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REMARK ON STABILIZATION OF TREE-SHAPED NETWORKS OF STRINGS

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Abstract. We consider a tree-shaped network of vibrating elastic strings, with feedback acting on the root of the tree. Using the d'Alembert representation formula, we show that the input-output map is bounded, i.e. this system is a well-posed system in the sense of G. Weiss (Trans. Am. Math. Soc. *342* (1994), 827–854). As a consequence we prove that the strings networks are not exponentially stable in the energy space. Moreover, we give explicit polynomial decay estimates valid for regular initial data.

Keywords: networks of strings, input-output map, well-posed system

MSC 2000: 35B37, 93B07, 93D15

1. Introduction

During the last years various models of multiple-link flexible structures have been given and developed. The structures which we have in mind consist of finitely many interconnected flexible elements like beams, plates, shells which represent trusses, frames, solar panels, antennae deformable mirrors; for more details concerning the models see [9]. The analysis of such models has in addition to its own mathematical interest applications control or stabilization problems, see [7], [8], [9] and [14].

First of all, we introduce some notation, which is simply that of [5], and refer to [5] for more details that are needed to formulate the problem under consideration. Let $\mathcal{A}$ be a tree. We call the root of $\mathcal{A}$ the exterior vertex and we denote it by $\mathcal{R}$. Moreover, we denote by $e_{\overline{\alpha}}$ and $\mathcal{O}_{\overline{\alpha}}$ the remaining edges and vertices, respectively, where $\overline{\alpha} = (\alpha_1, \ldots, \alpha_k)$ is a multi-index, possibly empty, of variable length k.

We choose the empty index for the edge containing the root $\mathcal{R}$. That edge is denoted by e and its vertex different from $\mathcal{R}$ is denoted by $\mathcal{O}$.

Assume there are $m_{\overline{\alpha}}$ edges, different from $e_{\overline{\alpha}}$, that branch out from $\mathcal{O}_{\overline{\alpha}}$. We denote these edges by $e_{\overline{\alpha}\circ\beta}$, $\beta = 1,\ldots,m_{\overline{\alpha}}$ and the other vertex of the edge $e_{\overline{\alpha}\circ\beta}$ by $\mathcal{O}_{\overline{\alpha}\circ\beta}$, i.e. the interior vertex $\mathcal{O}$, contained in the edge $e_{\overline{\alpha}}$, has multiplicity equal to $m_{\overline{\alpha}}+1$.

We denote by $\mathcal{M}$ the set of the interior vertices of $\mathcal{A}$ and by $\mathcal{S}$ the set of the exterior vertices except $\mathcal{R}$ and define

$$I_{\mathcal{M}} = \{\overline{\alpha}\colon\ \mathcal{O}_{\overline{\alpha}} \in \mathcal{M}\},\quad I_{\mathcal{S}} = \{\overline{\alpha}\colon\ \mathcal{O}_{\overline{\alpha}} \in \mathcal{S}\},$$

which are the sets of the indices of the interior and exterior vertices except $\mathcal{R}$, respectively.

We admit the empty multi-index in this notation, which corresponds to the vertex $\mathcal{O}$ and belongs to one of the sets $I_{\mathcal{M}}$ or $I_{\mathcal{S}}$. We denote also by $I = I_{\mathcal{M}} \cup I_{\mathcal{S}}$ the set of indices of all the vertices, except that of the root $\mathcal{R}$. We call the sets

$$\mathcal{A}_{\overline{\alpha}} = \{e_{\overline{\alpha}\circ\overline{\beta}}\colon\ \overline{\alpha}\circ\overline{\beta} \in I\}$$

for $\overline{\alpha} \in I_{\mathcal{M}}$, sub-trees of $\mathcal{A}$, where $\overline{\alpha}\circ\overline{\beta}$ denotes the multi-index of length $k+m$ defined by $\overline{\alpha}\circ\overline{\beta} = (\alpha_1,\ldots,\alpha_k,\beta_1,\ldots,\beta_m)$, with $\overline{\alpha} = (\alpha_1,\ldots,\alpha_k)$, $\overline{\beta} = (\beta_1,\ldots,\beta_m)$.

We study the one-node stabilization properties of the vibrations of a planar tree-shaped network of N strings, where $N \geqslant 3$, see [9] and [5] as concerns the model. That is, we analyze the possibility of quieting the motion of the tree shaped network, caused by a feedback deformation of its strings, by means of feedback applied through the nodes. More precisely, we consider the following initial and boundary value problem:

$$\frac{\partial^2 u_{\overline{\alpha}}}{\partial t^2}(x,t) - \frac{\partial^2 u_{\overline{\alpha}}}{\partial x^2}(x,t) = 0,\quad 0 < x < l_{\overline{\alpha}},\ t > 0,\ \overline{\alpha} \in I, \tag{1.1}$$

$$\frac{\partial u}{\partial t}(0,t) - \frac{\partial u}{\partial x}(0,t) = 0,\ t > 0;\quad u_{\overline{\alpha}}(l_{\overline{\alpha}},t) = 0,\ \overline{\alpha} \in I_{\mathcal{S}},\ t > 0, \tag{1.2}$$

$$u_{\overline{\alpha}\circ\beta}(0,t) = u_{\overline{\alpha}}(l_{\overline{\alpha}},t),\quad t > 0,\ \beta = 1,\ldots,m_{\overline{\alpha}},\ \overline{\alpha} \in I_{\mathcal{M}}, \tag{1.3}$$

$$\sum_{\beta=1}^{m_{\overline{\alpha}}} \frac{\partial u_{\overline{\alpha}\circ\beta}}{\partial x}(0,t) = \frac{\partial u_{\overline{\alpha}}}{\partial x}(l_{\overline{\alpha}},t),\quad t > 0,\ \overline{\alpha} \in I_{\mathcal{M}}, \tag{1.4}$$

$$u_{\overline{\alpha}}(x,0) = u^0_{\overline{\alpha}}(x),\ \frac{\partial u_{\overline{\alpha}}}{\partial t}(x,0) = u^1_{\overline{\alpha}}(x),\quad 0 < x < l_{\overline{\alpha}},\ \overline{\alpha} \in I, \tag{1.5}$$

where $u_{\overline{\alpha}}\colon\ [0,l_{\overline{\alpha}}] \times (0,+\infty) \to \mathbb{R}$, $\overline{\alpha} \in I$, is the transversal displacement of the string with index $\overline{\alpha}$ and of length $l_{\overline{\alpha}}$. With this notation the remaining elements related to the system (1.1)–(1.5) are defined exactly in subsection 2.2.2 of chapter 2 in [6] or in [4].

