Liselott Flodén; Marianne Olsson

Homogenization of some parabolic operators with several time scales

Applications of Mathematics, Vol. 52 (2007), No. 5, 431--446

Persistent URL: http://dml.cz/dmlcz/134687

HOMOGENIZATION OF SOME PARABOLIC OPERATORS WITH SEVERAL TIME SCALES

Liselott Flodén, Marianne Olsson, Östersund

(Received June 28, 2006, in revised version October 24, 2006)

Abstract. The main focus in this paper is on homogenization of the parabolic problem $\partial_t u^\varepsilon - \nabla \cdot (a(x/\varepsilon, t/\varepsilon, t/\varepsilon^r)\nabla u^\varepsilon) = f$. Under certain assumptions on a, there exists a G-limit b, which we characterize by means of multiscale techniques for $r > 0$, $r \neq 1$. Also, an interpretation of asymptotic expansions in the context of two-scale convergence is made.

Keywords: homogenization, G-convergence, multiscale convergence, parabolic, asymptotic expansion

MSC 2000: 35B27

1. Introduction

The main source of inspiration for the development of the modern theory of convergence for sequences of differential operators, is to find methods to compute effective properties of heterogenous materials. The most general strategies designed for this purpose are the two closely related concepts of G-convergence and H-convergence. The special case where the structure is built by small identical cubes is particularly well studied. This kind of problems is usually named periodic homogenization problems.

In this paper we study the homogenization of a parabolic problem with rapid oscillations in one spatial scale and two time scales. More precisely, we homogenize the equation

$$
\begin{aligned}
(1)\qquad \partial_t u^\varepsilon(x,t) - \nabla\cdot\Big(a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\nabla u^\varepsilon(x,t)\Big) &= f(x,t) \quad \text{in } \Omega\times(0,T),\\
u^\varepsilon(x,0) &= u_0(x) \quad \text{in } \Omega,\\
u^\varepsilon(x,t) &= 0 \quad \text{on } \partial\Omega\times(0,T),
\end{aligned}
$$

where Ω is an open bounded set in $\mathbb{R}^N$ and r and T are positive constants. For $\{\varepsilon\}$ tending to zero we get a sequence of equations. The homogenization problem consists in studying the asymptotic behavior of the corresponding sequence of solutions u^ε and finding a limit equation which admits the limit u of $\{u^\varepsilon\}$ as its unique solution. This so-called homogenized problem is governed by a coefficient matrix without rapid oscillations, which we characterize by means of multiscale convergence techniques for different values on r. We distinguish the three different cases; $0<r<2$ where $r\neq 1$, $r=2$ and $r>2$.

In Section 2 we study some features of asymptotic expansion. We discuss in which sense of convergence the expansion is valid, interpreting the significance of the first terms. A generalization of these observations will turn out to be useful in the homogenization procedure in Section 4.

Notation 1. We define

$$\Omega_T = \Omega\times(0,T)$$

and

$$\mathcal{Y}_{k,n} = Y^k\times(0,1)^n,$$

where $Y^k = Y_1\times\ldots\times Y_k$ and $Y_1=\ldots=Y_k=Y=(0,1)^N$.

Let $F(\mathbb{R}^{kN+n})$ be a space of real valued functions defined on $\mathbb{R}^{kN+n}$. By $F_\sharp(\mathcal{Y}_{k,n})$ we denote all functions in $F_{\mathrm{loc}}(\mathbb{R}^{kN+n})$ which are the periodic repetition of some function in $F(\mathcal{Y}_{k,n})$.

Furthermore, $L^2(0,T;X)$ denotes all functions $u\colon (0,T)\to X$ such that

$$\|u\|_{L^2(0,T;X)} = \Big(\int_0^T \|u(\cdot,t)\|_X^2\,\mathrm{d}t\Big)^{1/2} < \infty,$$

where X is a Banach space.

