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Ahmet Tekcan

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PROPER CYCLES OF INDEFINITE QUADRATIC FORMS  
AND THEIR RIGHT NEIGHBORS

AHMET TEKCAN, Bursa

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*Abstract.* In this paper we consider proper cycles of indefinite integral quadratic forms  $F = (a, b, c)$  with discriminant  $\Delta$ . We prove that the proper cycles of  $F$  can be obtained using their consecutive right neighbors  $R^i(F)$  for  $i \geq 0$ . We also derive explicit relations in the cycle and proper cycle of  $F$  when the length  $l$  of the cycle of  $F$  is odd, using the transformations  $\tau(F) = (-a, b, -c)$  and  $\chi(F) = (-c, b, -a)$ .

*Keywords:* quadratic form, indefinite form, cycle, proper cycle, right neighbor

*MSC 2000:* 11E04, 11E12, 11E16

## 1. INTRODUCTION

A real binary quadratic form (or just a form)  $F$  is a polynomial in two variables  $x$  and  $y$  of the type

$$F = F(x, y) = ax^2 + bxy + cy^2$$

with real coefficients  $a, b, c$ . We denote  $F$  briefly by  $F = (a, b, c)$ . The discriminant of  $F$  is defined by the formula  $b^2 - 4ac$  and is denoted by  $\Delta = \Delta(F)$ .  $F$  is an integral form if and only if  $a, b, c \in \mathbb{Z}$ , and is indefinite if and only if  $\Delta(F) > 0$ . An indefinite quadratic form  $F = (a, b, c)$  with discriminant  $\Delta$  is said to be reduced if

$$(1.1) \quad |\sqrt{\Delta} - 2|a|| < b < \sqrt{\Delta}.$$

Let  $\text{GL}(2, \mathbb{Z})$  be the multiplicative group of  $2 \times 2$  matrices  $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix}$  such that  $r, s, t, u \in \mathbb{Z}$  and  $\det g = \pm 1$ . Gauss (1777–1855) defined the group action of  $\text{GL}(2, \mathbb{Z})$  on the set of forms by the following formula: Let  $F = (a, b, c)$  be a form

and let  $g = \begin{pmatrix} r & s \\ t & u \end{pmatrix} \in \text{GL}(2, \mathbb{Z})$ . Then the form  $gF$  is defined by

$$gF(x, y) = a(rx + ty)^2 + b(rx + ty)(sx + uy) + c(sx + uy)^2.$$

That is,  $gF$  is obtained from  $F$  by making the substitution  $x \rightarrow rx + ty$ ,  $y \rightarrow sx + uy$ . Moreover,  $\Delta(F) = \Delta(gF)$  for all  $g \in \text{GL}(2, \mathbb{Z})$ . That is, the action of  $\text{GL}(2, \mathbb{Z})$  on forms leaves the discriminant invariant. If  $F$  is indefinite or integral, then so is  $gF$  for all  $g \in \text{GL}(2, \mathbb{Z})$ .

Let  $F$  and  $G$  be two forms. If there exists a  $g \in \text{GL}(2, \mathbb{Z})$  such that  $gF = G$ , then  $F$  and  $G$  are called equivalent. If  $\det g = 1$ , then  $F$  and  $G$  are called properly equivalent. If  $\det g = -1$ , then  $F$  and  $G$  are called improperly equivalent.

Let  $\varrho(F)$  denote the normalization (i.e., replacing  $F$  by its normalization, for further details see [1, p. 88]) of  $(c, -b, a)$ . To be more explicit, we set

$$(1.2) \quad \varrho(F) = (c, -b + 2cs, cs^2 - bs + a),$$

where

$$s = s(F) = \begin{cases} \text{sign}(c) \left\lfloor \frac{b}{2|c|} \right\rfloor & \text{for } |c| \geq \sqrt{\Delta}, \\ \text{sign}(c) \left\lfloor \frac{b + \sqrt{\Delta}}{2|c|} \right\rfloor & \text{for } |c| < \sqrt{\Delta}. \end{cases}$$

Note that, if  $F$  is reduced, then  $\varrho(F)$  is also reduced due to (1.1). In fact,  $\varrho$  is a permutation of the set of all reduced indefinite forms.

Now consider the transformations

$$(1.3) \quad \chi(F) = \chi(a, b, c) = (-c, b, -a)$$

and

$$(1.4) \quad \tau(F) = \tau(a, b, c) = (-a, b, -c).$$

If

$$\chi(F) = F,$$

that is,  $F = (a, b, -a)$  for the transformation  $\chi$  defined in (1.3), then  $F$  is called symmetric. We assume that  $F = (a, b, c)$  is indefinite and integral throughout the paper.