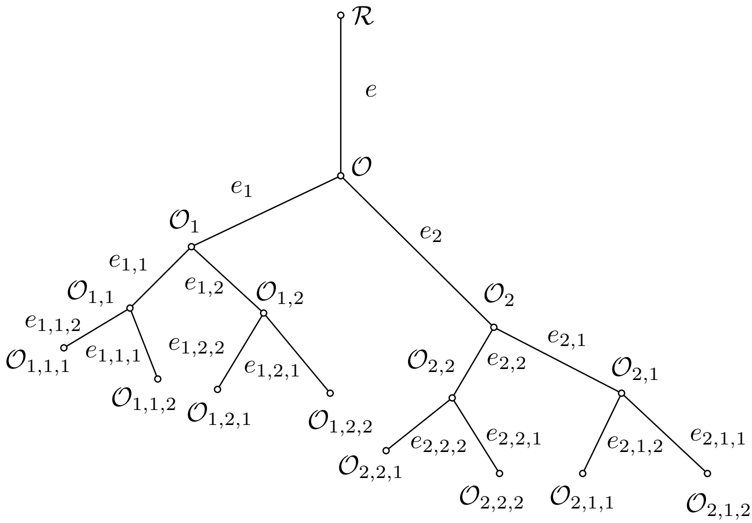

Figure 1. A tree-shaped network.

In the present paper we prove that the input-output map is bounded. As a consequence we show that the solutions of (1.1)–(1.5) are not uniformly stable in the energy space, thus, we give explicit decay estimates for regular initial data.

Our approach is based on the methodology introduced in Ammari and Tucsnak [3], where the exponential stability for the closed loop problem is reduced to an observability estimate for the corresponding uncontrolled system combined with a boundedness property of the transfer function of the associated open loop system, and on the d'Alembert representation formula.

The plan of the paper is as follows. In Section 2 we give precise statements of the main results. Section 3 contains the proof of the main result and the weighted observability inequality needed in the following sections. The proof of decay estimates results is given in Section 4. The last section is devoted to some comments and related questions.

2. Statement of the main results

The skew-adjoint operator corresponding to (3.2)–(3.7) can be diagonalized over the orthonormal basis of eigenvectors $\bar{\Phi}_n$, $n \in \mathbb{N}$. Let $\lambda_k = i w_k$, $k \in \mathbb{N}$ be the associated eigenvalues.

Let $\mathcal{A}_{\overline{\alpha}}$ be a sub-tree. We consider the eigenvalue problem inherited from the eigenvalue problem for the whole tree $\mathcal{A}$ with homogeneous Neumann boundary condition at the new root $\mathcal{O}_{\overline{\alpha}}$. This eigenvalue problem is similar to that for $\mathcal{A}$. Its spectrum will be called the spectrum of the $\mathcal{A}_{\overline{\alpha}}$ subtree.

The tree $\mathcal{A}$ is called a non-degenerate tree if the spectra of any two sub-trees $\mathcal{A}_{\bar{\alpha}\circ i}$, $\mathcal{A}_{\bar{\alpha}\circ j}$ of $\mathcal{A}$ with the common root $\mathcal{O}_{\bar{\alpha}}$ are disjoint.

We define the energy of $\bar{u} = u_{\bar{\alpha}}$ of (1.1)–(1.5) at an instant t by

$$E(t) = \sum_{\bar{\alpha}\in I} \frac{1}{2} \int_0^{l_{\bar{\alpha}}} \left(\left|\frac{\partial u_{\bar{\alpha}}}{\partial t}(x,t)\right|^2 + \left|\frac{\partial u_{\bar{\alpha}}}{\partial x}(x,t)\right|^2 \right) \mathrm{d}x. \tag{2.1}$$

We show that a sufficiently smooth solution (1.1)–(1.5) satisfies the energy identity

$$E(0) - E(t) = \int_0^t \left|\frac{\partial u}{\partial t}(0,s)\right|^2 \mathrm{d}s, \quad \forall\, t \geqslant 0. \tag{2.2}$$

The wellposedness space for (1.1)–(1.5) is $E = V \times X$, where

$$V = \Big\{ \bar{\varphi} \in \prod_{\bar{\alpha}\in I} H^1(0,l_{\bar{\alpha}})\colon\ \varphi_{\bar{\alpha}}(l_{\bar{\alpha}}) = 0,\ \ \bar{\alpha} \in I_{\mathcal{S}},$$
$$\varphi_{\bar{\alpha}\circ\beta}(0) = \varphi_{\bar{\alpha}}(l_{\bar{\alpha}}),\ \ \beta = 1,\dots,m_{\bar{\alpha}},\ \ \bar{\alpha} \in I_{\mathcal{M}} \Big\},$$

and

$$X = \prod_{\bar{\alpha}\in I} L^2(0,l_{\bar{\alpha}}).$$

The wellposedness and strong stability properties are summarized in the result below. The existence and uniqueness of finite energy solutions of (1.1)– (1.5) can be obtained by standard semigroup methods. For the strong stability see [5].

Proposition 2.1. *The following assertions hold true:*

1. *If* $(\bar{u}^0, \bar{u}^1) \in V \times X$, *then the problem* (1.1)–(1.5) *admits a unique solution*

$$\bar{u} = u_{\bar{\alpha}} \in C(0,T;V) \cap C^1(0,T;X)$$

such that $u(0,\cdot) \in H^1(0,T)$ *and*

$$\|u(0,\cdot)\|^2_{H^1(0,T)} \leqslant C\|(\bar{u}^0,\bar{u}^1)\|^2_{V\times X}, \tag{2.3}$$

where the constant $C > 0$ *depends only on* T. *Moreover,* $\bar{u}$ *satisfies the energy identity* (2.2).