Finally, for the evolution triple $H_0^1(\Omega)\subseteq L^2(\Omega)\subseteq H^{-1}(\Omega)$, we introduce the space $H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ of all functions u which belong to $L^2(0,T;H_0^1(\Omega))$ with $\partial_t u$ in $L^2(0,T;H^{-1}(\Omega))$. This space is equipped with the norm

$$\|u\|_{H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))} = \|u\|_{L^2(0,T;H_0^1(\Omega))} + \|\partial_t u\|_{L^2(0,T;H^{-1}(\Omega))}.$$

2. Two-scale convergence and asymptotic expansions

To enhance the comprehension of the two-scale convergence method and its relation to the multiple-scale expansion we first study a simple and illuminating homogenization example, namely the linear elliptic problem

$$
\begin{aligned}
-\nabla \cdot \Big(a\Big(\frac{x}{\varepsilon}\Big) \nabla u^\varepsilon(x) \Big) &= f(x) \quad \text{in } \Omega, \\
u^\varepsilon(x) &= 0 \qquad \text{on } \partial\Omega.
\end{aligned} \tag{2}
$$

Here $a(y)$ is periodic with period Y and $\Omega \subset \mathbb{R}^N$ is open and bounded. For $\varepsilon \to 0$ we have

$$u^\varepsilon(x) \rightharpoonup u(x) \quad \text{in } H_0^1(\Omega),$$

where u is the unique solution to the limit equation

$$
\begin{aligned}
-\nabla \cdot \big(b \nabla u(x) \big) &= f(x) \quad \text{in } \Omega, \\
u(x) &= 0 \qquad \text{on } \partial\Omega.
\end{aligned} \tag{3}
$$

The coefficient b is obtained from the local problem

$$-\nabla_y \cdot \big(a(y)(\nabla u(x) + \nabla_y u_1(x,y)) \big) = 0, \tag{4}$$

where u_1 is Y-periodic in its second argument.

The exact solution u^ε together with the solution u to the homogenized problem are found in Fig. 1 for a one dimensional example with $\Omega = (0,1)$, $f(x) = x^2$ and

$$a(y) = \frac{1}{2 + \sin(2\pi y)}. \tag{5}$$

To achieve the homogenization result above rigorously, one can use a method based on two-scale convergence. An adaptation of this technique to certain evolution cases is used to prove the homogenization result in Section 4. This concept was first introduced by Nguetseng in 1989, see [11].

Definition 2. A sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ is said to two-scale converge to a limit $u_0 \in L^2(\Omega \times Y)$ if

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) v\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega \int_Y u_0(x,y) v(x,y)\,\mathrm{d}y\,\mathrm{d}x$$

for all $v \in L^2(\Omega; C_\sharp(Y))$. We write

$$u^\varepsilon(x) \rightharpoonup\!\!\rightharpoonup u_0(x,y).$$

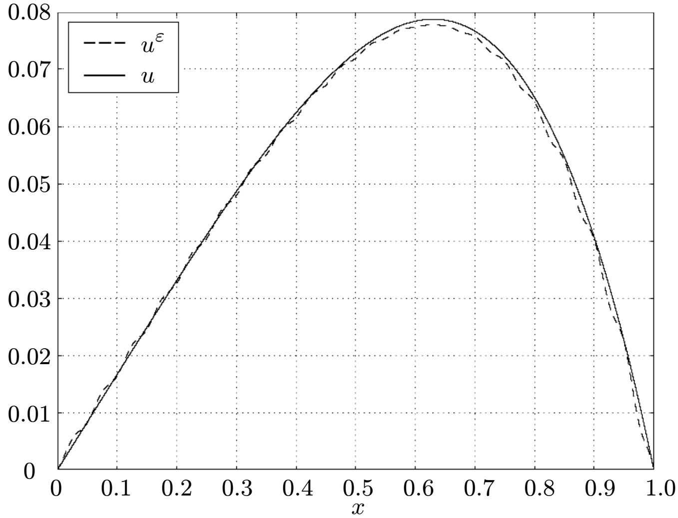

Figure 1. The solutions u^ε and u.

The definition is motivated by a compactness result which states that a sequence $\{u^\varepsilon\}$ bounded in $L^2(\Omega)$ possesses a two-scale convergent subsequence, see [11], [1] and [10]. The following theorem characterizes limits for sequences of gradients. This result allows us to pass to the limit in the weak form of (2) for certain choices of test functions and obtain the local problem (4) and the homogenized problem (3).

Theorem 3. *Let $\{u^\varepsilon\}$ be a sequence bounded in $H^1(\Omega)$. Then, up to a subsequence, it holds that*

$$u^\varepsilon(x) \to u(x) \quad \text{in } L^2(\Omega)$$

and

$$\nabla u^\varepsilon(x) \rightharpoonup\!\!\rightharpoonup \nabla u(x) + \nabla_y u_1(x, y),$$

where $u \in H^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_\sharp(Y)/\mathbb{R})$.