The cycle of  $F$  is the sequence  $((\tau\varrho)^i(G))$  for  $i \in \mathbb{Z}$ , where  $G = (k, l, m)$  is a reduced form with  $k > 0$  which is equivalent to  $F$ . Similarly, the proper cycle of  $F$  is the sequence  $(\varrho^i(G))$  for  $i \in \mathbb{Z}$ , where  $G$  is a reduced form which is properly equivalent

to  $F$ . The cycle and the proper cycle of  $F$  are invariants of the equivalence class of  $F$ . We represent the cycle or proper cycle of  $F$  by its period

$$F_0 \sim F_1 \sim \dots \sim F_{l-1}$$

of length  $l$ . We explain how to compute the cycle and proper cycle of  $F$  by the following lemma.

**Lemma 1.1.** *Let  $F = (a, b, c)$  be a reduced quadratic form of discriminant  $\Delta$ . Then the cycle of  $F$  is  $F_0 \sim F_1 \sim F_2 \sim \dots \sim F_{l-1}$  of length  $l$ , where  $F_0 = F = (a_0, b_0, c_0)$ ,*

$$(1.5) \quad s_i = |s(F_i)| = \left\lfloor \frac{b_i + \lfloor \sqrt{\Delta} \rfloor}{2|c_i|} \right\rfloor$$

and

$$(1.6) \quad F_{i+1} = (a_{i+1}, b_{i+1}, c_{i+1}) = (|c_i|, -b_i + 2s_i|c_i|, -(a_i + b_i s_i + c_i s_i^2))$$

for  $1 \leq i \leq l-2$ . The proper cycle of  $F$  is

$$(1.7) \quad F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \dots \sim \tau(F_{l-2}) \sim F_{l-1} \\ \sim \tau(F_0) \sim F_1 \sim \tau(F_2) \sim \dots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length  $2l$  if  $l$  is odd, and is

$$(1.8) \quad F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \dots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length  $l$  if  $l$  is even. In this case the equivalence class of  $F$  is the disjoint union of the proper equivalence class of  $F$  and the proper equivalence class of  $\tau(F)$  ([1]).

The right neighbor of  $F = (a, b, c)$ , denoted by  $R(F)$ , is the form  $(A, B, C)$  determined by the conditions

- (i)  $A = c$ ,
- (ii)  $b + B \equiv 0 \pmod{2A}$  and  $\sqrt{\Delta} - 2|A| < B < \sqrt{\Delta}$ ,
- (iii)  $B^2 - 4AC = \Delta$ .

It is clear from the definition that

$$(1.9) \quad R(F) = (A, B, C) = \begin{pmatrix} 1 & 0 \\ \delta & 1 \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} (a, b, c) = \begin{pmatrix} 0 & -1 \\ 1 & -\delta \end{pmatrix} (a, b, c),$$

where

$$(1.10) \quad b + B = 2c\delta.$$

Therefore  $F$  is properly equivalent to its right neighbor  $R(F)$  (see [2]).

2. PROPER CYCLES OF INDEFINITE QUADRATIC FORMS AND  
THEIR RIGHT NEIGHBORS

In this section we will consider the proper cycles of indefinite reduced quadratic forms  $F = (a, b, c)$ . We will show that the proper cycle of  $F$  can be given by using its consecutive right neighbors  $R^i(F)$  for  $i \geq 0$ . We also derive some relations in the cycle and proper cycle of  $F$ .

**Theorem 2.1.** *Let  $F_0 \sim F_1 \sim \dots \sim F_{l-1}$  be the cycle of  $F$  of length  $l$ , and let  $R^i(F_0)$  be the consecutive right neighbors of  $F_0$  for  $i \geq 0$ . Then*

(1) *If  $l$  is odd, then the proper cycle of  $F$  is*

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \dots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$$

*of length  $2l$ .*

(2) *If  $l$  is even, then the proper cycle of  $F$  is*

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \dots \sim R^{l-2}(F_0) \sim R^{l-1}(F_0)$$