2. *The estimate* $\lim_{t\to\infty} E(t) = 0$ *holds true for any finite energy solution of* (1.1)–(1.5) *if and only if the tree* $\mathcal{A}$ *is non-degenerate.*

We consider the open-loop problem associated with (1.1)–(1.5)

$$\frac{\partial^2 v_{\bar\alpha}}{\partial t^2}(x,t) - \frac{\partial^2 v_{\bar\alpha}}{\partial x^2}(x,t) = 0, \quad 0 < x < l_{\bar\alpha},\ t > 0,\ \bar\alpha \in I, \tag{2.4}$$

$$v_{\bar\alpha}(l_{\bar\alpha}, t) = 0, \quad t > 0,\ \bar\alpha \in I_{\mathcal{S}}, \tag{2.5}$$

$$v_{\bar\alpha\circ\beta}(0,t) = v_{\bar\alpha}(l_{\bar\alpha}, t), \quad t > 0,\ \beta = 1, \dots, m_k,\ \bar\alpha \in I_{\mathcal{M}}, \tag{2.6}$$

$$\sum_{\beta=1}^{m_{\bar\alpha}} \frac{\partial v_{\bar\alpha\circ\beta}}{\partial x}(0,t) = \frac{\partial v_{\bar\alpha}}{\partial x}(l_{\bar\alpha}, t), \quad \forall\, t > 0,\ \bar\alpha \in I_{\mathcal{M}}, \tag{2.7}$$

$$\frac{\partial v}{\partial x}(0,t) = k(t), \quad t > 0, \tag{2.8}$$

$$v_{\bar\alpha}(x,0) = 0,\ \frac{\partial v_{\bar\alpha}}{\partial t}(x,0) = 0, \quad 0 < x < l_{\bar\alpha},\ \bar\alpha \in I. \tag{2.9}$$

By using the transposition method, see [11], and the observation that the control operator satisfies an admissible condition, i.e. (3.9), we prove that for $k \in L^2(0,T)$ the problem (2.4)–(2.9) admits a unique solution having the regularity property

$$\bar v = v_{\bar\alpha} \in C(0,T;V) \cap C^1(0,T;X). \tag{2.10}$$

Our main result can now be stated as follows.

Theorem 2.2. *There exists a constant $C > 0$ depending only on T such that for $k \in L^2(0,T)$ and for a solution $\bar v \in C(0,T;V) \cap C^1(0,T;X)$ of (2.4)–(2.9) we have*

$$\|v(0,\cdot)\|_{H^1(0,T)} \leqslant C\|k\|_{L^2(0,T)}, \quad \forall\, k \in L^2(0,T). \tag{2.11}$$

Corollary 2.3.

1. *The system described by (1.1)–(1.5) is not exponentially stable in the energy space.*
2. *Let $\mathcal{A}$ be a non-degenerate tree. Then for all $t \geqslant 0$ we have*

$$E(t) \leqslant \frac{C}{t+1}\|(u^0, u^1)\|_Z^2, \quad \forall\, (\bar u^0, \bar u^1) \in Z, \tag{2.12}$$

where

$$Z = \left\{\left(\bar u^0 = \sum_{k\geqslant 0} a_k \bar\Phi_k,\ \bar u^1 = \sum_{k\geqslant 0} b_k \bar\Phi_k\right) \in V \times X\colon \sum_{k\geqslant 0} \frac{1}{d_k^2}(|w_k a_k|^2 + |b_k|^2) < \infty\right\},$$

$$\|(\bar u^0, \bar u^1)\|_Z^2 = \sum_{k\geqslant 0} \frac{1}{d_k^2}(|w_k a_k|^2 + |b_k|^2), \quad \forall\, (\bar u^0, \bar u^1) \in Z,$$

d_k is a sequence of strictly positive real numbers and $C > 0$ is a constant depending only on $l_{\bar\alpha}$, $\bar\alpha \in I$.

Remark 2.4.

1. The tree where $I_{\mathcal{M}} = \emptyset$, $I_{\mathcal{S}} = \{1, \ldots, N\}$, $m_{\bar\alpha} = N$, is non-degenerate if and only if $l_i/l_j \notin \mathbb{Q}$, for all $1 \leqslant i \neq j \leqslant N$, see [2], where $\mathbb{Q}$ denotes the set of all rational numbers.
2. Concerning the size of the set of non-degenerate trees, see [5].
3. In the case we are able to establish uniform lower estimates of d_n, for all n, we obtain an explicit characterization of the space Z. So, in the case when $\mathcal{A}$ is a simple tree (see [2] for details) corresponding to $I_{\mathcal{M}} = \emptyset$, $I_{\mathcal{S}} = \{1, \ldots, N\}$, $m_{\bar\alpha} = N$, we have $Z = \mathcal{D}(A^{(N+1)/2})$ if $l_i/l_j \notin \mathbb{Q}$, for all $1 \leqslant i \neq j \leqslant N$ and $l_i/\sum\limits_{i=1}^{N} l_i \in \mathcal{S}$,[1] for all $i = 1, \ldots, N$, where

$$\mathcal{D}(A) = \Big\{ (u, u_1, \ldots, u_N, v, v_1, \ldots, v_N) \in \Big[V \cap \Big(H^2(0,l) \times \prod_{i=1}^{N} H^1(0, l_i) \Big) \Big] \times V : \\ \frac{\mathrm{d}u}{\mathrm{d}x}(l) = \sum_{i=1}^{N} \frac{\mathrm{d}u_i}{\mathrm{d}x}(0),\ \frac{\mathrm{d}u}{\mathrm{d}x}(0) = v(0) \Big\}.$$

The corresponding operator A is defined by

$$A \begin{pmatrix} \bar u \\ \bar v \end{pmatrix} = \begin{pmatrix} \bar v \\ \mathrm{d}^2 \bar u / \mathrm{d}x^2 \end{pmatrix}, \quad \forall\, (\bar u, \bar v) \in \mathcal{D}(A).$$

Also in the case of the star-shaped network we can characterize the space Z. This has been done in [1] (see [1] for more details). However, it is unlikely to expect similar results in the case of general trees.

