4.32) \qquad \delta(v,\varrho) \leqslant \mathcal{C}(|H(v)| + |c(e,\varrho)|), \\
&(4.33) \qquad \|\varrho|_{\Omega_{2S}}\|_2 \leqslant \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)} + \widetilde{\tau}\mathcal{B}_e + 1)\delta(v,\varrho)^{1/2} \\
&\qquad\qquad + \mathcal{C}(\|H\|_\star + \|H\|_{-1}R^{-3/2}), \\
&(4.34) \qquad \|\varrho\|_2 \leqslant \mathcal{C}(\tau)R(\widetilde{\tau}\|z\|^{(h,R)} + \widetilde{\tau}\mathcal{B}_e + 1)\delta(v,\varrho)^{1/2} + \mathcal{C}\|H\|_{-1}.
\end{aligned}
$$

Let us now estimate $|H(v)|$, $\|H\|_\star$ and $\|H\|_{-1}$. To this end, we recall that $\operatorname{supp}(e) \subset \overline{\Omega_{2S}}$, $e|_{\partial\Omega} = (-1, 0, 0)$. Thus, using Theorem 3.3, (3.19) and Lemma 4.2 we obtain for $w \in Y_{h,R}$,

$$
\begin{aligned}
&(4.35) \qquad |a(e,w)| \leqslant \mathcal{C}(\tau)\mathcal{B}_e\|w\|^{(h,R)}, \quad |b(e,e,w)| \leqslant \mathcal{C}\widetilde{\tau}\mathcal{B}_e^2\|w\|^{(h,R)},\\
&(4.36) \qquad |F(w)| \leqslant \|F\|_{-1}\|w\|^{(h,R)},\\
&\qquad |A_1(\widetilde{\mathcal{R}}_1, \mathcal{R}_1)| \leqslant \mathcal{C}(\tau)\mathcal{B}_e(\widetilde{\tau}\|z\|^{(h,R)} + 1)h\delta(v,\varrho)^{1/2}.
\end{aligned}
$$

By (4.31), (4.35), (4.36) with $w = v$ and by the estimate $ab \leqslant (4\varepsilon)^{-1}a^2 + \varepsilon b^2$ we get

$$
(4.37) \qquad |H(v)| \leqslant \mathcal{C}(\tau)(\mathcal{B}_e(\widetilde{\tau}\|z\|^{(h,R)}h + \widetilde{\tau}\mathcal{B}_e + 1) + \|F\|_{-1})^2 + \tfrac{1}{4}\delta(v,\varrho).
$$

Due to (4.2) and the estimate of $\|\nabla\widetilde{\mathcal{R}}_1\|_2$ established by Lemma 4.2, we obtain

$$
\begin{aligned}
(4.38) \qquad |A_1(\widetilde{\mathcal{R}}_1, \mathcal{R}_1(\mathcal{S}(z,w)))| &= |A_1(\mathcal{R}_1(\mathcal{S}(z,w)), \widetilde{\mathcal{R}}_1)|\\
&\leqslant \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)} + 1)\max\{\|w\|_3, \|w\|_2\}\\
&\quad \times \mathcal{B}_e(\widetilde{\tau}\|z\|^{(h,R)}h + 1) \quad \forall\, w \in Y_{h,R}.
\end{aligned}
$$

Combining (4.9), (4.31), (4.35) and (4.38), we get

$$
(4.39) \qquad \|H\|_\star \leqslant \mathcal{C}(\tau)\mathcal{B}_e(\widetilde{\tau}\|z\|^{(h,R)} + \widetilde{\tau}\mathcal{B}_e + 1)(\widetilde{\tau}\|z\|^{(h,R)}h + 1) + \|F\|_{-1}.
$$

Arguing as in the proof of Theorem 4.2 in a similar situation, we find

$$
\max\{\|w\|_2, \|w\|_3\} \leqslant \mathcal{C}R\|w\|^{(h,R)}
$$

for $w \in Y_{h,R}$. When we insert this estimate into (4.38) and take into account (4.31) and (4.35), we arrive at the inequality

$$
(4.40) \qquad \|H\|_{-1} \leqslant \mathcal{C}(\tau)R\mathcal{B}_e(\widetilde{\tau}\|z\|^{(h,R)} + \widetilde{\tau}\mathcal{B}_e + 1)(\widetilde{\tau}\|z\|^{(h,R)}h + 1) + \|F\|_{-1}.
$$

