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SPACE-TIME DISCONTINUOS GALERKIN METHOD FOR SOLVING NONSTATIONARY CONVECTION-DIFFUSION-REACTION PROBLEMS

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Abstract. The paper presents the theory of the discontinuous Galerkin finite element method for the space-time discretization of a linear nonstationary convection-diffusion-reaction initial-boundary value problem. The discontinuous Galerkin method is applied separately in space and time using, in general, different nonconforming space grids on different time levels and different polynomial degrees p and q in space and time discretization, respectively. In the space discretization the nonsymmetric interior and boundary penalty approximation of diffusion terms is used. The paper is concerned with the proof of error estimates in "$L^2(L^2)$"- and "$\sqrt{\varepsilon}L^2(H^1)$"-norms, where $\varepsilon \geqslant 0$ is the diffusion coefficient. Using special interpolation theorems for the space as well as time discretization, we find that under some assumptions on the shape regularity of the meshes and a certain regularity of the exact solution, the errors are of order $O(h^p + \tau^q)$. The estimates hold true even in the hyperbolic case when $\varepsilon = 0$.

Keywords: nonstationary convection-diffusion-reaction equation, space-time discontinuous Galerkin finite element discretization, nonsymmetric treatment of diffusion terms, error estimates

MSC 2000: 65M60, 65M15, 65M12

1. Introduction

A number of complex problems from science and technology (aerospace engineering, turbomachinery, oil recovery, meteorology, environmental protection etc.) require to apply new efficient, robust, reliable and highly accurate numerical methods. It is necessary to develop techniques that allow to realize numerical approximations of

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strongly nonlinear singularly perturbed systems in domains with complex geometry whose solutions contain internal or boundary layers.

An excellent candidate to overcome these difficulties is the discontinuous Galerkin finite element (DGFE) method, which becomes more and more popular in the solution of a number of problems.

The DGFE method uses piecewise polynomial approximations of the sought solution on a finite element mesh without any requirement on continuity between neighbouring elements and can be considered a generalization of the finite volume and finite element methods. It allows to construct higher order schemes in a natural way and is suitable for approximation of discontinuous solutions of conservation laws or solutions of singularly perturbed convection-diffusion problems having steep gradients. This method exploites advantages of the finite element method and finite volume schemes with an approximate Riemann solver and can be applied on unstructured grids which are generated for most complex geometries.

The original DGFE method was introduced in [36] for the solution of a neutron transport linear equation and analyzed theoretically in [32] and later in [30]. Almost simultaneously the DGFE techniques were developed for the numerical solution of elliptic problems ([50]) and space semidiscretization of parabolic problems ([21], [1]), using the interior penalty Galerkin methods. In these works the symmetric approximation of the diffusion terms is used, called the SIPG (symmetric interior penalty Galerkin) method. Quite popular is the NIPG (nonsymmetric interior penalty Galerkin) method which was first introduced in [40]. Theoretical analysis of various types of the DGFE method applied to elliptic problems can be found, e.g., in [4], [2], [3] and [41].

The DGFE method found very soon a number of applications. Let us mention in particular the solution of nonlinear conservation laws ([11], [29], [19]) and compressible flow ([5], [6], [7], [13], [15], [17], [24], [26], [48]). A survey of DGFE methods, techniques and some applications can be found in [9] and [10].

In the discretization of nonstationary problems, one often uses the *space semidiscretization*, also called the *method of lines*. In this approach, the DGFE discretization with respect to space variables only is applied, whereas time remains continuous. This leads to a large system of ordinary differential equations which can be solved numerically by a suitable ODE solver. (See, e.g., [38], [39], [18], [20], [16], [45], [46], [47].) In CFD and conservation laws, usually Runge-Kutta methods are used, which however are conditionally stable. Therefore, it is suitable to apply implicit or semi-implicit methods. In [38] implicit θ-schemes are analyzed, [16] is concerned with the analysis of a semi-implicit linearized scheme for a nonlinear convection-diffusion problem and in [15] an efficient semi-implicit method for the solution of compressible Euler equations was developed. However, these methods have low order of accuracy in time.

Numerical simulation of strongly nonstationary transient problems requires the application of numerical schemes of high order of accuracy in space as well as in time. From this point of view, it appears suitable to use the discontinuous Galerkin discretization with respect to both space and time. The discontinuous Galerkin time discretization was introduced and analyzed, e.g., in [22] for the solution of ordinary differential equations. In [23], [43], [44] and [49], the discontinuous Galerkin time semidiscretization, combined with conforming space finite element discretization, is applied to linear parabolic problems. On the other hand, the works [29] and [48] apply the full DG discretization in the space-time domain. This requires to construct a mesh in the space-time cylinder, which may be a quite complicated task for 3D problems.

In this paper we are concerned with the space-time discontinuous Galerkin discretization applied separately in space and in time for the numerical solution of a nonstationary convection-diffusion-reaction equation. We follow and extend the results from [25], where the discontinuous Galerkin space semidiscretization of this problem was analyzed. In the present paper the time interval is split into subintervals and *on each time level a different space mesh* may be used in general. This approach is suitable particularly in the case when the space mesh adaptivity is performed in the course of increasing time. Moreover, *the triangulations used for the space discretization may be nonconforming with hanging nodes.* In the discontinuous Galerkin formulation we use the nonsymmetric version of discretization of the diffusion terms and the interior and boundary penalty (i.e., the NIPG method). For the space and time discretization, piecewise polynomial approximations of different orders p and q, respectively, are used. An important ingredient is the use of upwinding in the space discretization which consists, roughly speaking, in considering only the information that is brought from the position opposite to the streamwise direction. Under the assumption that the triangulations on all time levels are uniformly shape regular, $h \sim \tau$ (space and time steps are comparable) and the exact solution has some regularity properties, error estimates are derived for this space-time DGFE method which are *uniform with respect to the diffusion coefficient* $\varepsilon \to 0+$ *and are valid even in the hyperbolic case when* $\varepsilon = 0$.

The structure of the paper is as follows: First, the continuous problem is formulated and the main assumptions are introduced. Further, the discontinuous Galerkin discretization in space and time is described. In the next section, some auxiliary results concerning properties of forms appearing in the definition of the approximate solution are obtained. Then the error estimates of the DG space-time discretization are proved. These results are compared with numerical experiments. In Appendix, rather technical proofs of some estimates are presented.

2. Continuous problem

Let $\Omega \subset \mathbb{R}^d$ $(d = 2$ or $3)$ be a bounded polyhedral domain and $T > 0$. We consider the following initial-boundary value problem: Find $u\colon Q_T = \Omega \times (0,T) \to \mathbb{R}$ such that

$$\frac{\partial u}{\partial t} + \boldsymbol{v} \cdot \nabla u - \varepsilon \triangle u + cu = g \quad \text{in } Q_T, \tag{2.1}$$

$$u = u_D \quad \text{on } \partial\Omega^- \times (0,T), \tag{2.2}$$

$$\varepsilon \frac{\partial u}{\partial \boldsymbol{n}} = u_N \quad \text{on } \partial\Omega^+ \times (0,T), \tag{2.3}$$

$$u(x,0) = u^0(x), \quad x \in \Omega. \tag{2.4}$$

We assume that $\partial\Omega = \partial\Omega^- \cup \partial\Omega^+$ and

$$\boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) < 0 \quad \text{on } \partial\Omega^-, \tag{2.5}$$

$$\boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) \geqslant 0 \quad \text{on } \partial\Omega^+ \text{ for all } t \in [0,T]. \tag{2.6}$$

Here $\boldsymbol{n}(x)$ is the unit outer normal to the boundary $\partial\Omega$ of Ω, $\partial\Omega^-$ is the inflow boundary and $\partial\Omega^+$ is the outflow boundary. In the case $\varepsilon = 0$ we put $u_N = 0$ and ignore the Neumann condition (2.3).

We use the standard notation of function spaces. If ω is a bounded domain, then we define the Lebesgue spaces

$$L^\infty(\omega) = \{\text{measurable functions } \varphi\colon \|\varphi\|_{L^\infty(\omega)} = \{\operatorname{essup}_{x\in\omega}|\varphi(x)| < \infty\},$$
$$L^2(\omega) = \left\{\text{measurable functions } \varphi\colon \|\varphi\|_{L^2(\omega)} = \left(\int_\omega |\varphi|^2 \,\mathrm{d}x\right)^{1/2} < \infty\right\}$$

and the Sobolev space

$$H^k(\omega) = \left\{\varphi \in L^2(\omega)\colon \|\varphi\|_{H^k(\omega)} = \left(\sum_{|\alpha|\leqslant k} \|D^\alpha \varphi\|^2_{L^2(\omega)}\right)^{1/2} < \infty\right\}$$

with the seminorm

$$|\varphi|_{H^k(\omega)} = \left(\sum_{|\alpha|=k} \|D^\alpha \varphi\|^2_{L^2(\omega)}\right)^{1/2}.$$

We also use the Bochner spaces. Let X be a Banach space with a norm $\|\cdot\|_X$ and a seminorm $|\cdot|_X$ and let s be an integer. Then we define:

$$C([0,T];X) = \Big\{\varphi\colon\ [0,T] \to X,\ \text{continuous},\ \|\varphi\|_{C([0,T];X)} = \sup_{t\in[0,T]} \|\varphi(t)\|_X < \infty\Big\},$$

$$L^2(0,T;X) = \Big\{\varphi\colon\ (0,T) \to X,\ \text{strongly measurable},$$
$$\|\varphi\|^2_{L^2(0,T;X)} = \int_0^T \|\varphi(t)\|_X^2 \,\mathrm{d}t < \infty\Big\},$$

$$H^s(0,T;X) = \Big\{\varphi \in L^2(0,T;X)\colon\ \frac{\partial^\alpha \varphi}{\partial t^\alpha} \in L^2(0,T;X),\ \alpha = 1,\dots,s\Big\},$$

where the derivatives $\partial^\alpha\varphi/\partial t^\alpha$ are considered in the sense of distributions on $(0,T)$. The norm in $H^s(0,T;X)$ is defined as

$$\|\varphi\|_{H^s(0,T;X)} = \Big(\int_0^T \sum_{\alpha=0}^{s} \Big\|\frac{\partial^\alpha\varphi}{\partial t^\alpha}\Big\|_X^2 \,\mathrm{d}t\Big)^{1/2}. \tag{2.7}$$

Moreover, we set

$$|\varphi|_{C([0,T];X)} = \sup_{t\in[0,T]} |\varphi|_X,$$
$$|\varphi|_{L^2(0,T;X)} = \Big(\int_0^T |\varphi|_X^2 \,\mathrm{d}t\Big)^{1/2},$$
$$|\varphi|_{H^s(0,T;X)} = \Big(\int_0^T \Big|\frac{\partial^s\varphi}{\partial t^s}\Big|_X^2 \,\mathrm{d}t\Big)^{1/2}.$$

Assumptions on data (A)

We assume that the data satisfy the following conditions:

a) $g \in C([0,T];L^2(\Omega))$,

b) $u_0 \in L^2(\Omega)$,

c) u_D is the trace of some $u^* \in C([0,T];H^1(\Omega)) \cap L^\infty(Q_T)$ on $\partial\Omega^- \times (0,T)$,

d) $\boldsymbol{v} \in C([0,T];W^{1,\infty}(\Omega))$, $\boldsymbol{v}$, $\operatorname{div}\boldsymbol{v}$ bounded by a constant $C_{\boldsymbol{v}}$ a.e. in Q_T,

e) $c \in C([0,T];L^\infty(\Omega))$, $|c(x,t)| \leqslant C_c$ a.e. in Q_T,

f) $c - \frac{1}{2}\operatorname{div}\boldsymbol{v} \geqslant \gamma_0 > 0$ in Q_T with a constant γ_0,

g) $u_N \in C([0,T];L^2(\partial\Omega^+))$,

h) $\varepsilon \geqslant 0$.

With the aid of techniques from [37] and [33], it is possible to prove that there exists a unique weak solution. Moreover, it satisfies the condition $\partial u/\partial t \in L^2(Q_T)$. We will assume that the weak solution u is sufficiently regular, so that it satisfies problem (2.1)–(2.4) pointwise.

3. Discretization of the problem

Let $\mathcal{T}_h = \bigcup_{i \in i_h} K_i$ ($i_h \subset \{0,1,2,\dots\}$ is a suitable index set) be a triangulation of the closure of the domain Ω into a finite number of closed triangles ($d = 2$) or tetrahedra ($d = 3$). If two elements $K_i, K_j \in \mathcal{T}_h$ contain a nonempty open part of their sides, we call them *neighbours*. In this case we put $\Gamma_{ij} = \Gamma_{ji} = \partial K_i \cap \partial K_j$. For $i \in i_h$ we set $s_h(i) = \{j \in i_h;\ K_j \text{ is a neighbour of } K_i\}$. The boundary $\partial\Omega$ is formed by a finite number of faces of elements K_i adjacent to $\partial\Omega$. We denote all these boundary faces by S_j where $j \in i_{bh} \subset Z^- = \{-1,-2,\dots\}$ and set $\gamma_h(i) = \{j \in i_{bh};\ S_j \text{ is a face of } K_i\}$, $\Gamma_{ij} = S_j$ for $K_i \in \mathcal{T}_h$ such that $S_j \subset \partial K_i,\ j \in i_{bh}$. For K_i not containing any boundary face S_j we put $\gamma_h(i) = \emptyset$. Obviously, $s_h(i) \cap \gamma_h(i) = \emptyset$ for all $i \in i_h$. Now, writing $S_h(i) = s_h(i) \cup \gamma_h(i)$, we have

$$\partial K_i = \bigcup_{j \in S_h(i)} \Gamma_{ij}, \quad \partial K_i \cap \partial\Omega = \bigcup_{j \in \gamma_h(i)} \Gamma_{ij}. \tag{3.1}$$

In what follows, we shall call Γ_{ij} faces. As we see, we admit nonconforming triangulations with hanging nodes.