[1] $\mathcal{S}$ is the set of all numbers ϱ such that $\varrho \notin \mathbb{Q}$ and if $[0, a_1, \ldots, a_n, \ldots]$ is the expansion of ϱ as a continued fraction, then (a_n) is bounded. Let us notice that $\mathcal{S}$ is obviously uncountable and, by classical results on diophantine approximations, its Lebesgue measure is equal to zero. Roughly speaking the set $\mathcal{S}$ contains the irrationals which are "badly" approximable by rational numbers.

3. Proof of Theorem 2.2 and observability inequality

Proof of Theorem 2.2. We consider the following equation:

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad t > 0, \ x \in (0,l), \\ u(0,t) = h(t), \ u(l,t) = 0, \quad t > 0, \\ u(x,0) = 0, \quad \dfrac{\partial u}{\partial t}(x,0) = 0, \quad x \in (0,l), \end{cases}$$

where h is a regular function which vanishes in $(-\infty, 0)$. Our purpose is to determine an operator R_l such that

$$\frac{\partial u}{\partial x}(0,t) = R_l h'(t).$$

By the d'Alembert formula there exist F, G such that

$$u(x,t) = F(x+t) + G(x-t).$$

The equation $\partial u/\partial x(0,t) = R_l h'(t)$ can be written as $F'(t) + G'(-t) = R_l h'(t)$,

$$\begin{cases} u(x,0) = 0, \ 0 < x < l \\ \dfrac{\partial u}{\partial t}(x,0) = 0, \ 0 < x < l \end{cases} \Longrightarrow \begin{cases} F(x) + G(x) = 0, \ 0 < x < l \\ F'(x) - G'(x) = 0, \ 0 < x < l, \end{cases}$$

which implies

$$F'(t) = G'(t) = 0, \quad \forall\, 0 < t < l.$$

On the other hand,

$$u(0,t) = h(t) \Longrightarrow F'(t) - G'(-t) = h'(t) \quad \forall\, t > 0$$

and

$$u(l,t) = 0 \Longrightarrow F'(2l+t) - G'(-t) = 0 \quad \forall\, t > 0.$$

Thus

$$F'(t) = F'(t-2l) - h'(t-2l),$$

so

$$\frac{\partial u}{\partial x}(0,t) = 2F'(t) - h'(t), \quad \forall\, t > 0.$$

An explicit computation proves that R_l is given by the formula

$$R_l f(t) = -2\sum_{i=1}^{q} f(t-2il) - f(t), \quad \forall\, q \in \mathbb{N} \ \text{and} \ \forall\, t \in (2ql, 2(q+1)l)$$

with the convention $\sum\limits_{i=1}^{0} = 0$.

We consider now the problem

$$\begin{cases} \dfrac{\partial^2 u}{\partial t^2}(t,x) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad t>0,\ \ x \in (0,l), \\ u(l,t) = h(t),\ \ \dfrac{\partial u}{\partial x}(0,t) = k(t),\ \ t>0, \\ u(x,0) = 0,\ \ \dfrac{\partial u}{\partial t}(x,0) = 0,\ \ x \in (0,l), \end{cases}$$

and the same computation yields

$$\frac{\partial u}{\partial t}(0,t) = Q_l k(t) + M_l h'(t) \quad \forall\, t>0$$

and

$$\frac{\partial u}{\partial x}(l,t) = M_l k(t) - Q_l h'(t) \quad \forall\, t>0,$$

where

$$Q_l f(t) = 2\sum_{i=1}^{q} (-1)^{i+1} f(t-2il) - f(t), \quad \forall\, q \in \mathbb{N} \text{ and } \forall\, t \in (2ql, 2(q+1)l)$$

and

$$M_l f(t) = 2\sum_{i=1}^{q} (-1)^{i+1} f(t-(2i-1)l), \quad \forall\, q \in \mathbb{N} \text{ and } \forall\, t \in ((2q-1)l, (2q+1)l).$$

$\square$

Operators of type $\mathcal{T}$

Definition 3.1. An operator P is of type $\mathcal{T}$ if there exists a sequence $m = (m_k)_{k\in\mathbb{N}}$ strictly increasing and such that $m_0 = 0$, $\lim\limits_{k\to+\infty} m_k = +\infty$, and a complex sequence $\alpha = (\alpha_k)_{k\in\mathbb{N}}$ such that

$$Pf(t) = \sum_{i=0}^{q} \alpha_i f(t-m_i), \quad \text{a.e. } t \in (m_q, m_{q+1}) \text{ and } \forall\, q \in \mathbb{N}.$$

The sequences (m,α) are the parameters of P and we write $P \equiv (m,\alpha)$. The number α_0 is called the coefficient of P and we denote $\operatorname{coef}(P) = \alpha_0$.

R e m a r k 3.2. If $P \equiv (m, \alpha)$ is of type $\mathcal{A}$ and if $a \in \mathbb{R}_+ \setminus m$, then there exists $i_0 \in \mathbb{N}$ such that $m_{i_0} < a < m_{i_0+1}$. We denote by $(\tilde{m}, \tilde{\alpha})$ the sequence defined by

$$\begin{cases} \tilde{m}_i = m_i, \ \tilde{\alpha}_i = \alpha_i \ \text{ if } \ i \leqslant i_0, \\ \tilde{m}_{i_0+1} = a, \ \tilde{\alpha}_{i_0+1} = 0, \\ \tilde{m}_i = m_{i-1}, \ \tilde{\alpha}_i = \alpha_{i-1} \ \text{ if } \ i > i_0 + 1. \end{cases}$$

Then $P \equiv (\tilde{m}, \tilde{\alpha})$.

E x a m p l e s. The operators R_l, Q_l and M_l are of type $\mathcal{T}$ with $\operatorname{coef}(R_l) = \operatorname{coef}(Q_l) = -1$ and $\operatorname{coef}(M_l) = 0$.