We conclude from (4.33), (4.34), (4.39) and (4.40), after estimating the factor h by S,

$$
\begin{aligned}
(4.41) \qquad \|\varrho|_{\Omega_{2S}}\|_2 + R^{-1}\|\varrho\|_2 &\leqslant \mathcal{C}(\tau)\big((\widetilde{\tau}\|z\|^{(h,R)} + \widetilde{\tau}\mathcal{B}_e + 1)\delta(v,\varrho)^{1/2}\\
&\quad + \mathcal{B}_e(\widetilde{\tau}\|z\|^{(h,R)} + \widetilde{\tau}\mathcal{B}_e + 1)^2 + \|F\|_{-1}\big).
\end{aligned}
$$

In order to estimate the term $|c(e,\varrho)|$, we start with the inequality $|c(e,\varrho)| \leqslant \|\operatorname{div} e\|_2\|\varrho|_{\Omega_{2S}}\|_2$. Then we refer to (4.33) and evaluate $\|H\|_\star$ as in (4.39) and $\|H\|_{-1}$

as in (4.40), but we do not estimate the factor h everywhere by S. After some computations, we obtain

$$|c(e,\varrho)| \leqslant \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)}(\mathcal{B}_e h + \|\operatorname{div} e\|_2) + \widetilde{\tau}\mathcal{B}_e^2 + \mathcal{B}_e + \|F\|_{-1})^2 + \frac{1}{4}\delta(v,\varrho),$$

hence by virtue of (4.32) and (4.37) we conclude

$$\delta(v,\varrho) \leqslant \mathcal{C}(\tau)(\widetilde{\tau}\|z\|^{(h,R)}(\mathcal{B}_e h + \|\operatorname{div} e\|_2) + \widetilde{\tau}\mathcal{B}_e^2 + \mathcal{B}_e + \|F\|_{-1})^2. \tag{4.42}$$

Inequality (4.29) follows from (4.42); note that by Lemma 4.2, we have in the case $\kappa = 1$

$$\begin{aligned}\|\nabla\mathcal{R}_1(\mathcal{S}(z, v+e))\|_2 &\leqslant \|\nabla\mathcal{R}_1\|_2 + \|\nabla\widetilde{\mathcal{R}}_1\|_2 \\ &\leqslant \delta(v,\varrho)^{1/2} + \mathcal{C}(\tau)\mathcal{B}_e(\widetilde{\tau}\|z\|^{(h,R)}h + 1).\end{aligned}$$

Inequality (4.30) may be deduced from (4.41) and (4.42). □

We turn to the question of how to construct an extension $e \in V_{h,R}$ of the boundary data $(-1,0,0)$ such that inequality (4.23) is verified, $\|\operatorname{div} e\|_2$ is small, and $\operatorname{supp}(e) \subset \overline{\Omega_{2S}}$.

Theorem 4.4. *Let $\varepsilon \in (0,\infty)$. Then there is a function $\widetilde{e} = \widetilde{e}_{h,R,\varepsilon} \in V_{h,R}$ with $\operatorname{supp}(\widetilde{e}) \subset \overline{\Omega_{2S}}$, $\widetilde{e}|_{\partial\Omega} = (-1,0,0)$,*

$$\|\widetilde{e}\|_{1,2} + \|\widetilde{e}\|_{1,3} \leqslant \mathcal{C}(|\varphi_\varepsilon|_{1,3} + |\varphi_\varepsilon|_{2,3}); \quad \|\operatorname{div}\widetilde{e}\|_2 \leqslant \mathcal{C}h|\varphi_\varepsilon|_{2,2}; \tag{4.43}$$

$$\int_{P_{h,R}} \sum_{j,k,l,m=1}^{3} |v_j\partial_m v_k \widetilde{e}_l|\,\mathrm{d}x \leqslant (\mathcal{C}h^2|\varphi_\varepsilon|_{2,3} + \varepsilon)(\|v\|^{(h,R)})^2 \quad \forall\, v \in Y_{h,R}, \tag{4.44}$$

where φ_ε was introduced in Theorem 3.6. In addition, there is a function $\overline{e} = \overline{e}_{h,R} \in V_{h,R}$ with $\operatorname{supp}(\overline{e}) \subset \overline{\Omega_{2S}}$, $\overline{e}|_{\partial\Omega} = (-1,0,0)$ and $\|\overline{e}\|_{1,2} + \|\overline{e}\|_{1,3} \leqslant \mathcal{C}$.