*of length  $l$ .*

*Proof.* (1) Let  $l$  be odd. Then we know from (1.7) that the proper cycle of  $F$  is

$$F_0 \sim \tau(F_1) \sim F_2 \sim \tau(F_3) \sim \dots \sim F_{l-2} \sim \tau(F_{l-1})$$

of length  $2l$ . Let  $F_i$  and  $F_{i+1}$  be two forms in the cycle of  $F$ . Then

$$s_i = \left\lfloor \frac{b_i + \lfloor \sqrt{\Delta} \rfloor}{2|c_i|} \right\rfloor$$

by (1.5). For the right neighbor  $R(F_i)$  of  $F_i$  we have

$$\delta_i = \frac{b_i + B_i}{2c_i}$$

by (1.10). On the other hand,  $b_i + B_i \equiv 0 \pmod{2A_i}$  and  $\sqrt{\Delta} - 2|A_i| < B_i < \sqrt{\Delta}$  by the definition. Hence it is easily seen that

$$s_i = \begin{cases} -\delta_i & \text{if } i \text{ is even,} \\ \delta_i & \text{if } i \text{ is odd.} \end{cases}$$

Hence we can write  $s_i = |\delta_i|$ . Therefore  $|\delta_i|$  coincides with  $s_i$ . For the quadratic form  $F_0 = (a_0, b_0, c_0)$  we have from (1.5) and (1.6)

$$(2.1) \quad \begin{aligned} F_1 &= (|c_0|, -b_0 + 2s_0|c_0|, -a_0 - b_0s_0 - c_0s_0^2) \\ &= (-c_0, -b_0 - 2s_0c_0, -a_0 - b_0s_0 - c_0s_0^2). \end{aligned}$$

The first right neighbor of  $F_0 = (a_0, b_0, c_0)$  is

$$(2.2) \quad \begin{aligned} R^1(F_0) &= \begin{pmatrix} 0 & -1 \\ 1 & -\delta_0 \end{pmatrix} (a_0, b_0, c_0) \\ &= (c_0, -b_0 + 2\delta_0c_0, a_0 - b_0\delta_0 + c_0\delta_0^2) \end{aligned}$$

due to (1.9). Replacing  $\delta_0$  by  $-s_0$  in (2.2) we get

$$(2.3) \quad R^1(F_0) = (c_0, -b_0 - 2s_0c_0, a_0 + b_0s_0 + c_0s_0^2)$$

since  $s_0 = -\delta_0$ . Applying (1.4) we get from (2.1)

$$(2.4) \quad \tau(F_1) = (c_0, -b_0 - 2s_0c_0, a_0 + b_0s_0 + c_0s_0^2).$$

Consequently, (2.3) and (2.4) yield that

$$R^1(F_0) = (c_0, -b_0 - 2s_0c_0, a_0 + b_0s_0 + c_0s_0^2) = \tau(F_1).$$

Similarly it can be shown that

$$\begin{aligned} R^2(F_0) &= F_2, \\ R^3(F_0) &= \tau(F_3), \\ &\dots \\ R^{l-1}(F_0) &= F_{l-1}, \\ R^l(F_0) &= \tau(F_l), \\ R^{l+1}(F_0) &= F_1, \\ &\dots \\ R^{2l-2}(F_0) &= F_{l-2}, \\ R^{2l-1}(F_0) &= \tau(F_{l-1}). \end{aligned}$$

Therefore

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim \dots \sim R^{2l-2}(F_0) \sim R^{2l-1}(F_0)$$

is the proper cycle of  $F$  of length  $2l$ .

The second assertion can be proved in the same way. □

**Example 2.1.** The cycle of  $F = (1, 5, -4)$  is

$$\begin{aligned} F_0 = (1, 5, -4) &\sim F_1 = (4, 3, -2) \sim F_2 = (2, 5, -2) \\ &\sim F_3 = (2, 3, -4) \sim F_4 = (4, 5, -1) \end{aligned}$$

of length 5 which is an odd number. Therefore the proper cycle of  $F$  is

$$\begin{aligned} F_0 \sim R^1(F_0) \sim R^2(F_0) \sim R^3(F_0) \sim R^4(F_0) \sim R^5(F_0) \\ \sim R^6(F_0) \sim R^7(F_0) \sim R^8(F_0) \sim R^9(F_0) \end{aligned}$$

of length 10 since

$$\begin{aligned} R^1(F_0) &= (-4, 3, 2) = \tau(F_1), \\ R^2(F_0) &= (2, 5, -2) = F_2, \\ R^3(F_0) &= (-2, 3, 4) = \tau(F_3), \\ R^4(F_0) &= (4, 5, -1) = F_4, \\ R^5(F_0) &= (-1, 5, 4) = \tau(F_0), \\ R^6(F_0) &= (4, 3, -2) = F_1, \\ R^7(F_0) &= (-2, 5, 2) = \tau(F_2), \\ R^8(F_0) &= (2, 3, -4) = F_3, \\ R^9(F_0) &= (-4, 5, 1) = \tau(F_4). \end{aligned}$$