For $K \in \mathcal{T}_h$, by h_K and ϱ_K we denote the diameter of K and the diameter of the largest ball inscribed in K, respectively. We set $h = \max_{K \in \mathcal{T}_h} h_K$.

We introduce the so-called fractional Sobolev space

$$H^k(\Omega, \mathcal{T}_h) = \{\varphi;\ \varphi|_K \in H^k(K)\ \forall\, K \in \mathcal{T}_h\} \tag{3.2}$$

and define the seminorm

$$|\varphi|_{H^k(\Omega,\mathcal{T}_h)} = \Big(\sum_{K \in \mathcal{T}_h} |\varphi|^2_{H^k(K)}\Big)^{1/2}. \tag{3.3}$$

For $\varphi \in H^1(\Omega, \mathcal{T}_h)$ we introduce the following notation:

$$\varphi|_{\Gamma_{ij}} = \text{the trace of } \varphi|_{K_i} \text{ on } \Gamma_{ij}, \tag{3.4}$$

$$\varphi|_{\Gamma_{ji}} = \text{the trace of } \varphi|_{K_j} \text{ on } \Gamma_{ji}, \tag{3.5}$$

$$\langle\varphi\rangle_{\Gamma_{ij}} = \frac{1}{2}(\varphi|_{\Gamma_{ij}} + \varphi|_{\Gamma_{ji}}), \tag{3.6}$$

$$[\varphi]_{\Gamma_{ij}} = \varphi|_{\Gamma_{ij}} - \varphi|_{\Gamma_{ji}}, \tag{3.7}$$

$$\boldsymbol{n}_{ij} = \text{the unit outer normal to } \partial K_i \text{ on the face } \Gamma_{ij}. \tag{3.8}$$

Further, for $i \in i_h$ we set

$$\partial K_i^-(t) = \{x \in \partial K_i\colon\ \boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) < 0\}, \tag{3.9}$$

$$\partial K_i^+(t) = \{x \in \partial K_i\colon\ \boldsymbol{v}(x,t) \cdot \boldsymbol{n}(x) \geqslant 0\}. \tag{3.10}$$

($\boldsymbol{n}$ denotes here the unit outer normal to ∂K_i). In the sequel we often drop the dependence of ∂K_i^+ and ∂K_i^- on time in our notation.

In the paper [25] we derived and analyzed the following space-semidiscretization of our problem: Find $u_h \in C^1([0,T]; S_h^p)$ such that

$$\left(\frac{\partial u_h}{\partial t}, \varphi_h\right) + A_h(u_h(t), \varphi_h) = l_h(\varphi_h)(t) \quad \forall\, \varphi_h \in S_h^p \;\forall\, t \in (0,T), \tag{3.11}$$

$$(u_h(0), \varphi_h) = (u^0, \varphi_h) \quad \forall\, \varphi_h \in S_h^p, \tag{3.12}$$

where

$$A_h(u,\varphi) = a_h(u,\varphi) + b_h(u,\varphi) + c_h(u,\varphi) + \varepsilon J_h(u,\varphi), \tag{3.13}$$

$$S_h^p = \{\varphi \in L^2(\Omega);\; \varphi|_K \in \mathcal{P}^p(K)\; \forall K \in \mathcal{T}_h\}, \tag{3.14}$$

$$\mathcal{P}^p(K) = \text{ set of polynomials of degree at most } p \text{ on } K, \tag{3.15}$$

and $p \geqslant 1$ is an integer. The bilinear forms $(\cdot,\cdot)$, a_h, b_h, c_h, J_h are defined as follows:

$$(u,\varphi) = \int_\Omega u\varphi \,\mathrm{d}x, \tag{3.16}$$

$$\begin{aligned} a_h(u,\varphi) = {} & \varepsilon \sum_{i\in i_h} \int_{K_i} \nabla u \cdot \nabla\varphi \,\mathrm{d}x \\ & - \varepsilon \sum_{i\in i_h} \sum_{j\in s_h(i),\, j<i} \int_{\Gamma_{ij}} (\langle \nabla u\rangle \cdot \boldsymbol{n}_{ij}[\varphi] - \langle\nabla\varphi\rangle \cdot \boldsymbol{n}_{ij}[u]) \,\mathrm{d}S \\ & - \varepsilon \sum_{i\in i_h} \int_{\partial K_i^- \cap \partial\Omega} ((\nabla u\cdot \boldsymbol{n})\varphi - (\nabla\varphi\cdot\boldsymbol{n})u) \,\mathrm{d}S, \end{aligned} \tag{3.17}$$

$$\begin{aligned} b_h(u,\varphi) = {} & \sum_{i\in i_h} \int_{K_i} (\boldsymbol{v}\cdot\nabla u)\varphi \,\mathrm{d}x - \sum_{i\in i_h} \int_{\partial K_i^-\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})u\varphi \,\mathrm{d}S \\ & - \sum_{i\in i_h} \int_{\partial K_i^- \setminus \partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})[u]\varphi \,\mathrm{d}S, \end{aligned} \tag{3.18}$$

$$c_h(u,\varphi) = \int_\Omega cu\varphi \,\mathrm{d}x, \tag{3.19}$$

$$\begin{aligned} J_h(u,\varphi) = {} & \sum_{i\in i_h} \sum_{j\in s_h(i)} \operatorname{diam}(\Gamma_{ij})^{-1} \int_{\Gamma_{ij}} [u][\varphi] \,\mathrm{d}S \\ & + \sum_{i\in i_h} \sum_{j:\, \Gamma_{ij}\subset\partial\Omega^-} \operatorname{diam}(\Gamma_{ij})^{-1} \int_{\Gamma_{ij}} u\varphi \,\mathrm{d}S, \end{aligned} \tag{3.20}$$

$$
\begin{aligned}
(3.21)\qquad l_h(\varphi)(t) = {} & \int_\Omega g(t)\varphi\,\mathrm{d}x + \sum_{i\in i_h}\int_{\partial K_i^+\cap\partial\Omega} u_N(t)\varphi\,\mathrm{d}S \\
& + \varepsilon\sum_{i\in i_h}\sum_{j\colon \Gamma_{ij}\subset\partial\Omega^-} \mathrm{diam}(\Gamma_{ij})^{-1}\int_{\Gamma_{ij}} u_D(t)\varphi\,\mathrm{d}S \\
& + \varepsilon\sum_{i\in i_h}\int_{\partial K_i^-\cap\partial\Omega} u_D(t)(\nabla\varphi\cdot\boldsymbol{n})\,\mathrm{d}S \\
& - \sum_{i\in i_h}\int_{\partial K_i^-\cap\partial\Omega} (\boldsymbol{v}\cdot\boldsymbol{n})u_D(t)\varphi\,\mathrm{d}S.
\end{aligned}
$$

We discretize problem (3.11)–(3.12) also in time using the discontinuous Galerkin method (see, e.g., [44]). For this purpose, we consider a partition $0 = t_0 < t_1 < \ldots < t_M = T$ of the time interval $[0,T]$ and denote $I_m = (t_{m-1}, t_m)$, $\overline{I}_m = [t_{m-1}, t_m]$, $\tau_m = t_m - t_{m-1}$, $m = 1,\ldots,M$. We have

$$
(3.22)\qquad [0,T] = \bigcup_{i=1}^{M}\overline{I}_m, \quad I_m\cap I_n = \emptyset \ \text{ for } \ m\neq n.
$$

For a function φ defined on $\bigcup\limits_{i=1}^{M}(t_{m-1}, t_m)$ we introduce the notation

$$
(3.23)\qquad \varphi_m^{\pm} = \varphi(t_m\pm) = \lim_{t\to t_m\pm}\varphi(t),
$$

$$
(3.24)\qquad \{\varphi\}_m = \varphi_m^+ - \varphi_m^-.
$$

For each time interval I_m, $m = 1,\ldots,M$, we will consider, in general, a different triangulation $\mathcal{T}_{h,m} = \{K_i\}_{i\in i_{h,m}}$ of the domain Ω (in general, with hanging nodes). Therefore, for different intervals I_m we have different $s_{h,m}$, $S^p_{h,m}$, $a_{h,m}$, $b_{h,m}$, $J_{h,m}$, $l_{h,m}$, $A_{h,m}$, etc. Hence, we set

$$
(3.25)\qquad S^p_{h,m} = \{\varphi\in L^2(\Omega);\ \varphi|_K\in\mathcal{P}^p(K)\ \forall\, K\in\mathcal{T}_{h,m}\},
$$

$$
\begin{aligned}
(3.26)\qquad a_{h,m}(u,\varphi) = {} & \varepsilon\sum_{i\in i_{h,m}}\int_{K_i}\nabla u\cdot\nabla\varphi\,\mathrm{d}x \\
& - \varepsilon\sum_{i\in i_{h,m}}\sum_{j\in s_{h,m}(i), j<i}\int_{\Gamma_{ij}} (\langle\nabla u\cdot\boldsymbol{n}_{ij}[\varphi] - \langle\nabla\varphi\rangle\cdot\boldsymbol{n}_{ij}[u])\,\mathrm{d}S \\
& - \varepsilon\sum_{i\in i_{h,m}}\int_{\partial K_i^-\cap\partial\Omega} ((\nabla u\cdot\boldsymbol{n})\varphi - (\nabla\varphi\cdot\boldsymbol{n})u)\,\mathrm{d}S,
\end{aligned}
$$

$$(3.27)\quad b_{h,m}(u,\varphi)=\sum_{i\in i_{h,m}}\int_{K_i}(\boldsymbol{v}\cdot\nabla u)\varphi\,\mathrm{d}x-\sum_{i\in i_{h,m}}\int_{\partial K_i^-\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})u\varphi\,\mathrm{d}S$$
$$-\sum_{i\in i_{h,m}}\int_{\partial K_i^-\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})[u]\varphi\,\mathrm{d}S,$$

$$(3.28)\quad c_{h,m}(u,\varphi)=\int_\Omega cu\varphi\,\mathrm{d}x,$$

$$(3.29)\quad J_{h,m}(u,\varphi)=\sum_{i\in i_{h,m}}\sum_{j\in s_{h,m}(i)}\operatorname{diam}(\Gamma_{ij})^{-1}\int_{\Gamma_{ij}}[u][\varphi]\,\mathrm{d}S$$
$$+\sum_{i\in i_{h,m}}\sum_{j\colon\,\Gamma_{ij}\subset\partial\Omega^-}\operatorname{diam}(\Gamma_{ij})^{-1}\int_{\Gamma_{ij}}u\varphi\,\mathrm{d}S,$$

$$(3.30)\quad l_{h,m}(\varphi)(t)=\int_\Omega g(t)\varphi\,\mathrm{d}x+\sum_{i\in i_{h,m}}\int_{\partial K_i^+\cap\partial\Omega}u_N(t)\varphi\,\mathrm{d}S$$
$$+\varepsilon\sum_{i\in i_h}\sum_{j\colon\,\Gamma_{ij}\subset\partial\Omega^-}\operatorname{diam}(\Gamma_{ij})^{-1}\int_{\Gamma_{ij}}u_D(t)\varphi\,\mathrm{d}S$$
$$+\varepsilon\sum_{i\in i_{h,m}}\int_{\partial K_i^-\cap\partial\Omega}u_D(t)(\nabla\varphi\cdot\boldsymbol{n})\,\mathrm{d}S$$
$$-\sum_{i\in i_{h,m}}\int_{\partial K_i^-\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})u_D(t)\varphi\,\mathrm{d}S,$$

and

$$(3.31)\quad A_{h,m}(u,\varphi)=a_{h,m}(u,\varphi)+b_{h,m}(u,\varphi)+c_{h,m}(u,\varphi)+\varepsilon J_{h,m}(u,\varphi).$$

We set

$$(3.32)\quad h_m=\max_{K\in\mathcal{T}_{h,m}}h_K,\quad h=\max_{m=1,\dots,M}h_m\quad\text{and}\quad\tau=\max_{m=1,\dots,M}\tau_m.$$

Let $q\geqslant 1$ be an integer. The *approximate solution* is defined as a function

$$(3.33)\quad U(x,t)\in S^{p,q}_{h,\tau}$$
$$=\Big\{\varphi\in L^2(Q_T);\ \varphi|_{I_m}=\sum_{i=0}^q t^i\varphi_i,\ \text{with }\varphi_i\in S^p_{h,m},\ m=1,\dots,M\Big\}$$

satisfying

$$(3.34)\quad \sum_{m=1}^M\int_{I_m}((U',\varphi)+A_{h,m}(U,\varphi))\,\mathrm{d}t+\sum_{m=2}^M(\{U\}_{m-1},\varphi^+_{m-1})+(U_0^+,\varphi_0^+)$$
$$=\sum_{m=1}^M\int_{I_m}l_{h,m}(\varphi)\,\mathrm{d}t+(u_0,\varphi_0^+)\quad\forall\,\varphi\in S^{p,q}_{h,\tau}.$$

Here and in the sequel the symbols f' and f'' will mean the first and second order time derivatives of f. If we denote

$$(3.35)\quad B(u,v) = \sum_{m=1}^{M} \int_{I_m} ((u',v) + A_{h,m}(u,v))\,\mathrm{d}t + \sum_{m=2}^{M} (\{u\}_{m-1}, v_{m-1}^{+}) + (u_0^{+}, v_0^{+}),$$

$$L(v) = \sum_{m=1}^{M} \int_{I_m} l_{h,m}(v)\,\mathrm{d}t + (u_0, v_0^{+}),$$

we can write (3.34) in the form

$$(3.36)\qquad B(U,\varphi) = L(\varphi) \quad \forall\,\varphi \in S_{h,\tau}^{p,q}.$$

It is possible to show that the regular exact solution u satisfies the identity $B(u,\varphi) = L(\varphi)$ for all $\varphi \in S_{h,\tau}^{p,q}$ and, thus, we have

$$(3.37)\qquad B(U,\varphi) = B(u,\varphi) \quad \forall\,\varphi \in S_{h,\tau}^{p,q}.$$

By u and U we shall always denote the exact solution of problem (2.1)–(2.4) and the approximate solution obtained from (3.36), respectively. We denote

$$(3.38)\qquad e = U - u,$$

the error of the method. Our goal is the analysis of error estimates of the space-time DGFE method (3.36). To this end, we assume in the sequel that the system of triangulations $\mathcal{T}_{h,m}$, $m = 1,\dots,M$, $h \in (0,h_0)$, is *shape regular*: there exists a constant $C_{\mathcal{T}}$ independent of K, m and h such that

$$(3.39)\qquad \frac{h_K}{\varrho_K} \leqslant C_{\mathcal{T}}, \quad K \in \mathcal{T}_{h,m},\ m = 1,\dots,M,\ h \in (0,h_0).$$

Moreover, we assume that there exists a constant $C_D > 0$ such that

$$(3.40)\quad h_{K_i} \leqslant C_D\,\mathrm{diam}(\Gamma_{ij}), \quad K_i \in \mathcal{T}_{h,m},\ \Gamma_{ij} \subset \partial K_i,\ m = 1,\dots,M,\ h \in (0,h_0).$$

This means that Γ_{ij} do not degenerate with respect to h_{K_i} if $h \to 0+$. (This assumption will play an important role in the proof of (6.41)). It is obvious that $\mathrm{diam}(\Gamma_{ij}) \leqslant h_{K_i}$.