Proof. By a standard result of finite element theory (for example, see [45, Theorem 3.4.1], [5, Theorem 4.4.4]) and due to (A2), (A3) there is a linear operator $\Gamma_{h,R}\colon H^2(P_{h,R})^3 \mapsto V_{h,R}$ such that

$$\Gamma_{h,R}(w)(x) = w(x) \quad \text{for } w \in H^2(P_{h,R})^3,\ l \in \{1,\dots,k\}, \text{ and for any vertex } x \text{ of } K_l; \tag{4.45}$$

$$|(\Gamma_{h,R}(w) - w)|_{K_l}|_{r,p} \leqslant \mathcal{C}(\operatorname{diam} K_l)^{2-r}|w|_{K_l}|_{2,p} \tag{4.46}$$

for $w \in W^{2,p}(P_{h,R})^3$, $r \in \{0,1\}$, $l \in \{1,\dots,k\}$, $p \in \{2,3\}$. Now choose a function φ_ε with properties as stated in Theorem 3.6. Denote by $\widetilde{\varphi}_\varepsilon$ the zero extension of φ_ε to $\overline{P_{h,R}}$. Note that $\widetilde{\varphi}_\varepsilon \in C^\infty(\overline{P_{h,R}})^3$, $\widetilde{\varphi}_\varepsilon|_{\partial\Omega} = (-1,0,0)$ and $\operatorname{supp}(\widetilde{\varphi}_\varepsilon) \subset B_S$.

Put $\widetilde{e} := \Gamma_{h,R}(\widetilde{\varphi}_\varepsilon)$. Since (A3) implies $K_l \cap \overline{\Omega_S} = \emptyset$ for any $l \in \{1,\dots,k\}$ with $K_l \cap B_{2S}^c \neq \emptyset$, we may conclude that $\operatorname{supp}(\widetilde{e}) \subset \overline{\Omega_{2S}}$. This observation, inequality (4.46) and assumption (A3) yield for $r \in \{0,1\}$, $p \in \{2,3\}$ the estimate

$$|\widetilde{e} - \widetilde{\varphi}_\varepsilon|_{r,p} = \Bigg(\sum_{l=1,\, K_l \cap \overline{\Omega_{2S}} \neq \emptyset}^{k} |(\widetilde{e} - \widetilde{\varphi}_\varepsilon)|_{K_l}|_{r,p}^p \Bigg)^{1/p} \leqslant \mathcal{C} h^{2-r} |\varphi_\varepsilon|_{2,p}. \tag{4.47}$$

Next we observe that since $\operatorname{supp}(\widetilde{e}) \subset \overline{\Omega_{2S}}$, we have $\|\widetilde{e}\|^{(h,r)} = \|\nabla\widetilde{e}\|_2 \leqslant \mathcal{C}\|\nabla\widetilde{e}\|_3$, and $\|\widetilde{e}\|_2 \leqslant \mathcal{C}\|\widetilde{e}\|_3 \leqslant \mathcal{C}\|\widetilde{e}\|_6$. Thus the first inequality in (4.43) follows from Theorem 3.4 and (4.47). The second inequality in (4.43) is a consequence of (4.47) and the equation $\operatorname{div}\varphi_\varepsilon = 0$ (Theorem 3.6). Further, observe that

$$\begin{aligned}
\int_{P_{h,R}} |w_j \partial_m w_k \widetilde{e}_l| \,\mathrm{d}x &\leqslant \int_{P_{h,R}} |w_j \partial_m w_k (\widetilde{e} - \varphi_\varepsilon)_l| \,\mathrm{d}x + \int_{\Omega_S} |w_j \partial_m w_k \varphi_{\varepsilon,l}| \,\mathrm{d}x \\
&\leqslant \mathcal{C}\|w\|_6 \|\nabla w\|_2 \|\widetilde{e} - \widetilde{\varphi}_\varepsilon\|_3 + \varepsilon\|\nabla w\|_2^2 \\
&\qquad \forall\, w \in W_{h,R},\ \ \forall\, j,k,l,m \in \{1,2,3\},
\end{aligned}$$

where we have used Theorem 3.6 in the last inequality. Now inequality (4.44) follows from (4.47) and Theorem 3.4.