**Example 2.2.** The cycle of  $F = (1, 8, -5)$  is

$$\begin{aligned} F_0 = (1, 8, -5) &\sim F_1 = (5, 2, -4) \sim F_2 = (4, 6, -3) \\ &\sim F_3 = (3, 6, -4) \sim F_4 = (4, 2, -5) \sim F_5 = (5, 8, -1) \end{aligned}$$

of length 6 which is an even number. Therefore the proper cycle of  $F$  is

$$F_0 \sim R^1(F_0) \sim R^2(F_0) \sim R^3(F_0) \sim R^4(F_0) \sim R^5(F_0)$$

of length 6 since

$$\begin{aligned} R^1(F_0) &= (-5, 2, 4) = \tau(F_1), \\ R^2(F_0) &= (4, 6, -3) = F_2, \\ R^3(F_0) &= (-3, 6, 4) = \tau(F_3), \\ R^4(F_0) &= (4, 2, -5) = F_4, \\ R^5(F_0) &= (-5, 8, 1) = \tau(F_5). \end{aligned}$$

From Theorem 2.1 we can deduce the following corollary.

**Corollary 2.2.** Let  $F_0 \sim F_1 \sim \dots \sim F_{l-1}$  be the cycle of  $F$  of length  $l$ .

(1) If  $l$  is odd, then

$$R^i(F_0) = \begin{cases} F_i & \text{if } i \text{ is even,} \\ \tau(F_i) & \text{if } i \text{ is odd} \end{cases}$$

for  $1 \leq i \leq l-1$ , and

$$R^i(F_0) = \begin{cases} F_{i-l} & \text{if } i \text{ is even,} \\ \tau(F_{i-l}) & \text{if } i \text{ is odd} \end{cases}$$

for  $l \leq i \leq 2l-1$ .

(2) If  $l$  is even, then

$$R^i(F_0) = \begin{cases} F_i & \text{if } i \text{ is even,} \\ \tau(F_i) & \text{if } i \text{ is odd} \end{cases}$$

for  $1 \leq i \leq l-1$ .

**Theorem 2.3.** If  $l$  is odd, then in the cycle  $F_0 \sim F_1 \sim \dots \sim F_{l-1}$  of  $F$ ,

$$\chi(F_i) = F_{l-1-i}$$

for  $0 \leq i \leq l-1$  and  $F_{(l-1)/2}$  is a symmetric form.

*Proof.* Let  $F = (a, b, c)$  be a quadratic form. Then applying (1.5) and (1.6) we get

$$(2.5) \quad \begin{aligned} F_0 &= (a_0, b_0, c_0), \\ F_1 &= (a_1, b_1, c_1), \\ F_2 &= (a_2, b_2, c_2), \\ F_3 &= (a_3, b_3, c_3), \\ &\dots \\ F_{(l-3)/2} &= (a_{(l-3)/2}, b_{(l-3)/2}, c_{(l-3)/2}), \\ F_{(l-1)/2} &= (a_{(l-1)/2}, b_{(l-1)/2}, c_{(l-3)/2}), \\ F_{(l+1)/2} &= (-c_{(l-3)/2}, b_{(l-3)/2}, -a_{(l-3)/2}), \\ &\dots \\ F_{l-3} &= (-c_2, b_2, -a_2), \\ F_{l-2} &= (-c_1, b_1, -a_1), \\ F_{l-1} &= (-c_0, b_0, -a_0). \end{aligned}$$



It is clear from (2.5) that

$$\begin{aligned}
\chi(F_0) &= (-c_0, b_0, -a_0) = F_{l-1}, \\
\chi(F_1) &= (-c_1, b_1, -a_1) = F_{l-2}, \\
\chi(F_2) &= (-c_2, b_2, -a_2) = F_{l-3}, \\
&\dots \\
\chi(F_{(l-3)/2}) &= (-c_{(l-3)/2}, b_{(l-3)/2}, -a_{(l-3)/2}) = F_{(l+1)/2}, \\
\chi(F_{(l-1)/2}) &= (a_{(l-1)/2}, b_{(l-1)/2}, c_{(l-3)/2}) = F_{(l-1)/2}, \\
\chi(F_{(l+1)/2}) &= (a_{(l-3)/2}, b_{(l-3)/2}, c_{(l-3)/2}) = F_{(l-3)/2}, \\
&\dots \\
\chi(F_{l-3}) &= (a_2, b_2, c_2) = F_2, \\
\chi(F_{l-2}) &= (a_1, b_1, c_1) = F_1, \\
\chi(F_{l-1}) &= (a_0, b_0, c_0) = F_0.
\end{aligned}$$