Important tools in the derivation of the error estimates are the multiplicative trace inequality (see [18], Lemma 3.1)

$$(3.41)\qquad \|v\|_{L^2(\partial K)}^2 \leqslant C_M(\|v\|_{L^2(K)}|v|_{H^1(K)} + h_K^{-1}\|v\|_{L^2(K)}^2),$$
$$v \in H^1(K),\ K \in \mathcal{T}_{h,m},\ m = 1,\dots,M,\ h \in (0,h_0),$$

and the inverse inequality (see [8], proof of Theorem 3.2.6)

$$|v|_{H^1(K)} \leqslant C_I h_K^{-1} \|v\|_{L^2(K)}, \qquad v \in S^p_{h,m},\ \ K \in \mathcal{T}_{h,m},\ \ m = 1, \ldots, M,\ \ h \in (0, h_0). \tag{3.42}$$

In what follows, by C we shall denote a generic constant, independent of h and m, assuming in general different values at different places.

4. Properties of the forms A, B

In this section we prove some basic properties of the forms A and B (Lemmas 1, 2, 3).

Lemma 1. *We can express B in the form*

$$B(u,v) = \sum_{m=1}^{M} \int_{I_m} ((-u, v') + A_{h,m}(u,v))\,\mathrm{d}t - \sum_{m=1}^{M-1} (u_m^-, \{v\}_m) + (u_M^-, v_M^-). \tag{4.1}$$

P r o o f. Integration by parts yields (see (3.35))

$$\begin{aligned}
&\sum_{m=1}^{M} \int_{I_m} (u', v)\,\mathrm{d}t + \sum_{m=2}^{M} (\{u\}_{m-1}, v_{m-1}^+) + (u_0^+, v_0^+) \\
&\quad = \sum_{m=1}^{M} \int_{I_m} (-u, v')\,\mathrm{d}t + \sum_{m=1}^{M} ((u_m^-, v_m^-) - (u_{m-1}^+, v_{m-1}^+)) \\
&\qquad + \sum_{m=2}^{M} (u_{m-1}^+ - u_{m-1}^-, v_{m-1}^+) + (u_0^+, v_0^+) \\
&\quad = \sum_{m=1}^{M} \int_{I_m} (-u, v')\,\mathrm{d}t + \sum_{m=1}^{M-1} (u_m^-, v_m^- - v_m^+) + (u_M^-, v_M^-).
\end{aligned}$$

□

Lemma 2. *We have*

$$B(v,v) = \sum_{m=1}^{M} \int_{I_m} A_{h,m}(v,v)\,\mathrm{d}t + \|v\|_T^2, \tag{4.2}$$

where

$$\|v\|_T^2 = \frac{1}{2}\|v_0^+\|^2_{L^2(\Omega)} + \frac{1}{2}\sum_{m=1}^{M-1} \|\{v\}_m\|^2_{L^2(\Omega)} + \frac{1}{2}\|v_M^-\|^2_{L^2(\Omega)}. \tag{4.3}$$

P r o o f. If we put $u := v$ in (3.35) and in (4.1), we get

$$B(v,v) = \sum_{m=1}^{M} \int_{I_m} (v',v) + A_{h,m}(v,v)\,\mathrm{d}t + \sum_{m=1}^{M-1} (\{v\}_m, v_m^+) + (v_0^+, v_0^+),$$
$$B(v,v) = \sum_{m=1}^{M} \int_{I_m} (-v,v') + A_{h,m}(v,v)\,\mathrm{d}t + \sum_{m=1}^{M-1} (-v_m^-, \{v\}_m) + (v_M^-, v_M^-).$$

We arrive at (4.2) by adding these identities and dividing by two. $\square$

In the sequel we shall use the notation

$$\|v\|^2_{\boldsymbol{v},\Gamma} = \int_\Gamma |\boldsymbol{v}\cdot\boldsymbol{n}| v^2\,\mathrm{d}S \quad \text{for } \Gamma \subset \partial K, \ \ K \in \mathcal{T}_{h,m}.$$

Lemma 3. *The forms $A_{h,m}$ are coercive:*

$$A_{h,m}(v,v) \geqslant \|v\|^2_{E,m}, \quad v \in H^1(\Omega, \mathcal{T}_{h,m}), \tag{4.4}$$

where

$$\begin{aligned}\|v\|^2_{E,m} = {}& \varepsilon |v|^2_{H^1(\Omega,\mathcal{T}_{h,m})} + \gamma_0 \|v\|^2_{L^2(\Omega)} + \varepsilon J_{h,m}(v,v) \\ &+ \frac{1}{2} \sum_{i\in i_{h,m}} \big(\|v\|^2_{\boldsymbol{v},\partial K_i\cap\partial\Omega} + \|[v]\|^2_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega}\big).\end{aligned} \tag{4.5}$$

P r o o f. Using a process similar to that in (5.5)–(5.8) from [25] and following [14], we find that

$$\begin{aligned}A_{h,m}(v,v) = {}& \varepsilon |v|^2_{H^1(\Omega,\mathcal{T}_{h,m})} + \int_\Omega \Big(c - \frac{1}{2}\operatorname{div}\boldsymbol{v}\Big) v^2\,\mathrm{d}x + \varepsilon J_h(v,v) \\ &+ \frac{1}{2} \sum_{i\in i_{h,m}} \big(\|v\|^2_{\boldsymbol{v},\partial K_i\cap\partial\Omega} + \|[v]\|^2_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega}\big), \quad v \in H^1(\Omega,\mathcal{T}_{h,m}),\end{aligned} \tag{4.6}$$

which together with assumption (A) f) yields (4.4). $\square$

5. Abstract error estimate

In the derivation of error estimates we make use of a space-time interpolation of the exact solution, defined similarly to [43]:

$$\pi u \in S_{h,\tau}^{p,q}, \tag{5.1}$$

$$\int_{I_m} (\pi u - u, \varphi^*)\,\mathrm{d}t = 0 \quad \forall \varphi^* \in S_{h,\tau}^{p,q-1}, \quad m = 1,\ldots,M, \tag{5.2}$$

$$\pi u(t_m-) = \Pi_m u(t_m-), \quad m = 1,\ldots,M, \tag{5.3}$$

where Π_m is the L^2-projection to $S_{h,m}^p$ in space. This means that taking $v \in L^2(\Omega)$, we have $\Pi_m v \in S_{h,m}^p$ and $(\Pi_m v - v, \varphi) = 0$ for all $\varphi \in S_{h,m}^p$.

Lemma 4. *The projection πu is determined by* (5.1)–(5.3) *uniquely.*

Proof. Let $f, g \in S_{h,\tau}^{p,q}$ be two π-projections of u. Then for $\varphi = f - g$ we have

$$\int_{I_m} (\varphi, \varphi^*)\,\mathrm{d}t = 0 \quad \forall\, \varphi^* \in S_{h,\tau}^{p,q-1}, \quad \varphi(t_m-) = 0.$$

Since $\varphi' \in S_{h,\tau}^{p,q-1}$, integration by parts with respect to time yields

$$0 = \int_{I_m} (\varphi, \varphi')\,\mathrm{d}t = \frac{1}{2}\|\varphi\|_{L^2(\Omega)}^2\big|_{t_{m-1}+}^{t_m-}.$$

Hence, $\varphi(t_{m-1}+) = 0$. Since also $\varphi'' \in S_{h,\tau}^{p,q-1}$, we find that

$$0 = \int_{I_m} (\varphi, \varphi'')\,\mathrm{d}t = -\|\varphi'\|_{L^2(I_m\times\Omega)}^2.$$

This and the condition $\varphi(t_m-) = 0$ imply that $\varphi = 0$, which we wanted to prove. $\square$

In what follows, we derive error estimates in terms of the π-interpolation error (Lemmas 5, 6, 7, 8).

Lemma 5. *We have*

$$\begin{aligned} B(U - \pi u, U - \pi u) = {} & \sum_{m=1}^{M} \int_{I_m} A_{h,m}(u - \pi u, U - \pi u)\,\mathrm{d}t \\ & - \sum_{m=1}^{M-1} ((\pi u - u)_m^-, \{U - \pi u\}_m). \end{aligned} \tag{5.4}$$

P r o o f. From (3.37) and (4.1) we get

$$
\begin{aligned}
&B(U-\pi u, U-\pi u) = B(u-\pi u, U-\pi u)\\
&\quad = -\sum_{m=1}^{M}\int_{I_m}(u-\pi u,(U-\pi u)')\,\mathrm{d}t + \sum_{m=1}^{M}\int_{I_m}A_{h,m}(u-\pi u, U-\pi u)\,\mathrm{d}t\\
&\qquad -\sum_{m=1}^{M-1}((u-\pi u)_m^-,\{U-\pi u\}_m) + ((u-\pi u)_M^-,(U-\pi u)_M^-).
\end{aligned}
$$

The first term on the second line vanishes due to (5.2). The second term on the last line is also zero, because we have

$$((u-\pi u)_M^-,\varphi) = ((u-\Pi_M u)_M^-,\varphi) + ((\Pi_M u-\pi u)_M^-,\varphi) \tag{5.5}$$

for $\varphi\in S_{h,M}^p$ and both terms on the right-hand side of (5.5) vanish (the first term because of the properties of the L^2-projection and the other due to (5.3)). $\square$

The sum on the last line in (5.4) does not vanish because, in general, $\{U-\pi u\}_m\notin S_{h,m}^p$, as we use different triangulations of Ω on different time levels.

Under the notation

$$\xi = U-\pi u \quad (\in S_{h,\tau}^{p,q}), \tag{5.6}$$

$$\eta = \pi u - u \quad \text{(interpolation error)}, \tag{5.7}$$

we have $e=\xi+\eta$ and (5.4) can be rewritten as

$$B(\xi,\xi) = -\sum_{m=1}^{M}\int_{I_m}A_{h,m}(\eta,\xi)\,\mathrm{d}t - \sum_{m=1}^{M-1}(\eta_m^-,\{\xi\}_m). \tag{5.8}$$

The quantities u, U, η, ξ depend, of course, on x, t but usually we do not emphasize this dependence by notation, for the sake of simplicity.