Proposition 3.3. *If P is an operator of type $\mathcal{T}$ and $T > 0$, then there exists a constant $C = C(P, T) > 0$ such that for all $f \in L^2(0, T)$*

$$\|Pf\|_{L^2(0,T)} \leqslant C\|f\|_{L^2(0,T)}.$$

P r o o f. Let $P \equiv (m, \alpha)$ be an operator of type $\mathcal{A}$ and let $T > 0$. Let $q \in \mathbb{N}$ be such that $m_q < T \leqslant m_{q+1}$. Then

$$\begin{aligned} \int_0^T |Pf(t)|^2 \, \mathrm{d}t &\leqslant \sum_{k=0}^{q} \int_{m_k}^{m_{k+1}} |Pf(t)|^2 \, \mathrm{d}t = \sum_{k=0}^{q} \int_{m_k}^{m_{k+1}} \Big| \sum_{j=0}^{k} \alpha_j f(t - m_j) \Big|^2 \, \mathrm{d}t \\ &\leqslant \sum_{k=0}^{q} \int_{m_k}^{m_{k+1}} \Big(\sum_{j=0}^{k} |\alpha_j|^2 \Big) \Big(\sum_{j=0}^{k} |f(t - m_j)|^2 \Big) \, \mathrm{d}t \\ &= \sum_{k=0}^{q} \Big(\sum_{j=0}^{k} |\alpha_j|^2 \Big) \sum_{j=0}^{k} \int_{m_k}^{m_{k+1}} |f(t - m_j)|^2 \, \mathrm{d}t \\ &= \sum_{k=0}^{q} \Big(\sum_{j=0}^{k} |\alpha_j|^2 \Big) \sum_{j=0}^{k} \int_{m_k - m_j}^{m_{k+1} - m_j} |f(t)|^2 \, \mathrm{d}t \\ &\leqslant \sum_{k=0}^{q} \Big(\sum_{j=0}^{k} |\alpha_j|^2 \Big) \sum_{j=0}^{k} \int_0^T |f(t)|^2 \, \mathrm{d}t \\ &= \Big(\sum_{k=0}^{q} (k+1) \sum_{j=0}^{k} |\alpha_j|^2 \Big) \|f\|^2_{L^2(0,T)}. \end{aligned}$$

□

Proposition 3.4. *Let $P \equiv (m, \alpha)$ be an operator of type $\mathcal{T}$ and let g be a function defined a.e. in $[0, +\infty)$. Then the equation*

$$Pf(t) = g(t), \quad \text{a.e. } t \geqslant 0$$

admits a unique solution f if and only if $\operatorname{coef}(P) \neq 0$. *In this case $f = P^{-1}g$, P^{-1} is of type $\mathcal{T}$ and* $\operatorname{coef}(P^{-1}) = 1/\operatorname{coef}(P)$.

Proof. We construct by recurrence sequences (r, β) satisfying

$$f(t) = \sum_{i=0}^{k} \beta_i g(t - r_i) \quad \text{a.e. } t \in (r_q, r_{q+1}) \text{ and } \forall\, q \in \mathbb{N} \tag{3.1}$$

as follows:

$$\begin{cases} r_0 = 0 \text{ and } \beta_0 = \dfrac{1}{\alpha_0} \\ \text{for } n \in \mathbb{N}^*,\ r_n = \min\{m_i + r_j > r_{n-1} \colon j < n\},\ B_n = \{(i,j) \colon m_i + r_j = r_n\} \\ \text{and } \beta_n = -\dfrac{1}{\alpha_0} \sum\limits_{(i,j) \in B_n} \alpha_i \beta_j. \end{cases}$$

It is easy now to verify that f given by (3.1) is a solution of the equation $Pf = g$.

To prove uniqueness, it suffices to consider the sequence $r = (r_n)_{n \in \mathbb{N}}$ constructed above to show that the unique solution of the equation $Pf = 0$ is $f \equiv 0$ a.e. More precisely, for the solution f of $Pf = 0$ ($\operatorname{coef} P = \alpha_0 \neq 0$) we prove by recurrence in q that $f(t) = 0$, a.e. $t \in (0, m_q)$ for all $q \in \mathbb{N}^*$.

For $q = 1$ the result is true because

$$Pf(t) = \alpha_0 f(t), \quad \text{a.e. } t \in (0, m_1).$$

Suppose now that $f(t) = 0$ for a.e. $t \in (0, m_q)$ and let us prove that $f(t) = 0$ a.e. $t \in (m_q, m_{q+1})$.

Let

$$F = \Big\{ \sum k_i m_i \colon m_q < \sum k_i m_i < m_{q+1},\ k_i \in \mathbb{N} \Big\},$$

then F is a finite set of cardinality k_q. We denote by (r_k) the elements of F such that $r_0 = m_q < r_1 < r_2 < \ldots < r_{k_q} < m_{q+1} =: r_{k_q+1}$. We have

$$\forall\, r_j < t < r_{j+1},\ 0 \leqslant j \leqslant k_q,\ \ Pf(t) = \alpha_0 f(t) + \alpha_1 f(t - m_1) + \ldots + \alpha_q f(t - m_q) = 0,$$

and

$$\forall\, r_0 < t < r_1, \quad Pf(t) = \alpha_0 f(t) + \alpha_1 f(t - m_1) + \ldots + \alpha_q f(t - m_q) = 0.$$

Or,

$$\forall\, 1 \leqslant i \leqslant q, \quad r_0 - m - i < t - m_i < r_1 - m_i,$$

and $r_1 - m_i \leqslant m_q$ because if no $r_1 - m_i > m_q$, then $m_q < m_i + m_q < r_1$ which contradicts the construction of the sequence $(r_k)_{1 \leqslant k \leqslant k_q}$. Thus according to the hypotheses of recurrence

$$f(t - m_i) = 0, \ \forall\, 1 \leqslant i \leqslant q \Rightarrow f(t) = 0, \ \forall\, r_0 < t < r_1.$$

Suppose that $f(t) = 0$ a.e. $r_0 < t < r_j$, $1 \leqslant j \leqslant k_q$.