So  $\chi(F_i) = F_{l-1-i}$  for  $0 \leq i \leq l-1$  and  $F_{(l-1)/2}$  is a symmetric form since  $\chi(F_{(l-1)/2}) = F_{(l-1)/2}$  by (1.3).  $\square$

From Theorem 2.3, we can obtain the following result.

**Corollary 2.4.** *The cycle of  $F$  is*

$$\begin{aligned}
F_0 \sim F_1 \sim F_2 \sim \dots \sim F_{(l-3)/2} \sim F_{(l-1)/2} \sim \chi(F_{(l-3)/2}) \sim \dots \\
\sim \chi(F_2) \sim \chi(F_1) \sim \chi(F_0).
\end{aligned}$$

Now we can give the cycle of  $\chi(F)$  by the following theorem.

**Theorem 2.5.** *If  $l$  is odd, then the cycle of  $\chi(F)$  is*

$$\chi(F_l) \sim \chi(F_{l-1}) \sim \chi(F_{l-2}) \sim \dots \sim \chi(F_1)$$

of length  $l$ .

*Proof.* Let  $F_0 \sim F_1 \sim F_2 \sim \dots \sim F_{l-1}$  be the cycle of  $F$ . Then  $F_l = F_0, F_{l+1} = F_1, \dots, F_{2l} = F_{l-1}$ . We know from Theorem 2.3 that

$$\chi(F_i) = F_{l-1-i}$$

for  $0 \leq i \leq l-1$ . So  $\chi(F_{l-1}) = F_0$ . In particular,  $\chi(F_{l-2}) = F_1, \chi(F_{l-3}) = F_2, \dots, \chi(F_0) = F_{l-1}$ . Consequently, the cycle of  $\chi(F)$  is

$$\chi(F_l) \sim \chi(F_{l-1}) \sim \chi(F_{l-2}) \sim \dots \sim \chi(F_1)$$

of length  $l$ .  $\square$

**Example 2.3.** The cycle of  $F = (1, 7, -6)$  is

$$\begin{aligned} F_0 &= (1, 7, -6) \sim F_1 = (6, 5, -2) \sim F_2 = (2, 7, -3) \sim F_3 = (3, 5, -4) \\ &\sim F_4 = (4, 3, -4) \sim F_5 = (4, 5, -3) \sim F_6 = (3, 7, -2) \\ &\sim F_7 = (2, 5, -6) \sim F_8 = (6, 7, -1) \end{aligned}$$

of length 9. Note that

$$\begin{aligned} \chi(F_0) &= (6, 7, -1), \\ \chi(F_8) &= (1, 7, -6), \\ \chi(F_7) &= (6, 5, -2), \\ \chi(F_6) &= (2, 7, -3), \\ \chi(F_5) &= (3, 5, -4), \\ \chi(F_4) &= (4, 3, -4), \\ \chi(F_3) &= (4, 5, -3), \\ \chi(F_2) &= (3, 7, -2), \\ \chi(F_1) &= (2, 5, -6). \end{aligned}$$

Therefore the cycle of  $\chi(F) = (6, 7, -1)$  is

$$\begin{aligned} \chi(F_9) &= (6, 7, -1) \sim \chi(F_8) = (1, 7, -6) \sim \chi(F_7) = (6, 5, -2) \\ &\sim \chi(F_6) = (2, 7, -3) \sim \chi(F_5) = (3, 5, -4) \sim \chi(F_4) = (4, 3, -4) \\ &\sim \chi(F_3) = (4, 5, -3) \sim \chi(F_2) = (3, 7, -2) \sim \chi(F_1) = (2, 5, -6) \end{aligned}$$

of length 9.

#### *References*

- [1] *J. Buchmann*: Algorithms for Binary Quadratic Forms. Springer-Verlag, accepted.
- [2] *D. E. Flath*: Introduction to Number Theory. Wiley, New York, 1989.

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*Author's address:* A. Tekcan, Uludag University, Faculty of Science, Department of Mathematics, Görükle 16059, Bursa, Turkey, e-mail: [tekcan@uludag.edu.tr](mailto:tekcan@uludag.edu.tr).