Proof. See [1] or [10]. □

An earlier method for the determination of the local and the homogenized problem is the multiple-scale, or asymptotic expansion, method; see e.g. [4] and [5]. In this approach, one assumes that the solution u^ε can be expanded in a power series in ε of the form

$$u^\varepsilon(x) = u(x) + \varepsilon u_1\Big(x, \frac{x}{\varepsilon}\Big) + \varepsilon^2 u_2\Big(x, \frac{x}{\varepsilon}\Big) + \dots, \tag{6}$$

where u_k is Y-periodic in the second argument. Plugging (6) into equation (2) and equating equal powers of ε, we formally obtain both the homogenized problem (3) and the local problem (4), solved by u and u_1 respectively. Furthermore, u and u_1 are identical to the functions with the same names which appear in Theorem 3.

We know that

$$u^\varepsilon(x) \to u(x) \quad \text{in } L^2(\Omega)$$

but it remains to explore if also u_1 has significance in the sense of some suitable kind of limit. The truncated form

$$u^\varepsilon(x) \approx u(x) + \varepsilon u_1\Big(x, \frac{x}{\varepsilon}\Big)$$

of (6) can be written

$$\frac{u^\varepsilon(x) - u(x)}{\varepsilon} \approx u_1\Big(x, \frac{x}{\varepsilon}\Big), \tag{7}$$

which indicates that $\{(u^\varepsilon - u)/\varepsilon\}$ approaches u_1 in some sense similar to the two-scale convergence. According to Theorem 3.3 in [2] we can identify a class of smooth functions $v\colon\ \Omega \times Y \to \mathbb{R}$ such that

$$F^\varepsilon(\cdot) = \frac{1}{\varepsilon}\int_\Omega v\Big(x, \frac{x}{\varepsilon}\Big)(\cdot)\,\mathrm{d}x$$

is bounded in $(H^1(\Omega))'$. The key properties of these functions are that they are Y-periodic for any fixed $x \in \Omega$ and have integral mean value zero over Y in their second argument. Hence, for any bounded sequence $\{\alpha^\varepsilon\}$ in $H^1(\Omega)$ and

$$\beta^\varepsilon = \frac{1}{\varepsilon}\int_\Omega v\Big(x, \frac{x}{\varepsilon}\Big)\alpha^\varepsilon(x)\,\mathrm{d}x,$$

$\{\beta^\varepsilon\}$ converges up to a subsequence. This means that, still up to a subsequence, the limit

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{u^\varepsilon(x) - u(x)}{\varepsilon} v\Big(x, \frac{x}{\varepsilon}\Big)\,\mathrm{d}x$$

exists for suitable test functions v.

It turns out that the choice of test functions is crucial. Fig. 2 shows the left-hand side together with the right-hand side of (7), where u^ε and u are the solutions to (2) and (3), respectively, for the coefficient a given by (5).

Apparently, they have similar patterns of oscillations but differ in the global tendency. Fig. 3 displays the difference between these two functions, i.e.

$$g^\varepsilon(x) = \frac{u^\varepsilon(x) - u(x)}{\varepsilon} - u_1\Big(x, \frac{x}{\varepsilon}\Big).$$

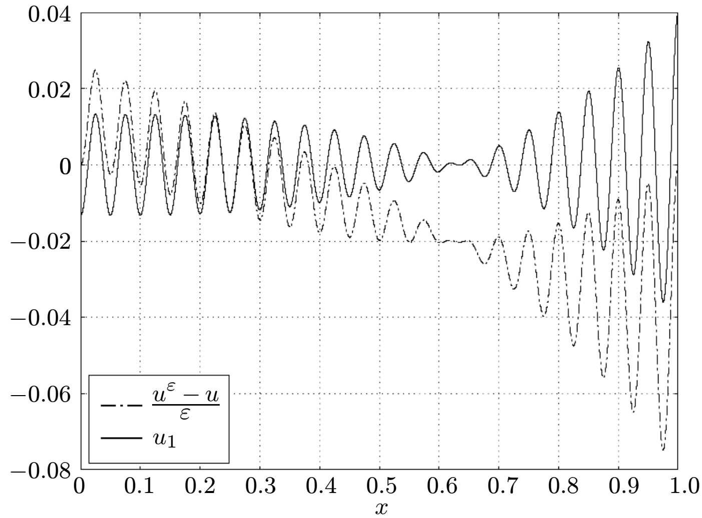

Figure 2. $(u^\varepsilon(x) - u(x))/\varepsilon$ together with $u_1(x, x/\varepsilon)$.