Let $r_j < t < r_{j+1}$, then for all $1 \leqslant i \leqslant q$, $t - m_i < r_{j+1} - m_i \leqslant r_j$ because if no $r_{j+1} - m_i > 0$, then $r_j < r_j + m_i < r_{j+1}$, which contradicts the construction of the sequence $(r_k)_{1 \leqslant k \leqslant k_q}$. Then

$$f(t - m_i) = 0, \quad \forall\, 1 \leqslant i \leqslant q \Rightarrow f(t) = 0 \text{ a.e. } r_j < t < r_{j+1}.$$

$\square$

E x a m p l e. We consider the operator $P = -R_l$ of parameters (m, α) given by

$$\begin{cases} m_k = 2kl & \forall\, k \in \mathbb{N}, \\ \alpha_0 = 1 \text{ and } \alpha_k = 2 & \forall\, k \geqslant 1. \end{cases}$$

Then the solution of the equation $Pf = g$ is given by

$$f(t) = g(t) + \sum_{j=1}^{q} (-1)^j g(t - 2jl) \quad \forall\, 2ql < t < 2(q+1)l \text{ and } \forall\, q \in \mathbb{N},$$

i.e. $f = -Q_l g$, which proves that $R_l^{-1} = Q_l$.

Proposition 3.5. *If P and Q are two operators of type $\mathcal{T}$, then $P + Q$ and PQ are of type $\mathcal{T}$ and*

$$\operatorname{coef}(P + Q) = \operatorname{coef}(P) + \operatorname{coef}(Q) \quad \textit{and} \quad \operatorname{coef}(PQ) = \operatorname{coef}(P) \cdot \operatorname{coef}(Q).$$

P r o o f. We suppose that $P \equiv (m, \alpha)$ and $Q \equiv (n, \beta)$.

1. For the sum it suffices to consider the sequence $r = m \cup n$, then $P \equiv (r, \tilde{\alpha})$ and $Q \equiv (r, \tilde{\beta})$ where

$$\tilde{\alpha}_i = \begin{cases} \alpha_j & \text{if } r_i = m_j \in m, \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad \tilde{\beta}_i = \begin{cases} \beta_j & \text{if } r_i = n_j \in n, \\ 0 & \text{otherwise,} \end{cases}$$

which implies $P + Q \equiv (r, \tilde{\alpha} + \tilde{\beta})$.

2. For the composition, we consider sequences defined by recurrence as follows:

$$\begin{cases} r_0 = 0 \ \text{ and } \ \gamma_0 = \alpha_0\beta_0, \\ \text{for } \ n \in \mathbb{N}^*, \ \ r_n = \min\{m_i + n_j\colon \ m_i + n_j > r_{n-1}, \ \ 0 \leqslant i, j < n\}, \\ \quad B_n = \{(i,j)\colon \ m_i + n_j = r_n\}, \\ \text{and } \ \gamma_n = \sum\limits_{(i,j)\in B_n} \alpha_i\beta_j; \end{cases}$$

then $PQ \equiv (r, \gamma)$. □

Case of a generic tree (i.e. a simple tree)

We consider the problem

$$\begin{cases} \dfrac{\partial^2 u_i}{\partial t^2}(x,t) - \dfrac{\partial^2 u_i}{\partial x^2}(x,t) = 0, \quad \forall\, t > 0, \ \ 0 < x < l_i, \ \ 1 \leqslant i \leqslant n, \\ \dfrac{\partial^2 u}{\partial t^2}(x,t) - \dfrac{\partial^2 u}{\partial x^2}(x,t) = 0, \quad \forall\, t > 0, \ \ 0 < x < l, \\ \sum\limits_{i=1}^{n} \dfrac{\partial u_i}{\partial x}(0,t) = \dfrac{\partial u}{\partial x}(l,t), \quad \forall\, t > 0, \\ u_i(l_i,t) = 0, \quad \forall\, t > 0, \ \ 1 \leqslant i \leqslant n, \\ u_i(0,t) = u(l,t), \quad 1 \leqslant i \leqslant n, \\ u_i(x,0) = 0, \ \ \dfrac{\partial u_i}{\partial t}(x,0) = 0, \ \ \forall\, x \in (0,l_i), \ \ 1 \leqslant i \leqslant n, \\ u(x,0) = 0, \ \ \dfrac{\partial u}{\partial t}(x,0) = 0, \ \ \forall\, x \in (0,l). \end{cases}$$

Proposition 3.6. *There exists an invertible operator $U\colon\ L^2(0,T) \to L^2(0,T)$ of type $\mathcal{T}$ satisfying*

$$\frac{\partial u}{\partial t}(0,t) = U\frac{\partial u}{\partial x}(0,t).$$

P r o o f. We denote $h(t) = u(t,l) = u_i(t,0)$ $(1 \leqslant i \leqslant n)$ and $k(t) = \partial u/\partial x(0,t)$. Then we have

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,0) = Q_l k(t) + M_l h'(t), \\ \dfrac{\partial u}{\partial x}(t,l) = M_l k(t) - Q_l h'(t), \end{cases}$$

which implies that for all $1 \leqslant i \leqslant n$, $\partial u_i/\partial x(0,t) = R_{l_i} h'(t)$ and the condition $\sum\limits_{i=1}^{n} \partial u_i/\partial x(0,t) = \partial u/\partial x(l,t)$ can be rewritten as

$$\Big(\sum_{i=1}^{n} R_{l_i}\Big) h'(t) = M_l k(t) - Q_l h'(t) \Longleftrightarrow \Big(\sum_{i=1}^{n} R_{l_i} + Q_l\Big) h'(t) = M_l k(t).$$

The operator $\left(\sum_{i=1}^{n} R_{l_i} + Q_l\right)$ is of type $\mathcal{T}$ and with the coefficient equal to $-(n+1)$, thus it is invertible and

$$h'(t) = \left(\sum_{i=1}^{n} R_{l_i} + Q_l\right)^{-1} M_l k(t),$$

hence

$$\frac{\partial u}{\partial t}(t,0) = \left[Q_l + M_l \left(\sum_{i=1}^{n} R_{l_i} + Q_l\right)^{-1} M_l\right] k(t) = Uk(t).$$

The operator U is of type $\mathcal{T}$ and $\operatorname{coef}(U) = -1$, thus U is invertible. □