In order to define a function $\overline{e}$ with the desired properties, we fix a function $\psi \in C_0^\infty(\mathbb{R}^3)$ with $\psi|_{\partial\Omega} = 1$, $\operatorname{supp}(\psi) \subset B_S$. Then we set $\overline{e} := (-\Gamma_{h,R}(\psi|_{P_{h,R}}), 0, 0)$. This function $\overline{e}$ verifies all the conditions stated in the theorem. $\square$

Now we can prove the desired linear estimates, which we state in the ensuing two corollaries.

Corollary 4.2. *Suppose that $\tau = \widetilde{\tau}$. Then there are constants $\widetilde{C}_1(\tau) \in (0, h_0]$, $\widetilde{C}_2(\tau) \in (0,\infty)$, only depending on τ, Ω, S, on the parameters σ_1, σ_2, φ_1 from* (A3), (A6) *and* (A8), *respectively, as well as on α_1, α_2, α_3 from* (S2), *such that the following holds:*

If $h \in (0, \widetilde{C}_1(\tau)]$, then there is a unique pair of functions $(v, \varrho) \in Y_{h,R} \times M_{h,R}$ which verifies the relations in (4.27) *and* (4.28), *with e replaced by $\widetilde{e}_{h,R,1/(4\tau)}$. (The latter function was introduced in Theorem 4.4.) Moreover, if $h \in (0, \widetilde{C}_1(\tau))$, then*

$$\begin{aligned}
&\mathcal{Z}(v, \varrho, z, \widetilde{e}_{h,R,1/(4\tau)}) \leqslant \widetilde{C}_2(\tau)(\|z\|^{(h,R)}h + 1 + \|F\|_{-1})^2, \\
&\|\varrho|_{\Omega_{2S}}\|_2 + R^{-1}\|\varrho\|_2 \leqslant \widetilde{C}_2(\tau)(\tau\|z\|^{h,R} + 1)^2(1 + \|F\|_{-1}).
\end{aligned} \tag{4.48}$$

Proof. Abbreviate $\widetilde{e} := \widetilde{e}_{h,R,1/(4\tau)}$. By Theorem 4.4 with $\varepsilon = 1/(4\tau)$, there is a constant $\mathcal{C}_1$, depending only on the quantities listed at the beginning of Corollary 4.2, such that for $w \in Y_{h,R}$,

$$|b(w,\widetilde{e},w)| = |b(w,w,\widetilde{e})| \leqslant \tau(\mathcal{C}_1 h^2 |\varphi_{1/4\tau}|_{2,3} + 1/(4\tau))(\|w\|^{(h,R)})^2.$$

Put $\widetilde{C}_1(\tau) := \min\{h_0, (\tau 8\mathcal{C}_1|\varphi_{1/(4\tau)}|_{2,3})^{-1/2}\}$, with h_0 fixed at the beginning of Section 3. Then, if $h \leqslant \widetilde{C}_1(\tau)$, we get the inequality $|b(w,\widetilde{e},w)| \leqslant \frac{1}{2}(\|w\|^{(h,R)})^2$ for $w \in Y_{h,R}$. According to (4.43), the estimates $\mathcal{B}_{\widetilde{e}} \leqslant \mathcal{C}(\tau)$ and $\|\operatorname{div}\widetilde{e}\|_2 \leqslant \mathcal{C}(\tau)h$ hold. Thus Corollary 4.2 follows from Theorem 4.3. $\square$

Corollary 4.3. *There are constants $\widetilde{C}_3$, $\widetilde{C}_4(\tau) > 0$ which depend on Ω, S, σ_1, σ_2, φ_1, α_1, α_2, α_3, with $\widetilde{C}_4(\tau)$ additionally being an increasing function of τ, such that the following is true:*