Lemma 6. *Let us denote*

$$
\begin{aligned}
\sigma_m(\eta;h,\varepsilon) = {}&\|\eta\|_{E,m} + \sqrt{\varepsilon}h|\eta|_{H^2(\Omega,\mathcal{T}_{h,m})}\\
&+\Big(\sum_{i\in i_m}h_{K_i}^{-2}\|\eta\|^2_{L^2(K_i)}\Big)^{1/2} + \Big(\sum_{i\in i_{h,m}}\|\eta^-\|^2_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega}\Big)^{1/2}.
\end{aligned}
\tag{5.9}
$$

Then there exists a constant C_A independent of u, U, h, ε such that

$$|A_{h,m}(\eta,\xi)| \leqslant C_A\|\xi\|_{E,m}\sigma_m(\eta;h,\varepsilon). \tag{5.10}$$

P r o o f. We use the estimates already derived in [14] and [25]. We begin with the form $a_{h,m}$ (see [14], proof of Lemma 3.8):

$$a_{h,m}(\eta,\xi) \leqslant \varepsilon|\eta|_{H^1(\Omega,\mathcal{T}_{h,m})}|\xi|_{H^1(\Omega,\mathcal{T}_{h,m})} + \varepsilon C|\xi|_{H^1(\Omega,\mathcal{T}_{h,m})}\sqrt{J_{h,m}(\eta,\eta)}$$
$$+ \varepsilon\sqrt{J_{h,m}(\xi,\xi)}\Big(\sum_{i\in i_{h,m}} h_{K_i}\|\nabla\eta\|^2_{L^2(\partial K_i)}\Big)^{1/2}.$$

Now, by the multiplicative trace inequality (3.41) and Young's inequality,

$$\sum_{i\in i_{h,m}} h_{K_i}\|\nabla\eta\|^2_{L^2(\partial K_i)} \leqslant C_M \sum_{i\in i_{h,m}} (h_{K_i}\|\nabla\eta\|_{L^2(K_i)}|\nabla\eta|_{H^1(K_i)} + \|\nabla\eta\|^2_{L^2(K_i)})$$
$$\leqslant \frac{3C_M}{2}\sum_{i\in i_{h,m}} |\eta|^2_{H^1(K_i)} + \frac{C_M}{2}\sum_{i\in i_{h,m}} h^2_{K_i}|\eta|^2_{H^2(K_i)}.$$

Hence,

$$\begin{aligned}(5.11)\quad & a_{h,m}(\eta,\xi)\\ &\leqslant C\Big\{\sqrt{\varepsilon}|\eta|_{H^1(\Omega,\mathcal{T}_{h,m})} + \sqrt{\varepsilon}\sqrt{J_{h,m}(\eta,\eta)} + \sqrt{\varepsilon}\Big(\sum_{i\in i_{h,m}} h^2_{K_i}|\eta|^2_{H^2(K_i)}\Big)^{1/2}\Big\}\\ &\quad\times\Big\{\sqrt{\varepsilon}|\xi|_{H^1(\Omega,\mathcal{T}_{h,m})} + \sqrt{\varepsilon}\sqrt{J_{h,m}(\xi,\xi)}\Big\}\\ &\leqslant C\|\xi\|_{E,m}(\|\eta\|_{E,m} + \sqrt{\varepsilon}h|\eta|_{H^2(\Omega,\mathcal{T}_{h,m})}).\end{aligned}$$

Due to [25], (5.18), for the form $b_{h,m}$ we have

$$(5.12)\qquad |b_{h,m}(\eta,\xi)| \leqslant \Big|\sum_{i\in i_{h,m}}\int_{K_i}\eta(\boldsymbol{v}\cdot\nabla\xi)\,\mathrm{d}x\Big| + \Big|\sum_{i\in i_{h,m}}\int_{K_i}\eta\xi\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x\Big|$$
$$+\Big|\sum_{i\in i_{h,m}}\Big(\int_{\partial K_i}(\boldsymbol{v}\cdot\boldsymbol{n})\xi\eta\,\mathrm{d}S - \int_{\partial K_i^-\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\xi\eta\,\mathrm{d}S - \int_{\partial K_i^-\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\xi[\eta]\,\mathrm{d}S\Big)\Big|.$$

The first term in (5.12) is estimated with the aid of the Cauchy inequality and the inverse inequality:

$$(5.13)\qquad \Big|\sum_{i\in i_{h,m}}\int_{K_i}\eta(\boldsymbol{v}\cdot\nabla\xi)\,\mathrm{d}x\Big| \leqslant C\Big(\sum_{i\in i_{h,m}} h^{-2}_{K_i}\|\eta\|^2_{L^2(K_i)}\Big)^{1/2}\|\xi\|_{L^2(\Omega)}.$$

The second term is estimated by

$$(5.14)\qquad \Big|\sum_{i\in i_{h,m}}\int_{K_i}\eta\xi\,\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x\Big| \leqslant C_{\boldsymbol{v}}\|\eta\|_{L^2(\Omega)}\|\xi\|_{L^2(\Omega)}$$

and the third term is rearranged as in [25], (5.20):

$$\begin{aligned}(5.15)\quad &\sum_{i\in i_{h,m}}\Big(\int_{\partial K_i^+}(\boldsymbol{v}\cdot\boldsymbol{n})\xi\eta\,\mathrm{d}S+\int_{\partial K_i^-\setminus\partial\Omega}\Big\{(\boldsymbol{v}\cdot\boldsymbol{n})\xi\eta-(\boldsymbol{v}\cdot\boldsymbol{n})\xi[\eta]\Big\}\,\mathrm{d}S\Big)\\ &\qquad=\sum_{i\in i_{h,m}}\Big(\int_{\partial K_i^+\cap\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\xi\eta\,\mathrm{d}S+\int_{\partial K_i^-\setminus\partial\Omega}(\boldsymbol{v}\cdot\boldsymbol{n})\eta^-[\xi]\,\mathrm{d}S\Big).\end{aligned}$$

We obtain

$$\begin{aligned}(5.16)\quad & b_{h,m}(\eta,\xi)\\ &\leqslant C\Big(\|\eta\|_{L^2(\Omega)}+\Big(\sum_{i\in i_m}h_{K_i}^{-2}\|\eta\|^2_{L^2(K_i)}\Big)^{1/2}\Big)\|\xi\|_{L^2(\Omega)}\\ &\quad+\sum_{i\in i_{h,m}}\|\xi\|_{\boldsymbol{v},\partial K_i^+\cap\partial\Omega}\|\eta\|_{\boldsymbol{v},\partial K_i^+\cap\partial\Omega}+\sum_{i\in i_{h,m}}\|[\xi]\|_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega}\|\eta^-\|_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega}\\ &\leqslant\frac{C}{\gamma_0}\Big(\sqrt{\gamma_0}\Big(\|\eta\|_{L^2(\Omega)}+\Big(\sum_{i\in i_m}h_{K_i}^{-2}\|\eta\|^2_{L^2(K_i)}\Big)^{1/2}\Big)\\ &\quad+\Big(\sum_{i\in i_{h,m}}\|\eta\|^2_{\boldsymbol{v},\partial K_i^+\cap\partial\Omega}\Big)^{1/2}+\Big(\sum_{i\in i_{h,m}}\|\eta^-\|^2_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega}\Big)^{1/2}\Big)\\ &\quad\times\Big(\sqrt{\gamma_0}\|\xi\|_{L^2(\Omega)}+\Big(\sum_{i\in i_{h,m}}\|\xi\|^2_{\boldsymbol{v},\partial K_i^+\cap\partial\Omega}\Big)^{1/2}+\Big(\sum_{i\in i_{h,m}}\|[\xi]\|^2_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega}\Big)^{1/2}\Big)\\ &\leqslant C\|\xi\|_{E,m}\Big(\|\eta\|_{E,m}+\Big(\sum_{i\in i_m}h_{K_i}^{-2}\|\eta\|^2_{L^2(K_i)}\Big)^{1/2}+\Big(\sum_{i\in i_{h,m}}\|\eta^-\|^2_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega}\Big)^{1/2}\Big).\end{aligned}$$

Further, we have

$$(5.17)\qquad c_h(\eta,\xi)\leqslant\frac{C_C}{\gamma_0}\sqrt{\gamma_0}\|\eta\|_{L^2(\Omega)}\sqrt{\gamma_0}\|\xi\|_{L^2(\Omega)},$$

$$(5.18)\qquad \varepsilon J_{h,m}(\eta,\xi)\leqslant\sqrt{\varepsilon J_{h,m}(\eta,\eta)}\sqrt{\varepsilon J_{h,m}(\xi,\xi)}.$$

On the basis of the above estimates for $a_{h,m}$, $b_{h,m}$, c_h and $J_{h,m}$ we deduce (5.10). □

From the estimates for $b_{h,m}$ and c_h we see that it is necessary to have $\gamma_0>0$ as assumed in (A) f). However, this assumption is not restrictive, as shown in [25].

Lemma 7. *The following estimate holds:*

$$(5.19)\quad \sum_{m=1}^{M}\int_{I_m}\|\xi\|^2_{E,m}\,\mathrm{d}t+\|\xi\|^2_T\leqslant 4C_A^2\sum_{m=1}^{M}\int_{I_m}\sigma_m^2(\eta;h,\varepsilon)\,\mathrm{d}t+8\sum_{m=1}^{M-1}\|\eta_m^-\|^2_{L^2(\Omega)}.$$

P r o o f. From Lemma 2 and formula (5.8) we get

$$\sum_{m=1}^{M}\int_{I_m} A_{h,m}(\xi,\xi)\,\mathrm{d}t+\|\xi\|_T^2 = B(\xi,\xi) = -\sum_{m=1}^{M}\int_{I_m} A_{h,m}(\eta,\xi)\,\mathrm{d}t-\sum_{m=1}^{M-1}(\eta_m^-,\{\xi\}_m). \tag{5.20}$$

By Lemma 3,

$$A_{h,m}(\xi,\xi)\geqslant\|\xi\|_{E,m}^2, \tag{5.21}$$

and by Lemma 6,

$$|A_{h,m}(\eta,\xi)|\leqslant C_A\|\xi\|_{E,m}\sigma_m(\eta;h,\varepsilon). \tag{5.22}$$

Using the Cauchy inequality, we get

$$\left|\sum_{m=1}^{M-1}(\eta_m^-,\{\xi\}_m)\right|\leqslant\left(2\sum_{m=1}^{M-1}\|\eta_m^-\|_{L^2(\Omega)}^2\right)^{1/2}\left(\frac12\sum_{m=1}^{M-1}\|\{\xi\}_m\|_{L^2(\Omega)}^2\right)^{1/2},$$

$$\sum_{m=1}^{M}\int_{I_m}\|\xi\|_{E,m}\sigma_m\,\mathrm{d}t\leqslant\left(\sum_{m=1}^{M}\int_{I_m}\sigma_m^2\,\mathrm{d}t\right)^{1/2}\left(\sum_{m=1}^{M}\int_{I_m}\|\xi\|_{E,m}^2\,\mathrm{d}t\right)^{1/2}.$$

From the above estimates, the definition (4.3) of the norm $\|\cdot\|_T$ and from (5.20) we get

$$\sum_{m=1}^{M}\int_{I_m}\|\xi\|_{E,m}^2\,\mathrm{d}t+\|\xi\|_T^2\leqslant\sqrt{2}\left(\sum_{m=1}^{M}\int_{I_m}\|\xi\|_{E,m}^2\,\mathrm{d}t+\|\xi\|_T^2\right)^{1/2}$$
$$\times\left(C_A\left(\sum_{m=1}^{M}\int_{I_m}\sigma_m^2\,\mathrm{d}t\right)^{1/2}+\left(2\sum_{m=1}^{M-1}\|\eta_m^-\|_{L^2(\Omega)}^2\right)^{1/2}\right).$$

This and the inequality $(a+b)^2\leqslant 2(a^2+b^2)$ already imply (5.19). □

From the inequality $\|e\|^2\leqslant 2(\|\xi\|^2+\|\eta\|^2)$, Lemma 7 and definitions (4.5), (5.9) of $\|\cdot\|_{E,m}$ and σ_m we deduce the following *abstract error estimate.*

Lemma 8. *We have*

$$\sum_{m=1}^{M}\int_{I_m}\|e\|_{E,m}^2\,\mathrm{d}t+\|e\|_T^2 \leqslant C\sum_{m=1}^{M}\int_{I_m}\sigma_m^2(\eta)\,\mathrm{d}t+C\sum_{m=1}^{M-1}\|\eta_m^-\|_{L^2(\Omega)}^2+2\|\eta\|_T^2, \tag{5.23}$$

$$\sum_{m=1}^{M}\int_{I_m}\|e\|_{E,m}^2\,\mathrm{d}t\leqslant C\sum_{m=1}^{M}\int_{I_m}\sigma_m^2(\eta)\,\mathrm{d}t+C\sum_{m=1}^{M-1}\|\eta_m^-\|_{L^2(\Omega)}^2. \tag{5.24}$$

6. Approximation properties of the interpolation π

In order to obtain an estimate of e in terms of h and τ, we shall investigate the approximation properties of the operator π, using the technique applied in [43].

Lemma 9. *We have*

$$\pi u|_{I_m} = \pi(\Pi_m u)|_{I_m}. \tag{6.1}$$

P r o o f. Since Π_m is the space L^2-projection, it follows from (5.2) that

$$\int_{I_m} (\pi u - \Pi_m u, \varphi^*)\,\mathrm{d}t = \int_{I_m} (\pi u - u, \varphi^*) = 0$$

for every $\varphi^* \in S_{h,\tau}^{p,q-1}$. This means that (5.2) is satisfied for $\Pi_m u$. Condition (5.3) is satisfied because $\Pi_m(\Pi_m u) = \Pi_m u$. $\square$

The analysis of approximation properties of the operator π with respect to time is based on the transformation of time integrals over the intervals I_m to integrals over $(-1,1)$ with the aid of an affine mapping Q_m of $(-1,1)$ onto I_m, and the use of approximations by Legendre orthogonal polynomials defined on $(-1,1)$.

Definition 1. The Legendre polynomials on $(-1,1)$ are defined by

$$\begin{aligned} L_0(\zeta) &= 1,\\ L_1(\zeta) &= \zeta,\\ L_{i+1}(\zeta) &= \frac{2i+1}{i+1}\zeta L_i(\zeta) - \frac{i}{i+1}L_{i-1}(\zeta). \end{aligned}$$

Polynomials L_i, $i = 0,1,\ldots$, form an orthogonal basis in $L^2(-1,1)$ and satisfy $L_i(1) = 1$.

Definition 2. Let us set

$$\hat{S}_{h,m}^{p,q} = \Big\{\varphi^* \in L^2(-1,1;\Omega);\ \varphi^* = \sum_{i=0}^{q} \zeta^i \varphi_i^* \text{ with } \varphi_i^* \in S_{h,m}^p\Big\}. \tag{6.2}$$

The projection defined in the same way as π, but on the reference interval $(-1,1)$, will be denoted by $\hat{\pi}_m$. More accurately,

$$\hat{\pi}_m w \in \hat{S}_{h,m}^{p,q}, \tag{6.3}$$

$$\int_{-1}^{1} (\hat{\pi}_m w - w, \varphi^*)\,\mathrm{d}\zeta = 0 \quad \forall\,\varphi^* \in \hat{S}_{h,m}^{p,q-1}, \tag{6.4}$$

$$\hat{\pi}_m w(1-) = \Pi_m w(1-) \tag{6.5}$$

for $w \in C([-1,1]; L^2(\Omega))$.