Case of a general tree

Suppose we construct $U_{\bar{\alpha}\circ i}$ $(1 \leqslant i \leqslant n_{\bar{\alpha}})$ satisfying $\operatorname{coef}(U_{\bar{\alpha}\circ i}) = -1$ and

$$\frac{\partial u_{\bar{\alpha}\circ i}}{\partial t}(0,t) = U_{\bar{\alpha}\circ i} \frac{\partial u_{\bar{\alpha}\circ i}}{\partial x}(0,t);$$

then

$$\begin{cases} \dfrac{\partial u_{\bar{\alpha}}}{\partial t}(0,t) = Q_{l_{\bar{\alpha}}} \dfrac{\partial u_{\bar{\alpha}}}{\partial x}(0,t) + M_{l_{\bar{\alpha}}} \dfrac{\partial u_{\bar{\alpha}}}{\partial t}(l_{\bar{\alpha}},t), \\ \dfrac{\partial u_{\bar{\alpha}}}{\partial x}(t,l_{\bar{\alpha}}) = M_{l_{\bar{\alpha}}} \dfrac{\partial u_{\bar{\alpha}}}{\partial x}(0,t) - Q_{l_{\bar{\alpha}}} \dfrac{\partial u_{\bar{\alpha}}}{\partial t}(l_{\bar{\alpha}},t). \end{cases}$$

Using the condition $\sum_{i=1}^{n_\alpha} \partial u_{\bar{\alpha}\circ i}/\partial x(0,t) = \partial u_{\bar{\alpha}}/\partial x(l_{\bar{\alpha}},t)$, we obtain by the same argument as that used for the generic tree case

$$\left(Q_{l_{\bar{\alpha}}} + \sum_{i=1}^{n_{\bar{\alpha}}} U_{\bar{\alpha}\circ i}^{-1}\right) \frac{\partial u_{\bar{\alpha}}}{\partial t}(l_{\bar{\alpha}},t) = M_{l_{\bar{\alpha}}} \frac{\partial u_{\bar{\alpha}}}{\partial x}(0,t).$$

The operator $\left(Q_{l_{\bar{\alpha}}} + \sum_{i=1}^{n_{\bar{\alpha}}} U_{\bar{\alpha}}^{-1}\right)$ is of type $\mathcal{T}$ and with the coefficient $-(1+n_{\bar{\alpha}})$, thus it is invertible and

$$\frac{\partial u_{\bar{\alpha}}}{\partial t}(t,0) = \left[Q_{l_{\bar{\alpha}}} + M_{l_{\bar{\alpha}}} \left(\sum_{i=1}^{n_{\bar{\alpha}}} U_{\bar{\alpha}\circ i}^{-1} + Q_{l_{\bar{\alpha}}}\right)^{-1} M_{l_{\bar{\alpha}}}\right] \frac{\partial u_{\bar{\alpha}}}{\partial x}(t,0) = U_{\bar{\alpha}} \frac{\partial u_{\bar{\alpha}}}{\partial x}(t,0).$$

The operator $U_{\bar{\alpha}}$ is of type $\mathcal{T}$ and $\operatorname{coef}(U_{\bar{\alpha}}) = -1$.

Conclusion: If $\bar{u}$ is a solution of (1.1)–(1.5) then there exists an operator U of type $\mathcal{A}$ such that

$$\frac{\partial u}{\partial t}(0,t) = U \frac{\partial u}{\partial x}(0,t), \quad t > 0,$$

which is exactly (2.11).

We consider the initial and boundary value problem

$$\frac{\partial^2 \varphi_{\bar\alpha}}{\partial t^2}(x,t) - \frac{\partial^2 \varphi_{\bar\alpha}}{\partial x^2}(x,t) = 0, \quad 0 < x < l_{\bar\alpha},\ t > 0,\ \bar\alpha \in I, \tag{3.2}$$

$$\varphi_{\bar\alpha}(l_{\bar\alpha}, t) = 0, \quad t > 0,\ \bar\alpha \in I_{\mathcal{S}}, \tag{3.3}$$

$$\varphi_{\bar\alpha\circ\beta}(0,t) = \varphi_{\bar\alpha}(l_{\bar\alpha}, t), \quad t > 0,\ \beta = 1, \ldots, m_{\bar\alpha},\ \bar\alpha \in I_{\mathcal{M}}, \tag{3.4}$$

$$\sum_{\beta=1}^{m_{\bar\alpha}} \frac{\partial \varphi_{\bar\alpha\circ\beta}}{\partial x}(0,t) = \frac{\partial \varphi_{\bar\alpha}}{\partial x}(l_{\bar\alpha}, t), \quad \forall\, t > 0,\ \bar\alpha \in I_{\mathcal{M}}, \tag{3.5}$$

$$\frac{\partial \varphi}{\partial x}(0,t) = 0, \quad t > 0, \tag{3.6}$$

$$\varphi_{\bar\alpha}(x,0) = u^0_{\bar\alpha}(x), \quad \frac{\partial \varphi_{\bar\alpha}}{\partial t}(x,0) = u^1_{\bar\alpha}(x), \quad 0 < x < l_{\bar\alpha},\ \bar\alpha \in I. \tag{3.7}$$

The following result, besides showing that the above problem is well posed in the natural energy space, gives an inequality for the trace of $\bar\varphi$ at the root of the tree. It is easy to see by the semi-group method [13] that the problem (3.2)–(3.7) is well-posed in the energy space. For the proof of the inequality (3.9) see [10] and [12].

Lemma 3.7. *Suppose that $(\bar u^0, \bar u^1) \in V \times X$. Then the initial and boundary value problem* (3.2)–(3.7) *admits a unique solution*

$$\bar\varphi \in C(0,T;V) \cap C^1(0,T;X) \tag{3.8}$$

satisfying

$$\varphi(0,\cdot) \in H^1(0,T).$$

Moreover, there exists a constant $C > 0$, depending only on T, such that

$$\|\varphi(0,\cdot)\|^2_{H^1(0,T)} \leqslant C \|(\bar u^0, \bar u^1)\|^2_{V\times X}. \tag{3.9}$$

Observability inequalities concerning the trace, at the root of the tree, of the solutions of (3.2)–(3.7) can be stated as follows.

Proposition 3.8.