Suppose that $\widetilde{\tau} \leqslant \widetilde{C_3}$. Then there is a unique pair of functions $(v,\varrho) \in Y_{h,R}\times M_{h,R}$ which verifies (4.27), (4.28) with e replaced by $\overline{e}_{h,R}$ (where $\overline{e}_{h,R}$ was defined in Theorem 4.4). Moreover,

$$\mathcal{Z}(v,\varrho,z,\overline{e}_{h,R}) \leqslant \widetilde{C}_4(\tau)(\widetilde{\tau}\|z\|^{(h,R)} + 1 + \|F\|_{-1})^2,$$
$$\|\varrho|_{\Omega_{2S}}\|_2 + R^{-1}\|\varrho\|_2 \leqslant \widetilde{C}_4(\tau)(\widetilde{\tau}\|z\|^{(h,R)} + 1)^2(1 + \|F\|_{-1}).$$

Proof. Abbreviate $\overline{e} := \overline{e}_{h,R}$. Then, using (3.18) and the fact that $\operatorname{supp}(\overline{e}) \subset B_{2S}$, we find for $w \in Y_{h,R}$

$$|b(w,\overline{e},w)| = |b(w,w,\overline{e})| \leqslant \mathcal{C}\widetilde{\tau}(\|w\|^{(h,R)})^2\|\overline{e}\|_3 \leqslant \mathcal{C}\widetilde{\tau}(\|w\|^{(h,R)})^2.$$

Now Corollary 4.3 follows from Theorem 4.3 and the last inequality in Theorem 4.4. $\square$

5. The nonlinear problem

In this section we use a fixed point argument in order to prove Theorems 2.2 and 2.3. Since we proceed in the same way as Rebollo [46, Appendix], we may be very brief. The following lemma pertains to the continuity of the fixed point operator we will use afterwards.

Lemma 5.1. *Let $y_1, \widetilde{y}_1, y_2, \widetilde{y}_2, e \in V_{h,R}$, $w \in Y_{h,R}$ and suppose that for any $v \in \{y_1, \widetilde{y}_1, y_2, \widetilde{y}_2\}$ the equality $v|_{\partial\Omega} = 0$ or $v|_{\partial\Omega} = (-1, 0, 0)$ holds. Then*

$$\begin{aligned}
&|b(y_1, y_2, w) - b(\widetilde{y}_1, \widetilde{y}_2, w)| + |A_1(\mathcal{R}_1(\mathcal{S}(y_1, y_2)), \mathcal{R}_1(\mathcal{S}(y_1, w))) \\
&\qquad\qquad\qquad\qquad\qquad\qquad - A_1(\mathcal{R}_1(\mathcal{S}(\widetilde{y}_1, \widetilde{y}_2)), \mathcal{R}_1(\mathcal{S}(\widetilde{y}_1, w)))| \\
&\leqslant \gamma\biggl(\mathcal{A}(y_1, \widetilde{y}_1) + \sum_{i=1}^{2} \|y_i - \widetilde{y}_i\|^{(h,R)}\biggr)\|w\|^{(h,R)}\biggl(1 + \sum_{i=1}^{2}(\|y_i\|^{(h,R)} + \|\widetilde{y}_i\|^{(h,R)})\biggr)^3,
\end{aligned}$$

with a constant $\gamma > 0$ which is independent of y_1, $\widetilde{y}_1$, y_2, $\widetilde{y}_2$, w, but may depend on all the other quantities involved, in particular on e, h and R. (See (S3) *for the definition of $\mathcal{A}(y_1, \widetilde{y}_1)$.)*