It is possible to find that

$$(\pi u)|_{I_m} = \pi_{I_m} u, \quad m = 1, \dots, M,$$

where

$$\pi_{I_m} u = (\hat{\pi}_m (u \circ Q_m)) \circ Q_m^{-1} \tag{6.6}$$

with the mapping $Q_m \colon (-1,1) \to I_m = (t_{m-1}, t_m)$ such that

$$Q_m(\zeta) = \frac{1}{2}(t_m + t_{m-1} + \zeta \tau_m) = t, \quad \zeta \in (-1,1), \tag{6.7}$$

$$Q_m^{-1}(t) = \frac{1}{\tau_m}(2t - t_m - t_{m-1}) = \zeta, \quad t \in I_m. \tag{6.8}$$

Lemma 10. *The projection $\hat{\pi}_m$ can be expressed with the aid of the Legendre polynomials in the following way. For $w \in C([-1,1]; L^2(\Omega))$ and any $m \in \{1, \dots, M\}$ we have*

$$\hat{\pi}_m w = \sum_{i=0}^{q-1} w_i L_i + \left(\Pi_m w(1-) - \sum_{i=0}^{q-1} w_i \right) L_q, \tag{6.9}$$

where

$$\Pi_m w = \sum_{i=0}^{\infty} w_i L_i, \quad w_i \in S^p_{h,m}. \tag{6.10}$$

Proof. a) Similarly to Lemma 4 it is possible to prove that there exists at most one function $\hat{\pi} w$ satisfying conditions (6.3)–(6.5). (Another proof can be found in [43], Lemma 1.1.)

b) Further, let $w \in C([-1,1]; L^2(\Omega))$. Then $\Pi_m w \in C([-1,1]; S^p_{h,m})$ can be expressed in the form (6.10). Let $\tilde{w} \in S^p_{h,m}$ and $k \leqslant q-1$. Then $\tilde{w} L_k \in \hat{S}^{p,q-1}_{h,m}$. By the definition of $\Pi_m w$ we have

$$(w, \tilde{w} L_k) = (\Pi_m w, \tilde{w} L_k). \tag{6.11}$$

Let us set

$$\overline{\pi}_m w := \sum_{i=0}^{q-1} w_i L_i + \left(\Pi_m w(1-) - \sum_{i=0}^{q-1} w_i \right) L_q. \tag{6.12}$$

It follows from (6.12) and (6.11) that

$$
\begin{aligned}
(6.13)\qquad & \int_{-1}^{1} (\overline{\pi} w - w, \tilde{w} L_k)\,\mathrm{d}\vartheta \\
&= \int_{-1}^{1} \Bigl(\sum_{i=0}^{q-1} w_i L_i + \Bigl(\Pi_m w(1-) - \sum_{i=0}^{q-1} w_i \Bigr) L_q - \Pi_m w, \tilde{w} L_k \Bigr)\,\mathrm{d}\vartheta.
\end{aligned}
$$

Now, denoting $\omega = \Pi_m w(1-) - \sum\limits_{i=0}^{q-1} w_i$ and using (6.10) and the orthogonality of the polynomials L_i, we find that

$$
(6.14)\qquad \int_{-1}^{1} (\overline{\pi} w - w, \tilde{w} L_k)\,\mathrm{d}\vartheta = \int_{-1}^{1} \Bigl(-\sum_{i=q}^{\infty} w_i L_i + \omega L_q, \tilde{w} L_k \Bigr)\,\mathrm{d}\vartheta = 0.
$$

This implies that condition (6.4), where we write $\overline{\pi}$ instead of $\hat{\pi}$, is satisfied.

Moreover, by (6.10) and (6.12) and the relation $L_i(1) = 1$ we easily show that (6.5), where $\hat{\pi}$ is replaced by $\overline{\pi}$, holds.

From both parts a) and b) of the proof we conclude that $\hat{\pi} = \overline{\pi}$, which we wanted to prove. $\square$

Now we shall seek estimates of the norms $\|\eta\|_{L^2(I_m;X)}$, where $\eta = u - \pi u$ and X denotes the spaces appearing on the right-hand side of (5.24), e.g. $X = L^2(K_i)$, $X = H^1(\Omega, \mathcal{T}_{h,m})$, etc.—see definitions (4.5), (5.9). Let us set $v_m = \Pi_m u \circ Q_m$. (Due to the regularity assumption (6.24) on the exact solution u introduced later, $v_m \in C([-1,1]; L^2(\Omega)$. Moreover, $\Pi_m v_m = v_m$.) The relation

$$
\eta|_{I_m} = (\pi u - u)|_{I_m} = \pi_{I_m} u - \Pi_m u + \Pi_m u - u|_{I_m} = \pi \Pi_m u - \Pi_m u + \Pi_m u - u|_{I_m}
$$

and the substitution theorem imply that

$$
\begin{aligned}
(6.15)\qquad \|\eta\|_{L^2(I_m;X)} &\leqslant \|u - \Pi_m u\|_{L^2(I_m;X)} + \|\Pi_m u - \pi \Pi_m u\|_{L^2(I_m;X)} \\
&= \|u - \Pi_m u\|_{L^2(I_m;X)} \\
&\quad + \|\Pi_m u - (\hat{\pi}_m(\Pi_m u \circ Q_m)) \circ Q_m^{-1}\|_{L^2(I_m;X)} \\
&= \|u - \Pi_m u\|_{L^2(I_m;X)} + \sqrt{\frac{\tau_m}{2}} \|v_m - \hat{\pi}_m v_m\|_{L^2(-1,1;X)}.
\end{aligned}
$$

Let us denote by $\hat{P}$ the L^2-time-projection to $\mathcal{P}^q(-1,1)$ (space of all polynomials depending on $\zeta \in (-1,1)$ of degree at most q). Let us expand the function v_m using Legendre polynomials as $v_m = \sum\limits_{i=0}^{\infty} \tilde{v}_i L_i$, $\tilde{v}_i \in S^p_{h,m}$. Since

$$
\int_{-1}^{1} \Bigl(\sum_{i=q+1}^{\infty} \tilde{v}_i L_i \Bigr) L_k \,\mathrm{d}t = 0, \quad k = 0, \dots, q,
$$

we have

$$v_m - \hat{P}v_m = \sum_{i=q+1}^{\infty} \tilde{v}_i L_i.$$

Now, by Lemma 10 applied to $w = v_m$, we find that

$$\begin{aligned} v_m - \hat{\pi}_m v_m &= \sum_{i=q}^{\infty} \tilde{v}_i L_i - \Bigl(\Pi_m v_m(1) - \sum_{i=0}^{q-1} \tilde{v}_i\Bigr) L_q \\ &= \sum_{i=q+1}^{\infty} \tilde{v}_i L_i - \Bigl(\sum_{i=q+1}^{\infty} \tilde{v}_i\Bigr) L_q \\ &= v_m - \hat{P}v_m - \Bigl(\sum_{i=q+1}^{\infty} \tilde{v}_i\Bigr) L_q. \end{aligned} \tag{6.16}$$

Lemma 11. *The following inequality holds:*

$$\Bigl\| \sum_{i=q+1}^{\infty} \tilde{v}_i \Bigr\|_X \leqslant C_q \|v_m'\|_{L^2(-1,1;X)}. \tag{6.17}$$

P r o o f. See [43], p. 23. □

Lemma 12. *We have*

$$\|v_m'\|_{L^2(-1,1;X)} = \sqrt{\frac{\tau_m}{2}} \|(\Pi_m u)'\|_{L^2(I_m;X)}. \tag{6.18}$$

P r o o f. Using the substitution $t = Q_m(\zeta)$, we find that

$$\begin{aligned} \|v_m'\|^2_{L^2(-1,1;X)} &= \|(\Pi_m u \circ Q_m)'\|^2_{L^2(-1,1;X)} \\ &= \frac{\tau_m}{2} \int_{I_m} \|(\Pi_m u)'(t)\|_X^2 \,\mathrm{d}t = \biggl(\sqrt{\frac{\tau_m}{2}} \|(\Pi_m u)'\|_{L^2(I_m;X)}\biggr)^2. \end{aligned}$$

□

Lemma 13. *Let us define P_m as the L^2-time-projection to $\mathcal{P}^q(I_m)$. Then*

$$\begin{aligned} \|u - \pi u\|_{L^2(I_m;X)} &\leqslant \|u - \Pi_m u\|_{L^2(I_m;X)} + \|\Pi_m u - P_m \Pi_m u\|_{L^2(I_m;X)} \\ &\quad + C_q \frac{\tau_m}{2} \|(\Pi_m u)' - \psi'\|_{L^2(I_m;X)} \quad \forall\, \psi \in S^{p,q}_{h,\tau}. \end{aligned} \tag{6.19}$$

P r o o f. For any $v \in L^2(I_m)$ we have $P_m v = (\hat{P}(v \circ Q_m)) \circ Q_m^{-1}$. We successively get

$$\begin{aligned}
&\sqrt{\frac{\tau_m}{2}}\|v_m - \hat{P}v_m\|_{L^2(-1,1;X)} \\
&\quad = \sqrt{\frac{\tau_m}{2}}\|\Pi_m u \circ Q_m - \hat{P}(\Pi_m u \circ Q_m)\|_{L^2(-1,1;X)} \\
&\quad = \sqrt{\frac{\tau_m}{2}}\|\Pi_m u \circ Q_m - ((\hat{P}(\Pi_m u \circ Q_m)) \circ Q_m^{-1}) \circ Q_m\|_{L^2(-1,1;X)} \\
&\quad = \left(\frac{\tau_m}{2}\int_{-1}^{1} \|(\Pi_m u - P_m \Pi_m u) \circ Q_m\|_X^2 \,\mathrm{d}\zeta\right)^{1/2} \\
&\quad = \|\Pi_m u - P_m \Pi_m u\|_{L^2(I_m;X)}.
\end{aligned}$$

This and (6.15)–(6.18) imply that

$$\begin{aligned}
\|u - \pi u\|_{L^2(I_m;X)} \leqslant \|u - \Pi_m u\|_{L^2(I_m;X)} &+ \|\Pi_m u - P_m \Pi_m u\|_{L^2(I_m;X)} \\
&+ C_q \frac{\tau_m}{2}\|(\Pi_m u)'\|_{L^2(I_m;X)}.
\end{aligned} \tag{6.20}$$

Substituting $u := u - \psi$ in (6.20), $\psi \in S_{h,\tau}^{p,q}$, and taking into account that

$$\Pi_m \psi = \psi|_{I_m} = P_m \Pi_m \psi, \tag{6.21}$$

$$(\Pi_m \psi)' = (\psi|_{I_m})', \tag{6.22}$$

$$\pi\psi = \psi, \tag{6.23}$$

we arrive at (6.19). □

We will assume that the exact solution u satisfies the regularity condition

$$u \in \mathcal{H} = H^{q+1}(0,T;H^1(\Omega)) \cap C(0,T;H^{p+1}(\Omega)) \tag{6.24}$$

for some integers $p, q \geqslant 1$.

Lemma 14. *There exist constants C_Π and C_P such that*

$$\|v(t) - \Pi_m v(t)\|_{L^2(K)} \leqslant C_\Pi h_m^{p+1}|v(t)|_{H^{p+1}(K)} \ \forall\, K \in \mathcal{T}_h, \text{ a.a. } t \in (0,T), \tag{6.25}$$

$$|v(t) - \Pi_m v(t)|_{H^1(K)} \leqslant C_\Pi h^p |v(t)|_{H^{p+1}(K)} \ \forall\, K \in \mathcal{T}_h, \text{ a.a. } t \in (0,T), \tag{6.26}$$

$$\|v(t) - \Pi_m v(t)\|_{L^2(\Omega)} \leqslant C_\Pi h_m^{p+1}|v(t)|_{H^{p+1}(\Omega)} \text{ for a.a. } t \in (0,T), \tag{6.27}$$

$$|v(t) - \Pi_m v(t)|_{H^1(\Omega,\mathcal{T}_{h,m})} \leqslant C_\Pi h^p |v(t)|_{H^{p+1}(\Omega)} \text{ for a.a. } t \in (0,T), \tag{6.28}$$

$$|v(t) - \Pi_m v(t)|_{H^2(\Omega,\mathcal{T}_{h,m})} \leqslant C_\Pi h^{p-1} |v(t)|_{H^{p+1}(\Omega)} \text{ for a.a. } t \in (0,T), \tag{6.29}$$

$$\|v(x) - P_m v(x)\|_{L^2(I_m)} \leqslant C_P \tau_m^{q+1} |v(x)|_{H^{q+1}(I_m)} \text{ for a.a. } x \in \Omega, \tag{6.30}$$

$$|v(x) - P_m v(x)|_{H^1(I_m)} \leqslant C_P \tau_m^{q} |v(x)|_{H^{q+1}(I_m)} \text{ for a.a. } x \in \Omega \tag{6.31}$$

for $m = 1, \ldots, M$ and $v \in \mathcal{H}$.

Proof. For the proof of estimates (6.25)–(6.29), see [25], Lemma 4.1. The last two estimates (6.30), (6.31) can be proved in a similar way using the transformations Q_m and standard scaling arguments. $\square$

Lemma 15. *We have*

$$\|\Pi_m v\|_{L^2(K)} \leqslant \|v\|_{L^2(K)}, \tag{6.32}$$

$$|\Pi_m v|_{H^1(K)} \leqslant C_B |v|_{H^1(K)} \tag{6.33}$$

for $v \in H^1(\Omega, \mathcal{T}_{h,m})$, $K \in \mathcal{T}_{h,m}$, $m = 1, \ldots, M$, with a constant C_B independent of v, K, m.