1. *For all $l_{\bar\alpha}$, $\bar\alpha \in I$ and for all $T > 0$ there exists no constant $C > 0$ such that the solutions $\bar\varphi$ of* (3.2)–(3.7) *satisfy*

$$\int_0^T \left|\frac{\partial \varphi}{\partial t}(0,t)\right|^2 \mathrm{d}t \geqslant C \|(\bar u^0, \bar u^1)\|^2_{V\times X}, \quad \forall\, (\bar u^0, \bar u^1) \in V \times X. \tag{3.10}$$

2. *There exist $T > 0$ and positive numbers d_k, $k \in \mathbb{N}$, such that the solution $\bar\varphi$ of (3.2)– (3.7) satisfies*

$$\int_0^T \Big|\frac{\partial \varphi}{\partial t}(0,t)\Big|^2 \,\mathrm{d}t \geqslant C\|(\bar u^0, \bar u^1)\|_Y^2, \quad \forall\,(\bar u^0, \bar u^1) \in V \times X, \tag{3.11}$$

where

$$Y = \Big\{\Big(\bar u^0 = \sum_{k\geqslant 0} a_k \bar\Phi_k,\ \bar u^1 = \sum_{k\geqslant 0} b_k \bar\Phi_k\Big) \in V \times X,$$
$$\sum_{k\geqslant 0} d_k^2(|w_k a_k|^2 + |b_k|^2) < \infty\Big\},$$
$$\|(\bar u^0, \bar u^1)\|_Y^2 = \sum_{k\geqslant 0} d_k^2(|w_k a_k|^2 + |b_k|^2), \quad \forall\,(\bar u^0, \bar u^1) \in Y,$$

and $C > 0$ is a constant depending only on $l_{\overline{\alpha}}$, $\overline{\alpha} \in I$.

Proof. The first assertion is proved in [6]; for the sake of completeness, we give a proof. For the tree $I_{\mathcal{S}} = \{1,2\}$, $I_{\mathcal{M}} = \emptyset$, $m_{\overline{\alpha}} = 2$ there exists no constant $C > 0$ such that

$$\Big|\sin\Big(k\pi\frac{l_i}{L}\Big)\Big| \geqslant C, \quad \forall\, k \in \mathbb{Z}^*, \ \ i = 1,2,$$

where $L = l + l_1 + l_2$. So we get the existence of a sequence $(p_m) \subset \mathbb{N}$, $\lim\limits_{m\to\infty} p_m = \infty$ such that

$$\lim_{m\to\infty} \sin\Big[p_m\pi\frac{l_1}{L}\Big] = 0. \tag{3.12}$$

If we denote by $\varphi_{i,m}$ the solution of (3.2)–(3.7) with the initial data

$$\varphi_m(x,0) = \cos(w_{p_m}x),$$
$$\varphi_{1,m}(x,0) = -\frac{\cos(w_{p_m}l)}{\sin(w_{p_m}l_1)}\sin(w_{p_m}(x-l_1)),$$
$$\varphi_{2,m}(x,0) = -\frac{\cos(w_{p_m}l)}{\sin(w_{p_m}l_2)}\sin(w_{p_m}(x-l_2)),$$
$$\frac{\partial \varphi_{i,m}}{\partial t}(x,0) = 0, \quad \forall\, x \in (0,l_i), \ \ i = 1,2,$$
$$\frac{\partial \varphi_m}{\partial t}(x,0) = 0, \quad \forall\, x \in (0,l),$$

a simple calculation using (3.12) implies that

$$\lim_{m\to\infty} \frac{\int_0^T |\partial\varphi_m/\partial t(0,t)|^2\,\mathrm{d}t}{\|(\bar\varphi_m(x,0),\partial\bar\varphi_m/\partial t(x,0))\|^2_{V\times X}} = \lim_{m\to\infty} \sin^2(w_{p_m} l_1) = \lim_{m\to\infty} \sin^2\Bigl(\frac{p_m\pi}{L} l_1\Bigr) = 0,$$

so (3.10) is false for any l, l_1, l_2.

From [5] we obtain that for all $T > 2\sum_{\bar\alpha\in I} l_{\bar\alpha}$ there exists a constant $C_T > 0$ and a sequence $(d_n)_{n\in\mathbb{N}}$ of real numbers such that

$$\int_0^T \Bigl|\frac{\partial\varphi}{\partial t}(0,t)\Bigr|^2\,\mathrm{d}t \geqslant C_T \sum_{n=0}^{+\infty} d_n^2[w_n^2 a_n^2 + b_n^2], \tag{3.13}$$

which is exactly (3.11). The weights d_n depend on the lengths $l_{\bar\alpha}$ of the strings and the choice of the root. □

We note that (3.13) holds for all trees, regardless of whether they are degenerate or not. Moreover, if the tree $\mathcal{A}$ is non-degenerate, all the coefficients d_n are different from zero.

4. Proof of Corollary 2.3

According to Theorem 2.2 in [3] the solutions of (1.1)–(1.5) satisfy the estimate

$$E(t) \leqslant M\mathrm{e}^{-\omega t}E(0), \quad \forall\, t \geqslant 0, \tag{4.1}$$

where $M,\omega > 0$ are constants depending only on $l_{\bar\alpha}$ if and only if the solution $\bar\varphi$ of (3.2)–(3.7) satisfies

$$\int_0^T \Bigl|\frac{\partial\varphi}{\partial t}(0,s)\Bigr|^2\,\mathrm{d}s \geqslant \frac{C}{4}E(0), \quad \forall\,(\bar u^0,\bar u^1) \in V\times X.$$

The above inequality clearly contradicts assertion 1 in Proposition 3.8. So the assumption (4.1) is false. We complete in this way the proof of the first assertion of Corollary 2.3.

We pass now to the proof of the second assertion of this corollary. Let $\mathcal{A}$ be a non-degenerate tree. By Proposition 3.8, the solution $\bar\varphi$ of (1.1)–(1.5) satisfies the inequality

$$\int_0^T \Bigl|\frac{\partial\varphi}{\partial t}(0,t)\Bigr|^2\,\mathrm{d}t \geqslant K_1\|(\bar u^0,\bar u^1)\|^2_Y.$$

The conclusion (2.12) follows now by simply using Theorem 2.4 in [3].

5. Comments and related questions

Simple examples illustrating the meaning of these results are investigated in [1] concerning the star-shaped networks case of strings and in [2] concerning generic trees.

A question related to the problem studied in this paper is the stabilization problem for nonlinear dynamic networks of strings, see [9] for the models.

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