P r o o f. In this proof we have to indicate the additional function variable which appears in the choice of A_1 and in the definition of $\mathcal{R}_1$, and which we have omitted up to now in view of convention (2.3). Abbreviate $\mathcal{R} := \mathcal{R}_1(y_1, \mathcal{S}(y_1, y_2))$, $\mathcal{G} := \mathcal{R}_1(y_1, \mathcal{S}(y_1, w))$, and let $\widetilde{\mathcal{R}}$ and $\widetilde{\mathcal{G}}$ be defined as $\mathcal{R}$ and $\mathcal{G}$, respectively, but with y_i replaced by $\widetilde{y}_i$, for $i \in \{1, 2\}$. Then

$$|A_1(y_1, \mathcal{R}, \mathcal{G}) - A_1(\widetilde{y}_1, \widetilde{\mathcal{R}}, \widetilde{\mathcal{G}})| \leqslant \sum_{i=1}^{5} \mathcal{M}_1,$$

where

$$\begin{aligned}
\mathcal{M}_1 &:= |A_1(y_1, \mathcal{R}, \mathcal{G}) - A_1(\widetilde{y}_1, \mathcal{R}_1(\widetilde{y}_1, \mathcal{S}(\widetilde{y}_1, y_2)), \mathcal{G})|, \\
\mathcal{M}_2 &:= |A_1(\widetilde{y}_1, \mathcal{R}_1(\widetilde{y}_1, \mathcal{S}(\widetilde{y}_1, y_2)), \mathcal{G}) - A_1(\widetilde{y}_1, \widetilde{\mathcal{R}}, \mathcal{G})|, \\
\mathcal{M}_3 &:= |A_1(\widetilde{y}_1, \widetilde{\mathcal{R}}, \mathcal{G}) - A_1(y_1, \widetilde{\mathcal{R}}, \mathcal{G})|, \\
\mathcal{M}_4 &:= |A_1(y_1, \widetilde{\mathcal{R}}, \mathcal{G}) - A_1(\widetilde{y}_1, \widetilde{\mathcal{R}}, \mathcal{R}_1(\widetilde{y}_1, \mathcal{S}(y_1, w)))|, \\
\mathcal{M}_5 &:= |A_1(\widetilde{y}_1, \widetilde{\mathcal{R}}, \mathcal{R}_1(\widetilde{y}_1, \mathcal{S}(y_1, w))) - A_1(\widetilde{y}_1, \widetilde{\mathcal{R}}, \widetilde{\mathcal{G}})|.
\end{aligned}$$

Note that $\mathcal{M}_4 = 0$, as follows by the symmetry of A_1 and the definition of $\mathcal{R}_1$ in Theorem 2.1. The terms $\mathcal{M}_i$ for $i \in \{1, 2, 3, 5\}$ may be estimated by using (2.4), Theorem 3.4, Lemma 3.2 and (4.3). A suitable estimate of $|b(y_1, y_2, w) - b(\widetilde{y}_1, \widetilde{y}_2, w)|$ easily follows from (3.19). □

With an argument similar to that in [46], we may now carry out the

P r o o f of Theorem 2.2. Choose $\widetilde{C}_1(\tau)$ and $\widetilde{C}_2(\tau)$ as in Corollary 4.2, and suppose that $h \leqslant \min\{\widetilde{C}_1(\tau), (2\widetilde{C}_2(\tau)^{1/2})^{-1}\}$. Abbreviate $\widetilde{e} := \widetilde{e}_{h,R,1/(4\tau)}$ with $\widetilde{e}_{h,R,1/(4\tau)}$ introduced in Theorem 4.4. Define

$$\mathcal{M} := \{\alpha \in Y_{h,R} \colon\ \|\alpha + \widetilde{e}\|^{(h,R)} \leqslant 2\widetilde{C}_2(\tau)^{1/2}(1 + \|F\|_{-1}\}.$$

Obviously $\mathcal{M}$ is a convex subset of $Y_{h,R}$, and it is closed with respect to the norm $\|\cdot\|^{(h,R)}$ on $Y_{h,R}$. Since $Y_{h,R}$ is finite dimensional, the set $\mathcal{M}$ is even compact with respect to this norm. For $\alpha \in \mathcal{M}$, let $(v_\alpha, \varrho_\alpha) \in Y_{h,R} \times M_{h,R}$ be the solution of (4.27), (4.28) with z, e replaced by $\alpha + \widetilde{e}$ and $\widetilde{e}$, respectively. Put $\mathcal{T}(\alpha) = v_\alpha$ $(\alpha \in \mathcal{M})$. The above assumption on h and inequality (4.48) imply that $\mathcal{T}(\mathcal{M}) \subset \mathcal{M}$. Moreover, the mapping $\mathcal{T}$ is continuous with respect to the norm $\|\cdot\|^{(h,R)}$, as follows from Lemma 5.1 and (S3). Now we may apply Brower's fixed point theorem, which yields existence of a fixed point α_0 of $\mathcal{T}$. Then the pair $(u_{h,R}, \varrho_{h,R}) := (v_{\alpha_0} + \widetilde{e}, \pi_{\alpha_0})$ solves (2.5)–(2.7), and Theorem 2.2 follows from Corollary 4.2. □