Proof is easily carried out on the basis of results from [8]. $\square$

Now we attack the derivation of the error estimates. We start from estimate (5.24). Thus, we need to estimate the expression

$$\sum_{m=1}^{M} \int_{I_m} \sigma_m^2(\eta; h, \varepsilon)\,\mathrm{d}t + \sum_{m=1}^{M-1} \|\eta_m^-\|^2_{L^2(\Omega)}, \tag{6.34}$$

where

$$\begin{aligned} \sigma_m = {} & \Big\{ \varepsilon |\eta|^2_{H^1(\Omega, \mathcal{T}_{h,m})} + \gamma_0 \|\eta\|^2_{L^2(\Omega)} + \varepsilon J_{h,m}(\eta, \eta) \\ & + \frac{1}{2} \sum_{i \in i_{h,m}} \big(\|\eta\|^2_{\boldsymbol{v}, \partial K_i \cap \partial\Omega} + \|[\eta]\|^2_{\boldsymbol{v}, \partial K_i^- \setminus \partial\Omega} \big) \Big\}^{1/2} \\ & + \sqrt{\varepsilon} h |\eta|_{H^2(\Omega, \mathcal{T}_{h,m})} + \Big\{ \sum_{i \in i_{h,m}} \|\eta^-\|^2_{\boldsymbol{v}, \partial K_i^- \setminus \partial\Omega} \Big\}^{1/2}. \end{aligned} \tag{6.35}$$

Taking into account inequality (6.19) where we set $\psi = P_m \Pi_m u$, we need to estimate the norms of the expressions

$$u - \Pi_m u,\ \Pi_m u - P_m \Pi_m u,\ (\Pi_m u - P_m \Pi_m u)'. \tag{6.36}$$

Lemma 16. *Let $u \in \mathcal{H}$ and assume that there exist constants $C_S, \hat{C}_S$ such that*

$$\frac{1}{\hat{C}_S} h_K \leqslant \tau_m \leqslant C_S h_K, \quad K \in \mathcal{T}_{h,m},\ \ m = 1, \ldots, M. \tag{6.37}$$

We denote $\tilde{C}_q = 1 + \frac{1}{4}C_q^2$ *with* C_q *from Lemma 11. Then*

$$\int_{I_m} |\eta|^2_{H^1(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \leqslant 3C_\Pi^2 h^{2p}|u|^2_{L^2(I_m;H^{p+1}(\Omega))} + 3C_B^2 C_P^2 \tilde{C}_q \tau_m^{2q+2}|u|^2_{H^{q+1}(I_m;H^1(\Omega))}, \tag{6.38}$$

$$\int_{I_m} \|\eta\|^2_{L^2(\Omega)}\,\mathrm{d}t \leqslant 3C_\Pi^2 h^{2p+2}|u|^2_{L^2(I_m;H^{p+1}(\Omega))} + 3C_P^2 \tilde{C}_q \tau_m^{2q+2}|u|^2_{H^{q+1}(I_m;L^2(\Omega))}, \tag{6.39}$$

$$\int_{I_m} \sum_{i\in i_{h,m}} h_{K_i}^{-2}\|\eta\|^2_{L^2(K_i)}\,\mathrm{d}t \leqslant 3C_\Pi^2 h^{2p}|u|^2_{L^2(I_m;H^{p+1}(\Omega))} + 3C_S^2 C_P^2 \tilde{C}_q \tau_m^{2q}|u|^2_{H^{q+1}(I_m;L^2(\Omega))}, \tag{6.40}$$

$$\begin{aligned}\int_{I_m} J_{h,m}(\eta,\eta)\,\mathrm{d}t \leqslant{}& 6C_M C_\Pi^2 C_D h^{2p}|u|^2_{L^2(I_m;H^{p+1}(\Omega))}\\ &+ 3C_M C_P^2 \tilde{C}_q C_D \tau_m^{2q}(3C_S^2|u|^2_{H^{q+1}(I_m;L^2(\Omega))}\\ &+ C_B^2\tau_m^2|u|^2_{H^{q+1}(I_m;H^1(\Omega))}),\end{aligned} \tag{6.41}$$

$$\begin{aligned}\int_{I_m} \sum_{i\in i_{h,m}} (\|\eta\|^2_{\boldsymbol{v},\partial K_i\cap\partial\Omega} &+ \|[\eta]\|^2_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega})\,\mathrm{d}t\\ &\leqslant 3C_M C_\Pi^2 C_{\boldsymbol{v}} h^{2p+1}|u|^2_{L^2(I_m;H^{p+1}(\Omega))}\\ &+ \frac{3}{2}C_M C_P^2 \tilde{C}_q C_{\boldsymbol{v}} \tau_m^{2q+1}\{3C_S|u|^2_{H^{q+1}(I_m;L^2(\Omega))}\\ &+ C_B^2 h\tau_m|u|^2_{H^{q+1}(I_m;H^1(\Omega))}\},\end{aligned} \tag{6.42}$$

$$\int_{I_m} |\eta|^2_{H^2(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \leqslant 3C_\Pi^2 h^{2(p-1)}|u|^2_{L^2(I_m;H^{p+1}(\Omega))} + 3C_B^2 C_I^2 C_P^2 \tilde{C}_q C_S^2 \tau_m^{2q}|u|^2_{H^{q+1}(I_m;H^1(\Omega))}, \tag{6.43}$$

$$\sum_{m=1}^{M-1} \|\eta_m^-\|^2_{L^2(\Omega)} \leqslant C_\Pi^2 \hat{C}_S T h^{2p+1}\|u\|^2_{C([0,T];H^{p+1}(\Omega))}. \tag{6.44}$$

P r o o f of this lemma is rather technical. For the sake of completeness of the exposition, we will prove it in Appendix.

Now, using the abstract error estimate from Lemma 8, relation (6.35) and estimates from Lemma 16, we finally arrive (for $h, \tau < 1$) at the *main result*:

Theorem 1. *Let assumptions* (A) a)–h), (3.39), (3.40) *and* (6.37) *be satisfied. Let* u *be the exact solution of problem* (2.1)–(2.4) *satisfying the condition* $u \in \mathcal{H}$, *where the space* $\mathcal{H}$ *is defined by* (6.24), *and let* U *denote the approximate solution obtained with the aid of the method* (3.36). *Then there exists a constant* C *independent of* h,

τ and ε such that the error $e = U - u$ satisfies the estimate

$$(6.45)\qquad \sum_{m=1}^{M} \int_{I_m} \|e\|_{E,m}^2 \,\mathrm{d}t \leqslant Ch^{2p}\{|u|^2_{L^2(0,T;H^{p+1}(\Omega))} + |u|^2_{C([0,T];H^{p+1}(\Omega))}\} + C\tau^{2q}\{|u|^2_{H^{q+1}(0,T;L^2(\Omega))} + |u|^2_{H^{q+1}(0,T;H^1(\Omega))}\}.$$

This estimate is also valid for $\varepsilon = 0$, i.e. in the hyperbolic case.

7. Numerical experiments

In this section we present some numerical experiments with the space-time DGFE method described and analyzed in the previous sections. We solve equation (2.1) in $Q_T = (0,1)^2 \times (0,1)$ with $v_1 = v_2 = 1$, $c = 0.5$ and two choices of ε: $\varepsilon = 0.005$ (parabolic case) and $\varepsilon = 0$ (hyperbolic case). The right-hand side g and the boundary and initial conditions are chosen in such a way that they conform to the exact solution

$$u_{\mathrm{ex}}(x_1, x_2, t) = (1 - \mathrm{e}^{-t})(2x + 2y - xy + 2(1 - \mathrm{e}^{v_1(x_1-1)/\nu})(1 - \mathrm{e}^{v_2(x_2-1)/\nu})),$$

where $\nu = 0.05$ is the constant determining the steepness of the boundary layer in the exact solution. The problem is solved on a sequence of non-nested nonuniform space meshes $\mathcal{T}_{h_1}$, $\mathcal{T}_{h_2}, \ldots$. On all time levels, the same triangulations are used. Figs. 1 and 2 show the coarsest mesh $\mathcal{T}_{h_1}$ and the mesh $\mathcal{T}_{h_4}$, respectively. We inspect

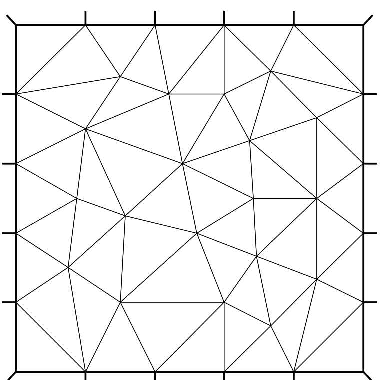

Figure 1. Coarse mesh.

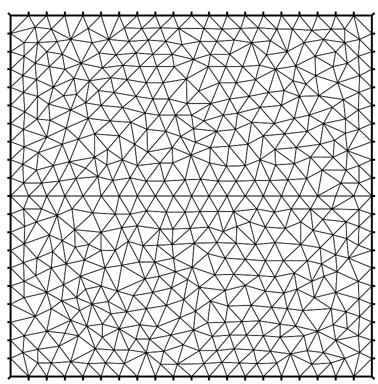

Figure 2. Fine mesh.

the experimental order of convergence (EOC) with respect to τ and h, which vary simultaneously due to condition (6.37). For successive pairs (τ, h) and (τ', h') we evaluate the experimental order of convergence in space and time defined as

$$\mathrm{EOC}_{\mathrm{space}} = \frac{\log(\|e_{\tau' h'}\|) - \log(\|e_{\tau h}\|)}{\log h' - \log h}, \quad \mathrm{EOC}_{\mathrm{time}} = \frac{\log(\|e_{\tau' h'}\|) - \log(\|e_{\tau h}\|)}{\log \tau' - \log \tau},$$

where $e_{\tau h} = u_{\rm ex} - U$ is the error of the method when the exact solution $u_{\rm ex}$ is approximated by the DG approximate solution U computed with the aid of a space triangulation of size h and a time interval partition of size τ. As $\|\cdot\|$ we use the norm $\|\cdot\|_{L^2(Q_T)}$-norm (i.e., $L^2(L^2)$-norm) and the seminorm $\sqrt{\varepsilon}|\cdot|_{L^2(0,T;H^1(\Omega,\mathcal{T}_h))}$ (i.e., $\sqrt{\varepsilon}L^2(H^1)$-seminorm). Moreover, we compute the global orders of convergence. We approximate

$$\|e_{\tau h}\| \approx C_1 h^r + C_2 \tau^s$$

by the method of least squares, using the `MINPACK` package [34]. The results are shown in Tabs. 1–4, where $\mathrm{EOC}^0_{\rm space}$, $\mathrm{EOC}^0_{\rm time}$ and $\mathrm{EOC}^1_{\rm space}$, $\mathrm{EOC}^1_{\rm time}$ denote the experimental orders of convergence in $L^2(L^2)$-norm and $\sqrt{\varepsilon}L^2(H^1)$-seminorm, respectively.

h	τ	$\|e_{\tau h}\|_{L^2(L^2)}$	$\|e_{\tau h}\|_{\sqrt{\varepsilon}L^2(H^1)}$	$\mathrm{EOC}^0_{\rm space}$	$\mathrm{EOC}^0_{\rm time}$	$\mathrm{EOC}^1_{\rm space}$	$\mathrm{EOC}^1_{\rm time}$
0.2838	0.2500	$4.5853E-02$	$1.9970E-01$	–	–	–	–
0.2172	0.2000	$3.5474E-02$	$1.5372E-01$	0.96	1.15	0.98	1.17
0.1540	0.1667	$2.2387E-02$	$1.2782E-01$	1.34	2.52	0.54	1.01
0.1035	0.1000	$1.2945E-02$	$8.9991E-02$	1.38	1.07	0.88	0.69
0.0768	0.0769	$5.3557E-03$	$5.7493E-02$	2.95	3.36	1.50	1.71
0.0532	0.0526	$2.3742E-03$	$3.7567E-02$	2.22	2.14	1.16	1.12
0.0398	0.0400	$1.3345E-03$	$2.6438E-02$	1.98	2.10	1.21	1.28
0.0270	0.0270	$5.2577E-04$	$1.6779E-02$	2.40	2.38	1.17	1.16
Global order of convergence				2.07	2.11	1.07	1.11

Table 1. $\varepsilon = 0.005$, $p = 1$, $q = 1$ (parabolic case).

h	τ	$\|e_{\tau h}\|_{L^2(L^2)}$	$\|e_{\tau h}\|_{\sqrt{\varepsilon}L^2(H^1)}$	$\mathrm{EOC}^0_{\rm space}$	$\mathrm{EOC}^0_{\rm time}$	$\mathrm{EOC}^1_{\rm space}$	$\mathrm{EOC}^1_{\rm time}$
0.2838	0.2500	$2.0470E-02$	$8.7193E-02$	–	–	–	–
0.2172	0.2000	$1.0103E-02$	$5.5539E-02$	2.64	3.16	1.69	2.02
0.1540	0.1667	$4.3992E-03$	$3.4110E-02$	2.42	4.56	1.42	2.67
0.1035	0.1000	$1.6821E-03$	$1.7835E-02$	2.42	1.88	1.63	1.27
0.0768	0.0769	$4.9668E-04$	$7.7827E-03$	4.08	4.65	2.78	3.16
0.0532	0.0526	$1.6550E-04$	$3.3350E-03$	3.00	2.90	2.31	2.23
0.0398	0.0400	$7.7630E-05$	$1.8029E-03$	2.61	2.76	2.12	2.24
0.0270	0.0270	$2.7654E-05$	$7.0749E-04$	2.66	2.63	2.41	2.39
Global order of convergence				2.89	2.78	2.05	2.41

Table 2. $\varepsilon = 0.005$, $p = 2$, $q = 2$ (parabolic case).

h	τ	$\|e_{\tau h}\|_{L^2(L^2)}$	$\mathrm{EOC}^0_{\mathrm{space}}$	$\mathrm{EOC}^0_{\mathrm{time}}$
0.2838	0.2500	$4.9212E-02$	–	–
0.2172	0.2000	$3.8843E-02$	0.89	1.06
0.1540	0.1667	$2.5997E-02$	1.17	2.20
0.1035	0.1000	$1.5581E-02$	1.29	1.00
0.0768	0.0769	$6.9089E-03$	2.72	3.10
0.0532	0.0526	$3.2904E-03$	2.02	1.95
0.0398	0.0400	$1.8620E-03$	1.96	2.07
0.0270	0.0270	$7.5458E-04$	2.32	2.30
Global order of convergence			1.95	1.99

Table 3. $\varepsilon = 0$, $p = 1$, $q = 1$ (hyperbolic case).

h	τ	$\|e_{\tau h}\|_{L^2(L^2)}$	$\mathrm{EOC}^0_{\mathrm{space}}$	$\mathrm{EOC}^0_{\mathrm{time}}$
0.2838	0.2500	$2.3451E-02$	–	–
0.2172	0.2000	$1.2484E-02$	2.36	2.83
0.1540	0.1667	$6.1746E-03$	2.05	3.86
0.1035	0.1000	$2.6342E-03$	2.14	1.67
0.0768	0.0769	$8.0848E-04$	3.95	4.50
0.0532	0.0526	$2.6400E-04$	3.05	2.95
0.0398	0.0400	$1.0761E-04$	3.09	3.27
0.0270	0.0270	$2.7962E-05$	3.47	3.44
Global order of convergence			2.87	2.98

Table 4. $\varepsilon = 0$, $p = 2$, $q = 2$ (hyperbolic case).