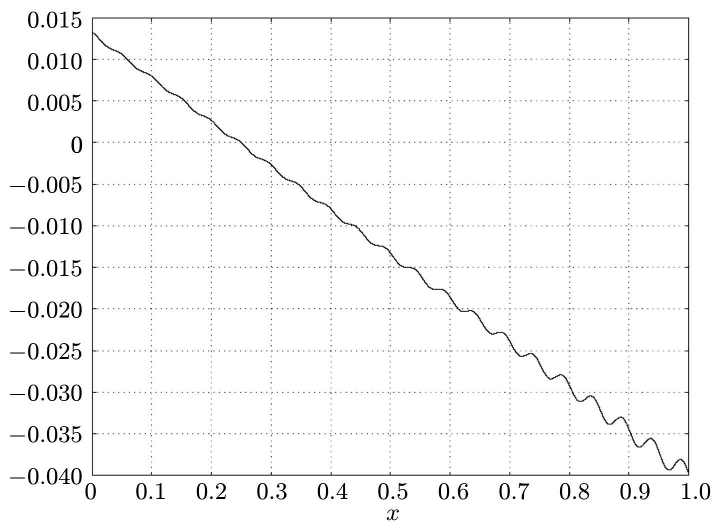

Figure 3. The function $g^\varepsilon(x)$.

For

$$v(x, y) = x^2 \cos(2\pi y)$$

the graph of $g^\varepsilon(x)v(x, x/\varepsilon)$ is found in Fig. 4. The integral of $g^\varepsilon(x)v(x, x/\varepsilon)$ over Ω is close to zero for small ε. The high frequency variations of g^ε are negligible. Furthermore, the function v has mean value zero in its second variable and replacing y by x/ε with ε small the slower global tendency of g^ε is filtered away by the oscillations.

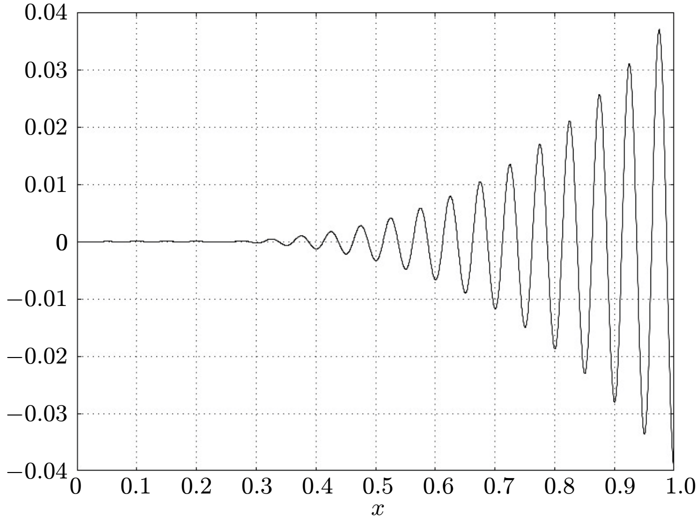

Figure 4. The product $g^{\varepsilon}(x)v(x,x/\varepsilon)$, $\int_Y v\,(x,y)\,\mathrm{d}y = 0$.

For this choice of v and $\varepsilon = 0.05$ we obtain

$$\int_\Omega \frac{u^\varepsilon(x)-u(x)}{\varepsilon} v\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x \approx 0.00224 \approx \int_\Omega\int_Y u_1(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x \approx 0.00221,$$

and hence a limit of two-scale type consistent with the approach of asymptotic expansion seems to be at hand for this kind of test functions. If we omit the requirement that v have integral mean value zero over Y and choose e.g.

$$v(x,y) = x^2(3+\cos(2\pi y)),$$

we obtain the function illustrated in Fig. 5, whose integral over Ω does obviously not vanish.