Theorem 2.3 is proved by a similar argument, with the smallness condition on h replaced by one on $\widetilde{\tau}$, and by referring to Corollary 4.3 instead of 4.2.

Appendix

In this section we indicate a proof of some results from analysis stated in Section 3. We begin by the

Proof of Theorem 3.4. Take v as in Theorem 3.4. This means in particular that v is constant on $\partial\Omega$. We extend v by this constant value to $P_{h,R} \cup \overline{\Omega}$. This extension, which we also denote by v, belongs to $H^1(P_{h,R} \cup \overline{\Omega})^3$. Put $\widetilde{v}(x) := v(R \cdot x)$ for $x \in R^{-1} \cdot (P_{h,R} \cup \overline{\Omega})$. By (3.2), the set $R^{-1} \cdot (P_{h,R} \cup \overline{\Omega})$ verifies an internal cone condition in the sense of [1, 5.3]. In fact, according to (3.2), the cone C mentioned in [1, 5.3] may be chosen with vertex angle $\pi/12$ and height 1. It follows by [1, 5.4] that $\|\widetilde{v}\|_6 \leqslant \gamma \|\widetilde{v}\|_{1,2}$, where γ is a numerical constant. By a scaling argument, we may conclude that $\|v\|_6 \leqslant \mathcal{C}(\|\nabla v\|_2 + R^{-1}\|v\|_2)$. On the other hand, by a slight modification of the proof of [12, Theorem 3.4] we get $\|v\|_2 \leqslant \mathcal{C}R\|v\|^{(h,R)}$; also compare the proof of [13, Lemma 7.1]. Combining the preceding inequalities, we obtain Theorem 3.4. □

Proof of Theorem 3.5. We use the approach from the proof of [23, Theorem III.3.4]. Let $g \in L^2(\Omega_{2S})^3$ and put

$$\psi(x) := -(4\pi)^{-1} \int_{\Omega_{2S}} |x-y|^{-1} g(y)\,\mathrm{d}y \quad \forall\, x \in \mathbb{R}^3.$$

It is well known that $\psi \in H^2_{\mathrm{loc}}(\mathbb{R}^3)$, $\|\nabla\psi\|_6 \leqslant \mathcal{C}\|g\|_2$, $\|\partial_l\partial_m\psi\|_2 \leqslant \|g\|_2$ $(1 \leqslant l$, $m \leqslant 3)$, $\Delta\psi = \overline{g}$, where $\overline{g}$ denotes the zero extension of g to $\mathbb{R}^3$. (These estimates follow from the Hardy-Littlewood-Sobolev and the Calderon-Zygmund inequality, respectively.) In particular, we have $\|\nabla\psi|_{\Omega_{4S}}\|_2 \leqslant \mathcal{C}\|g\|_2$. Obviously $|\nabla\psi(x)| \leqslant \mathcal{C}\|g\|_2|x|^{-2}$ and $|\partial_l\partial_m\psi(x)| \leqslant \|g\|_2|x|^{-3}$ for $x \in B^c_{4S}$, $1 \leqslant l$, $m \leqslant 3$. Observing that