Our DGFE computations were carried out with the aid of the `FreeFEM++` modelling environment from [27] which was adapted to the DGFE space-time discretization. The time integrals were evaluated by quadrature formulae exact for polynomials of degree 5 and 9 in the case of elements linear in time and quadratic in time, respectively. The quadrature formulae used for integration over the triangles and their sides were exact for polynomials of degree 5 both for the linear and quadratic elements. The nonsymmetric linear problem was solved in each time step by the multifrontal direct solver `UMFPACK` ([12]).

8. Conclusion

Under assumptions (A), a)–h) from the first section, the assumption of uniform shape regularity of meshes on individual time levels, nondegeneracy of faces Γ_{ij}, some relations between time steps and space mesh sizes and a certain regularity of the exact solution, we have derived $L^2(L^2)$ and $\sqrt{\varepsilon}L^2(H^1)$ estimates for the error of the approximate solution by the separated space-time discontinuous Galerkin method, using polynomials of degree p in space and q in time and, in general, different space grids on different time levels. The error is then of order $O(h^p + \tau^q)$. The estimate holds true even if $\varepsilon = 0$ (hyperbolic case).

Computational results shown in Section 7 indicate that the error estimates are suboptimal in the $L^2(L^2)$-norm, but optimal in $\sqrt{\varepsilon}L^2(H^1)$-seminorm. On the other hand, they are uniform with respect to the diffusion coefficient $\varepsilon \to 0+$. Moreover, the method allows us to use different nonconforming space meshes with hanging nodes on different time levels, which plays an importnat role in time dependent adaptive mesh refinement in the space domain Ω. From this point of view it will be suitable to carry out numerical experiments for this situation in order to examine the influence of the use of time dependent triangulations in the domain Ω. Another question is the choice of $q = 0$, when our method can be treated as backward Euler's scheme. The technique applied here does not allow to analyze this case. This was carried out in [16], but the error estimate obtained blows up for $\varepsilon \to 0+$.

From the analysis presented here one can see that further modifications and generalizations are possible, namely the use of quadrilateral space elements and/or the application of the *hp*-version of the DGFE discretization. The analysis of this case can be performed following the above results and estimates from, e.g., [28] and [43]. Further open problems are the derivation of optimal error estimates in the case when the SIPG variant of the DG space discretization of the diffusion terms is used, the analysis of the effect of numerical integration, extension to nonlinear convection-diffusion problems (solved in [20] and [14] by the method of lines), derivation of a posteriori error estimates, and last but not least, the application of the DG space-time discretization to the solution of some nonstationary technically relevant problems, as, e.g., the compressible viscous flow described by the system of Navier-Stokes equations.

9. Appendix

P r o o f of Lemma 16. For each estimate we shall first make use of Lemma 13 (estimate (6.19) where we set $\psi := P_m\Pi_m u$) and in estimating individual terms coming from (6.19) use Lemma 14 (estimates (6.25)–(6.31)), Lemma 15 (estimates (6.32) and (6.33)), assumption (6.37) and standard tools (Young's inequality, multiplicative trace inequality (3.41), inverse inequality (3.42)).

By Lemma 13, for the $H^1(\Omega, \mathcal{T}_{h,m})$-norm we have

$$\begin{aligned}(9.1)\quad &\int_{I_m} |\eta|^2_{H^1(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \\ &\leqslant 3\Big\{\int_{I_m} |u-\Pi_m u|^2_{H^1(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t + \int_{I_m} |\Pi_m u - P_m\Pi_m u|^2_{H^1(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \\ &\quad + \Big(\frac{C_q\tau_m}{2}\Big)^2 \int_{I_m} |(\Pi_m u - P_m\Pi_m u)'|^2_{H^1(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t\Big\}.\end{aligned}$$

Then we estimate the individual terms on the right-hand side of (9.1) using (6.26), (6.30) and (6.33):

$$\begin{aligned}(9.2)\quad \int_{I_m} |u-\Pi_m u|^2_{H^1(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t &\leqslant C_\Pi^2 h^{2p}\int_{I_m} |u|^2_{H^{p+1}(\Omega)}\,\mathrm{d}t \\ &= C_\Pi^2 h^{2p} |u|^2_{L^2(I_m;H^{p+1}(\Omega))},\end{aligned}$$

$$\begin{aligned}(9.3)\quad &\int_{I_m} |\Pi_m u - P_m\Pi_m u|^2_{H^1(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \\ &= \sum_{i\in i_{h,m}} \int_{I_m}\int_{K_i} |\nabla(\Pi_m u - P_m\Pi_m u)|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= \sum_{i\in i_{h,m}} \int_{K_i}\int_{I_m} \sum_{j=1}^{d}\Big(\frac{\partial}{\partial x_j}\Pi_m u - P_m\Big(\frac{\partial}{\partial x_j}\Pi_m u\Big)\Big)^2\,\mathrm{d}t\,\mathrm{d}x \\ &\leqslant C_P^2 \sum_{i\in i_{h,m}} \int_{K_i} \tau_m^{2q+2}|\nabla\Pi_m u|^2_{H^{q+1}(I_m)}\,\mathrm{d}x \\ &= C_P^2\tau_m^{2q+2} \sum_{i\in i_{h,m}} \int_{K_i}\int_{I_m} \Big|\frac{\partial^{q+1}}{\partial t^{q+1}}(\nabla\Pi_m u)\Big|^2\,\mathrm{d}t\,\mathrm{d}x \\ &= C_P^2\tau_m^{2q+2} \sum_{i\in i_{h,m}} \int_{I_m}\int_{K_i} \Big|\nabla\Pi_m\Big(\frac{\partial^{q+1}u}{\partial t^{q+1}}\Big)\Big|^2\,\mathrm{d}x\,\mathrm{d}t \\ &= C_P^2\tau_m^{2q+2} \int_{I_m}\sum_{i\in i_{h,m}} \Big|\Pi_m\Big(\frac{\partial^{q+1}u}{\partial t^{q+1}}\Big)\Big|^2_{H^1(K_i)}\,\mathrm{d}t \\ &\leqslant C_B^2 C_P^2\tau_m^{2q+2} \int_{I_m}\sum_{i\in i_{h,m}} \Big|\frac{\partial^{q+1}u}{\partial t^{q+1}}\Big|^2_{H^1(K_i)}\,\mathrm{d}t \\ &= C_B^2 C_P^2\tau_m^{2q+2} |u|^2_{H^{q+1}(I_m;H^1(\Omega))}.\end{aligned}$$

Similarly to the above, using also (6.31), we obtain

$$
\begin{aligned}
(9.4) \qquad & \int_{I_m} |(\Pi_m u - P_m \Pi_m u)'|^2_{H^1(\Omega, \mathcal{T}_{h,m})} \, \mathrm{d}t \\
&= \int_{I_m} \sum_{i \in i_{h,m}} \int_{K_i} |\nabla (\Pi_m u - P_m \Pi_m u)'|^2 \, \mathrm{d}x \, \mathrm{d}t \\
&= \sum_{i \in i_{h,m}} \int_{K_i} \int_{I_m} |(\nabla \Pi_m u - P_m (\nabla \Pi_m u))'|^2 \, \mathrm{d}t \, \mathrm{d}x \\
&\leqslant C_P^2 \sum_{i \in i_{h,m}} \int_{K_i} \tau_m^{2q} |\nabla \Pi_m u|^2_{H^{q+1}(I_m)} \, \mathrm{d}x \\
&\leqslant C_B^2 C_P^2 \tau_m^{2q} |u|^2_{H^{q+1}(I_m; H^1(\Omega))}.
\end{aligned}
$$

Estimates (9.1)–(9.4) imply (6.38).

For the $L^2(\Omega)$-norm we have

$$
\begin{aligned}
(9.5) \qquad & \int_{I_m} \|\eta\|^2_{L^2(\Omega)} \, \mathrm{d}t \\
&\leqslant 3 \int_{I_m} \|u - \Pi_m u\|^2_{L^2(\Omega)} \, \mathrm{d}t + 3 \int_{I_m} \|\Pi_m u - P_m \Pi_m u\|^2_{L^2(\Omega)} \, \mathrm{d}t \\
&\quad + 3 \Big(\frac{C_q \tau_m}{2} \Big)^2 \int_{I_m} \|(\Pi_m u - P_m \Pi_m u)'\|^2_{L^2(\Omega)} \, \mathrm{d}t,
\end{aligned}
$$

and

$$
\begin{aligned}
(9.6) \qquad \int_{I_m} \|u - \Pi_m u\|^2_{L^2(\Omega)} \, \mathrm{d}t &\leqslant C_\Pi^2 \int_{I_m} \sum_{i \in i_{h,m}} h^{2p+2} |u|^2_{H^{p+1}(K_i)} \, \mathrm{d}t \\
&= C_\Pi^2 h^{2p+2} |u|^2_{L^2(I_m; H^{p+1}(\Omega))},
\end{aligned}
$$

$$
\begin{aligned}
(9.7) \qquad & \int_{I_m} \|\Pi_m u - P_m \Pi_m u\|^2_{L^2(\Omega)} \, \mathrm{d}t \\
&= \int_\Omega \int_{I_m} |\Pi_m u - P_m \Pi_m u|^2 \, \mathrm{d}t \, \mathrm{d}x \\
&\leqslant C_P^2 \tau_m^{2q+2} \int_\Omega |\Pi_m u|^2_{H^{q+1}(I_m)} \, \mathrm{d}x \\
&= C_P^2 \tau_m^{2q+2} \int_{I_m} \int_\Omega \Big(\Pi_m \Big(\frac{\partial^{q+1} u}{\partial t^{q+1}} \Big) \Big)^2 \, \mathrm{d}x \, \mathrm{d}t \\
&\leqslant C_P^2 \tau_m^{2q+2} \int_{I_m} \Big\| \frac{\partial^{q+1} u}{\partial t^{q+1}} \Big\|^2_{L^2(\Omega)} \, \mathrm{d}t \\
&= C_P^2 \tau_m^{2q+2} |u|^2_{H^{q+1}(I_m; L^2(\Omega))},
\end{aligned}
$$

$$
\begin{aligned}
(9.8)\qquad \int_{I_m} \|(\Pi_m u - P_m\Pi_m u)'\|^2_{L^2(\Omega)}\,\mathrm{d}t &= \int_\Omega\int_{I_m} |(\Pi_m u - P_m\Pi_m u)'|^2\,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant C_P^2\tau_m^{2q}\int_\Omega |\Pi_m u|^2_{H^{q+1}(I_m)}\,\mathrm{d}x \\
&\leqslant C_P^2\tau_m^{2q}|u|^2_{H^{q+1}(I_m;L^2(\Omega))}.
\end{aligned}
$$

Estimate (9.8) is obtained in a way similar to (9.7). From (9.5)–(9.8) we get (6.39).

Estimate (6.40) is obtained in a way similar to (6.39), writing K_i instead of Ω and using assumption (6.37).