In this case

$$\int_\Omega \frac{u^\varepsilon(x)-u(x)}{\varepsilon} v\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x \approx -0.02441 \neq \int_\Omega\int_Y u_1(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x \approx 0.00221$$

for $\varepsilon = 0.05$. This shows that the class of test functions must be more restricted than for the usual two-scale convergence. Our investigation above reveals that $\{(u^\varepsilon - u)/\varepsilon\}$ is approaching u_1 only in a certain weak sense, namely

$$\lim_{\varepsilon\to 0}\int_\Omega \frac{u^\varepsilon(x)-u(x)}{\varepsilon} v\Big(x,\frac{x}{\varepsilon}\Big)\,\mathrm{d}x = \int_\Omega\int_Y u_1(x,y)v(x,y)\,\mathrm{d}y\,\mathrm{d}x,$$

where the delicate question is to identify the appropriate class of test functions.

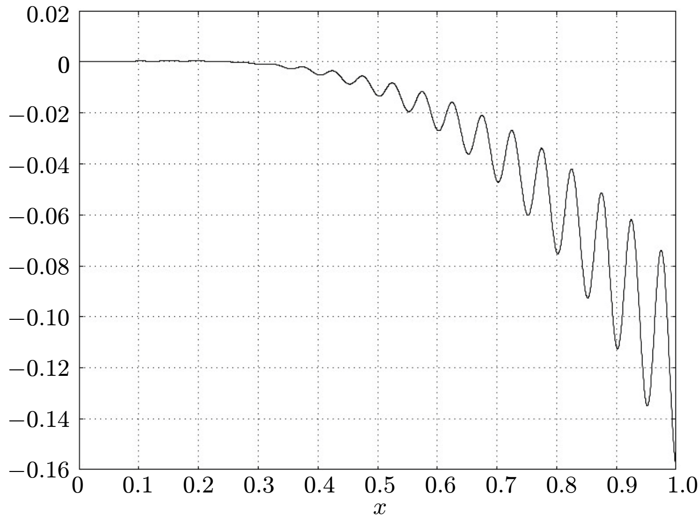

Figure 5. The product $g^{\varepsilon}(x)v(x,x/\varepsilon)$, $\int_Y v(x,y)\,\mathrm{d}y \neq 0$.

In [9] it is proven that for a bounded sequence $\{u^{\varepsilon}\}$ in $H^1(\Omega)$ and with u and u_1 defined as in Theorem 3 it holds that, up to a subsequence,

$$\lim_{\varepsilon\to 0}\int_{\Omega}\frac{u^{\varepsilon}(x)-u(x)}{\varepsilon}v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)\,\mathrm{d}x=\int_{\Omega}\int_{Y}u_1(x,y)v_1(x)v_2(y)\,\mathrm{d}y\,\mathrm{d}x$$

for all $v_1\in D(\Omega)$ and $v_2\in C_{\sharp}^{\infty}(Y)/\mathbb{R}$. Thus u_1 appears as a limit of two-scale type to $\{(u^{\varepsilon}-u)/\varepsilon\}$ which contributes to the interpretation of the asymptotic expansion (6). A generalization of this result, see Theorem 9, will be used to prove the homogenization result in Section 4.

3. Multiscale convergence

In [2], Allaire and Briane introduced so-called multiscale convergence. This generalization of two-scale convergence is suitable when studying problems with multiple scales of periodic oscillations. We give the following definition.

Definition 4. A sequence $\{u^{\varepsilon}\}$ in $L^2(\Omega)$ is said to $(n+1)$-scale converge to $u_0\in L^2(\Omega\times Y^n)$ if

$$\begin{aligned}(8)\qquad &\lim_{\varepsilon\to 0}\int_{\Omega}u^{\varepsilon}(x)v\Big(x,\frac{x}{\varepsilon_1},\dots,\frac{x}{\varepsilon_n}\Big)\,\mathrm{d}x\\ &=\int_{\Omega}\int_{Y^n}u_0(x,y_1,\dots,y_n)v(x,y_1,\dots,y_n)\,\mathrm{d}y_n\dots\mathrm{d}y_1\,\mathrm{d}x\end{aligned}$$

for all $v \in L^2(\Omega; C_\sharp(Y^n))$, where $\varepsilon_k(\varepsilon) \to 0$ when $\varepsilon \to 0$. We write

$$u^\varepsilon(x) \overset{n+1}{\rightharpoonup} u_0(x, y_1, \ldots, y_n).$$

In [2] it is proven that a bounded sequence $\{u^\varepsilon\}$ in $L^2(\Omega)$ has a subsequence which $(n+1)$-scale converges to some limit $u_0 \in L^2(\Omega \times Y^n)$ if the scales obey certain separation requirements.