$\int_{\partial\Omega} \nabla\psi n^{(\Omega)}\,\mathrm{d}o = \int_\Omega \Delta\psi\,\mathrm{d}x = 0$, with $n^{(\Omega)}$ denoting the outward unit normal to Ω, we see that the boundary value $\psi|_{\partial\Omega}$ satisfies the assumptions of [23, Exercise III.3.4]. Thus, according to that reference, there is a function $w \in H^1(\Omega_{2S})^3$ with

$$w|_{\partial\Omega} = -\nabla\psi|_{\partial\Omega}, \quad w|_{\partial B_{2S}} = 0, \quad \operatorname{div} w = 0, \quad \|w\|_{1,2} \leqslant \mathcal{C}\|\nabla\psi|_{\partial\Omega}\|_{1/2,2},$$

where $\|\cdot\|_{1/2,2}$ denotes the intrinsic norm of the fractional order Sobolev space $W^{1/2,2}(\partial\Omega)^3$; see [23, Section II.3], for example. By a standard trace theorem, it follows that $\|w\|_{1,2} \leqslant \mathcal{C}\|\nabla\psi|_{\Omega_{2S}}\|_{1,2}$, hence $\|w\|_{1,2} \leqslant \mathcal{C}\|g\|_2$ according to the estimates of ψ stated above. Let us denote the zero extension of w to $P_{h,R}$ by $\widetilde{w}$. Then the function $\mathcal{F}(g) := \nabla\psi + \widetilde{w}$ has the properties stated in Theorem 3.5. □

For the proof of Theorem 3.6, we need

Theorem A1. *There is $w \in C^\infty(\mathbb{R}^3)$ and an open set $U \subset \mathbb{R}^3$ such that $\partial\Omega \subset U \subset B_S$, $\operatorname{div} w = 0$, $\operatorname{supp}(\operatorname{rot} w) \subset B_S$ and $\operatorname{rot} w|_U = (-1,0,0)$.*

Proof. First we choose an open set $V \subset \mathbb{R}^3$ with $\partial\Omega \subset V$, $\overline{V} \subset B_S$, and a function $\varphi \in C_0^\infty(B_S)^3$ with $\varphi|_V = (-1,0,0)$. By the latter equation, and because $\operatorname{supp}(\varphi) \subset B_S$, it follows by [23, Theorem III.3.2] that there are functions $\gamma_i \in C_0^\infty(\Omega)^3$, $\gamma_e \in C_0^\infty(\Omega_S)^3$ with $\operatorname{div}\gamma_i = -\operatorname{div}(\varphi|_\Omega)$, $\operatorname{div}\gamma_e = -\operatorname{div}(\varphi|_{\Omega_S})$. Now put $v := \varphi + \widetilde{\gamma}_i + \widetilde{\gamma}_e$, where $\widetilde{\gamma}_i, \widetilde{\gamma}_e$ denote the zero extensions of γ_i, γ_e, respectively, to B_S. Observe that $v \in C_0^\infty(B_S)^3$, $\operatorname{div} v = 0$, and there is an open neighbourhood U of $\partial\Omega$ with $v|_U = \varphi|_U = (-1,0,0)$. In particular, the zero extension of v to $\mathbb{R}^3$, denoted by $\widetilde{v}$, belongs to $C_0^\infty(\mathbb{R}^3)^3$, with $\operatorname{div}\widetilde{v} = 0$. But in this situation it is well known that there is $w \in C^\infty(\mathbb{R}^3)^3$ with $\operatorname{rot} w = \widetilde{v}$ (see [24, Exercise VIII.4.1], for example). This function w possesses the required properties. □

Once Theorem A.1 is available, a proof of Theorem 3.6 may be carried out by arguments which are more or less standard. Thus we may restrict ourselves to some indications. Let $\mu \in (0,\infty)$, and choose a function $\psi_\mu \in C^\infty(\overline{\Omega_S})$ as in [23, Lemma III.6.2] ("regularized distance"), with the domain Ω in the latter reference replaced by Ω_S; compare [25, Lemma III.4.2], where a similar function (denoted by θ_μ) is considered. Since $\psi_\mu = 1$ in a vicinity of $\partial\Omega_S = \partial\Omega \cup \partial B_S$, we have $\nabla\psi_\mu = 0$ in that vicinity. Choose a function w as in Theorem A.1. As $\operatorname{rot} w \in C_0^\infty(B_S)^3$, we get $\operatorname{supp}(\operatorname{rot}(\psi_\mu w)) \subset B_S$. These observations and the proof of [25, Lemma IV.2.3] show that a suitable function φ_ε is given by $\varphi_\varepsilon := \operatorname{rot}(\psi_\varepsilon w)$, with a sufficiently small value of ε.

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