For the $J_{h,m}$-terms we have

$$
\begin{aligned}
(9.9)\ &\int_{I_m} J_{h,m}(\eta,\eta)\,\mathrm{d}t \\
&\quad\leqslant 3\int_{I_m} J_{h,m}(u-\Pi_m u, u-\Pi_m u)\,\mathrm{d}t \\
&\qquad + 3\int_{I_m} J_{h,m}(\Pi_m u - P_m\Pi_m u, \Pi_m u - P_m\Pi_m u)\,\mathrm{d}t \\
&\qquad + 3\Big(\frac{C_q\tau_m}{2}\Big)^2\int_{I_m} J_{h,m}((\Pi_m u - P_m\Pi_m u)', (\Pi_m u - P_m\Pi_m u)')\,\mathrm{d}t.
\end{aligned}
$$

By (3.20), (3.40), the multiplicative trace inequality (3.41) and Lemma 14,

$$
\begin{aligned}
(9.10)\quad &\int_{I_m} J_{h,m}(u-\Pi_m u, u-\Pi_m u)\,\mathrm{d}t \\
&\quad\leqslant 2C_D\int_{I_m}\sum_{i\in i_{h,m}}\int_{\partial K_i\setminus\partial\Omega} h_{K_i}^{-1}(u-\Pi_m u)^2\,\mathrm{d}S\,\mathrm{d}t \\
&\qquad + C_D\int_{I_m}\sum_{i\in i_{h,m}}\int_{\partial K_i\cap\partial\Omega} h_{K_i}^{-1}(u-\Pi_m u)^2\,\mathrm{d}S\,\mathrm{d}t \\
&\quad\leqslant 2C_D\int_{I_m}\sum_{i\in i_{h,m}}\int_{\partial K_i} h_{K_i}^{-1}(u-\Pi_m u)^2\,\mathrm{d}S\,\mathrm{d}t \\
&\quad\leqslant 2C_MC_D\int_{I_m}\sum_{i\in i_{h,m}} h_{K_i}^{-1}\{\|u-\Pi_m u\|_{L^2(K_i)}|u-\Pi_m u|_{H^1(K_i)} \\
&\qquad\qquad + h_{K_i}^{-1}\|u-\Pi_m u\|^2_{L^2(K_i)}\}\,\mathrm{d}t \\
&\quad\leqslant 2C_MC_D\int_{I_m}\sum_{i\in i_{h,m}} h_{K_i}^{-1}C_\Pi^2\Big(h_{K_i}^{p+1}h_{K_i}^{p} + \frac{1}{h_{K_i}}h_{K_i}^{2p+2}\Big)|u|^2_{H^{p+1}(K_i)}\,\mathrm{d}t \\
&\quad\leqslant 2C_MC_\Pi^2C_Dh^{2p}|u|^2_{L^2(I_m;H^{p+1}(\Omega))}.
\end{aligned}
$$

Further, again the use of (3.41), Lemma 14, assumption (6.37) and a process similar to that in the proof of (9.6)–(9.8) and (9.3) yield

$$
\begin{aligned}
(9.11)\quad & \int_{I_m} J_{h,m}(\Pi_m u - P_m\Pi_m u, \Pi_m u - P_m\Pi_m u)\,\mathrm{d}t \\
&\leqslant 2C_D \int_{I_m} \sum_{i\in i_{h,m}} \frac{1}{h_{K_i}} \|\Pi_m u - P_m\Pi_m u\|^2_{L^2(\partial K_i)}\,\mathrm{d}t \\
&\leqslant 2C_D \int_{I_m} \sum_{i\in i_{h,m}} \frac{C_M}{h_{K_i}} \|\Pi_m u - P_m\Pi_m u\|_{L^2(K_i)} \Pi_m u - P_m\Pi_m u|_{H^1(K_i)}\,\mathrm{d}t \\
&\quad + 2C_D \int_{I_m} \sum_{i\in i_{h,m}} \frac{C_M}{h^2_{K_i}} \|\Pi_m u - P_m\Pi_m u\|^2_{L^2(K_i)}\,\mathrm{d}t \\
&\leqslant C_M C_D \sum_{i\in i_{h,m}} \frac{3}{h^2_{K_i}} \int_{K_i}\int_{I_m} (\Pi_m u - P_m\Pi_m u)^2\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + C_M C_D \sum_{i\in i_{h,m}} \int_{K_i}\int_{I_m} |\nabla(\Pi_m u - P_m\Pi_m u)|^2\,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant C_M C_P^2 C_D \sum_{i\in i_{h,m}} \Big\{ \frac{3\tau_m^{2q+2}}{h^2_{K_i}} \int_{K_i} |\Pi_m u|^2_{H^{q+1}(I_m)}\,\mathrm{d}x \\
&\quad + \tau_m^{2q+2} \int_{K_i} |\nabla \Pi_m u|^2_{H^{q+1}(I_m)}\,\mathrm{d}x \Big\} \\
&\leqslant C_M C_P^2 C_D \sum_{i\in i_{h,m}} \tau_m^{2q} \Big(\frac{3\tau_m^2}{h^2_{K_i}} |u|_{H^{q+1}(I_m;L^2(K_i))} + C_B^2\tau_m^2 |u|^2_{H^{q+1}(I_m;H^1(K_i))} \Big) \\
&\leqslant C_M C_P^2 C_D \tau_m^{2q} (3C_S^2 |u|^2_{H^{q+1}(I_m;L^2(\Omega))} + C_B^2\tau_m^2 |u|^2_{H^{q+1}(I_m;H^1(\Omega))}).
\end{aligned}
$$

Similarly, using the process from (9.4) and (9.8), we get

$$
\begin{aligned}
(9.12)\quad & \int_{I_m} J_{h,m}((\Pi_m u - P_m\Pi_m u)', (\Pi_m u - P_m\Pi_m u)')\,\mathrm{d}t \\
&\leqslant 2C_D \int_{I_m} \sum_{i\in i_{h,m}} \frac{1}{h_{K_i}} \|(\Pi_m u - P_m\Pi_m u)'\|^2_{L^2(\partial K_i)}\,\mathrm{d}t \\
&= C_M C_D \sum_{i\in i_{h,m}} \frac{3}{h^2_{K_i}} \int_{K_i}\int_{I_m} ((\Pi_m u - P_m\Pi_m u)')^2\,\mathrm{d}t\,\mathrm{d}x \\
&\quad + C_M C_D \sum_{i\in i_{h,m}} \int_{K_i}\int_{I_m} |\nabla(\Pi_m u - P_m\Pi_m u)'|^2\,\mathrm{d}t\,\mathrm{d}x \\
&\leqslant C_M C_P^2 C_D \sum_{i\in i_{h,m}} \Big\{ \frac{3\tau_m^{2q}}{h^2_{K_i}} \int_{K_i} |\Pi_m u|^2_{H^{q+1}(I_m)}\,\mathrm{d}x + \tau_m^{2q} \int_{K_i} |\nabla \Pi_m u|^2_{H^{q+1}(I_m)}\,\mathrm{d}x \Big\} \\
&\leqslant C_M C_P^2 C_D \tau^{2q-2} (3C_S^2 |u|^2_{H^{q+1}(I_m;L^2(\Omega))} + C_B^2\tau_m^2 |u|^2_{H^{q+1}(I_m;H^1(\Omega))}).
\end{aligned}
$$

Estimates (9.9)–(9.12) imply (6.41).

For the boundary $\boldsymbol{v}$-norm we have

$$\begin{aligned}
\int_{I_m} \sum_{i\in i_{h,m}} (\|\eta\|^2_{\boldsymbol{v},\partial K_i\cap\partial\Omega} + \|[\eta]\|^2_{\boldsymbol{v},\partial K_i^-\setminus\partial\Omega})\,\mathrm{d}t \\
&\leqslant 2\sum_{i\in i_{h,m}} \int_{I_m} \|\eta\|^2_{\boldsymbol{v},\partial K_i}\,\mathrm{d}t \\
&\leqslant 2C_{\boldsymbol{v}} \sum_{i\in i_{h,m}} \int_{I_m} \|\eta\|^2_{L^2(\partial K_i)}\,\mathrm{d}t \\
&\leqslant 6C_{\boldsymbol{v}} \sum_{i\in i_{h,m}} \int_{I_m} \|u-\Pi_m u\|^2_{L^2(\partial K_i)}\,\mathrm{d}t \\
&\quad + 6C_{\boldsymbol{v}} \sum_{i\in i_{h,m}} \int_{I_m} \|\Pi_m u - P_m\Pi_m u\|^2_{L^2(\partial K_i)}\,\mathrm{d}t \\
&\quad + 6C_{\boldsymbol{v}} \Big(\frac{C_q\tau_m}{2}\Big)^2 \sum_{i\in i_{h,m}} \int_{I_m} \|(\Pi_m u - P_m\Pi_m u)'\|^2_{L^2(\partial K_i)}\,\mathrm{d}t,
\end{aligned}$$

where

$$\sum_{i\in i_{h,m}} \int_{I_m} \|u-\Pi_m u\|^2_{L^2(\partial K_i)}\,\mathrm{d}t \leqslant C_M C_\Pi^2 h^{2p+1} |u|^2_{L^2(I_m;H^{p+1}(\Omega))}, \tag{9.13}$$

$$\begin{aligned}
\sum_{i\in i_{h,m}} \int_{I_m} \|\Pi_m u - P_m\Pi_m u\|^2_{L^2(\partial K_i)}\,\mathrm{d}t \\
\leqslant \frac{1}{2} C_M C_P^2 \tau_m^{2q+1} (3C_S |u|^2_{H^{q+1}(I_m;L^2(\Omega))} + C_B^2 h\tau_m |u|^2_{H^{q+1}(I_m;H^1(\Omega))}),
\end{aligned} \tag{9.14}$$

$$\begin{aligned}
\sum_{i\in i_{h,m}} \int_{I_m} \|(\Pi_m u - P_m\Pi_m u)'\|^2_{L^2(\partial K_i)}\,\mathrm{d}t \\
\leqslant \frac{1}{2} C_M C_P^2 \tau_m^{2q-1} (3C_S |u|^2_{H^{q+1}(I_m;L^2(\Omega))} + C_B^2 h\tau_m |u|^2_{H^{q+1}(I_m;H^1(\Omega))}).
\end{aligned} \tag{9.15}$$

Here, in (9.13) we proceeded in the same way as in (9.10), in (9.14) in the same way as in (9.11) and in (9.15) in the same way as in (9.12). These estimates imply (6.42).

For the $H^2(\Omega,\mathcal{T}_{h,m})$-norm we have

$$\begin{aligned}
\int_{I_m} |\eta|^2_{H^2(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \\
\leqslant 3\Big\{ \int_{I_m} |u-\Pi_m u|^2_{H^2(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t + \int_{I_m} |\Pi_m u - P_m\Pi_m u|^2_{H^2(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \\
+ \Big(\frac{C_q\tau_m}{2}\Big)^2 \int_{I_m} |(\Pi_m u - P_m\Pi_m u)'|^2_{H^2(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \Big\},
\end{aligned} \tag{9.16}$$

where

$$\int_{I_m} |u - \Pi_m u|^2_{H^2(\Omega,\mathcal{T}_{h,m})} \leqslant C_\Pi^2 h^{2(p-1)} |u|^2_{L^2(I_m;H^{p+1}(\Omega))}, \tag{9.17}$$

$$\begin{aligned}
&\int_{I_m} |\Pi_m u - P_m \Pi_m u|^2_{H^2(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \\
&\quad\leqslant C_P^2 \tau_m^{2q+2} \int_{I_m} \sum_{i\in i_{h,m}} \Big|\Pi_m\Big(\frac{\partial^{q+1}u}{\partial t^{q+1}}\Big)\Big|^2_{H^2(K_i)}\,\mathrm{d}t \\
&\quad\leqslant C_P^2 \tau_m^{2q+2} \int_{I_m} \sum_{i\in i_{h,m}} \frac{C_I^2}{h_{K_i}^2}\Big|\Pi_m\Big(\frac{\partial^{q+1}u}{\partial t^{q+1}}\Big)\Big|^2_{H^1(K_i)}\,\mathrm{d}t \\
&\quad\leqslant C_B^2 C_I^2 C_P^2 C_S^2 \tau_m^{2q} \int_{I_m} \sum_{i\in i_{h,m}} \Big|\frac{\partial^{q+1}u}{\partial t^{q+1}}\Big|^2_{H^1(K_i)}\,\mathrm{d}t \\
&\quad= C_B^2 C_I^2 C_P^2 C_S^2 \tau_m^{2q} |u|^2_{H^{q+1}(I_m;H^1(\Omega))},
\end{aligned} \tag{9.18}$$

$$\begin{aligned}
&\int_{I_m} |(\Pi_m u - P_m \Pi_m u)'|^2_{H^2(\Omega,\mathcal{T}_{h,m})}\,\mathrm{d}t \\
&\quad\leqslant C_B^2 C_I^2 C_P^2 C_S^2 \tau_m^{2q-2} |u|^2_{H^{q+1}(I_m;H^1(\Omega))}.
\end{aligned} \tag{9.19}$$

Here our argumentation is more brief, because the estimates are almost the same as in the case of the $H^1(\Omega,\mathcal{T}_{h,m})$-norm. The only difference is that in (9.18) and (9.19) we have used the inverse inequality (3.42) for the components of $\nabla\Pi_m\big(\frac{\partial^{q+1}u}{\partial t^{q+1}}\big)$. Estimates (9.16)–(9.19) yield (6.43).

For the last term in (6.34) we have

$$\begin{aligned}
\sum_{m=1}^{M-1} \|\eta_m^-\|^2_{L^2(\Omega)} &= \sum_{m=1}^{M-1} \|(u-\pi u)(t_m^-)\|^2_{L^2(\Omega)} \\
&= \sum_{m=1}^{M-1} \|(u-\Pi_m u)(t_m^-)\|^2_{L^2(\Omega)} \\
&\leqslant C_\Pi^2 \sum_{m=1}^{M-1} h_m^{2p+2} |u(t_m)|^2_{H^{p+1}(\Omega)} \\
&\leqslant C_\Pi^2 \hat{C}_S h^{2p+1} \sum_{m=1}^{M-1} \tau_m |u(t_m)|^2_{H^{p+1}(\Omega)} \\
&\leqslant C_\Pi^2 \hat{C}_S T h^{2p+1} \|u\|^2_{C([0,T];H^{p+1}(\Omega))},
\end{aligned}$$

which is (6.44). □

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