Adapting to our problem (1) we define $2,3$-scale convergence which includes two spatial scales and three time scales.

Definition 5. A sequence $\{u^\varepsilon\}$ in $L^2(\Omega_T)$ is said to 2,3-scale converge to $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{1,2})$ if

$$\lim_{\varepsilon \to 0} \int_{\Omega_T} u^\varepsilon\Big(x, t\Big) v\Big(x, t, \frac{x}{\varepsilon_1}, \frac{t}{\varepsilon_1'}, \frac{t}{\varepsilon_2'}\Big) \,\mathrm{d}x \,\mathrm{d}t$$
$$= \int_{\Omega_T} \int_{\mathcal{Y}_{1,2}} u_0(x, t, y, s_1, s_2) v(x, t, y, s_1, s_2) \,\mathrm{d}y \,\mathrm{d}s_2 \,\mathrm{d}s_1 \,\mathrm{d}x \,\mathrm{d}t$$

for all $v \in L^2(\Omega_T; C_\sharp(\mathcal{Y}_{1,2}))$. This is denoted by

$$u^\varepsilon(x, t) \overset{2,3}{\rightharpoonup} u_0(x, t, y, s_1, s_2).$$

The following compactness result holds true.

Theorem 6. *Let $\{u^\varepsilon\}$ be a bounded sequence in $L^2(\Omega_T)$ and assume that $\varepsilon_1 = \varepsilon$, $\varepsilon_1' = \varepsilon$ and $\varepsilon_2' = \varepsilon^r$, where $r > 0$ and $r \neq 1$. Then there exists a function $u_0 \in L^2(\Omega_T \times \mathcal{Y}_{1,2})$ such that, up to a subsequence, $\{u^\varepsilon\}$ 2,3-scale converges to u_0.*

Proof. This is an immediate consequence of Theorem 2.4 in [2]. □

Remark 7. We omit the case $r = 1$ to obtain separation between the scales, see [2] for details.

Theorem 8 deals with $2,3$-scale convergence of gradients. The proof follows directly along the lines of the proof of Theorem 3.1 in [8].

Theorem 8. *Let $\{u^\varepsilon\}$ be a sequence bounded in $H^1(0, T; H_0^1(\Omega), H^{-1}(\Omega))$ and assume that $\varepsilon_1 = \varepsilon$, $\varepsilon_1' = \varepsilon$ and $\varepsilon_2' = \varepsilon^r$, where $r > 0$ and $r \neq 1$. Then it holds up to a subsequence that*

$$u^\varepsilon(x, t) \to u(x, t) \quad \text{in } L^2(\Omega_T)$$

and

$$\nabla u^\varepsilon(x, t) \overset{2,3}{\rightharpoonup} \nabla u(x, t) + \nabla_y u_1(x, t, y, s_1, s_2),$$

where $u \in L^2(0, T; H_0^1(\Omega))$ and $u_1 \in L^2(\Omega_T \times (0,1)^2; H_\sharp^1(Y)/\mathbb{R})$.

The following theorem generalizes our observations in Section 2 to a certain evolution case.

Theorem 9. *Let $\{u^\varepsilon\}$ be a bounded sequence in $H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))$, and let u and u_1 be defined as in Theorem 8. Then, up to a subsequence,*

$$\lim_{\varepsilon\to 0}\int_{\Omega_T}\frac{u^\varepsilon(x,t)-u(x,t)}{\varepsilon}v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t$$
$$=\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}}u_1(x,t,y,s_1,s_2)v_1(x)v_2(y)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}x\,\mathrm{d}t$$

for all $v_1\in D(\Omega)$, $v_2\in L^2_\sharp(Y)/\mathbb{R}$, $c_1\in D(0,T)$ and $c_2,c_3\in L^2_\sharp(0,1)$ if $r>0$, $r\neq 1$.

Proof. The proof is performed in exactly the same way as the proof of Corollary 3.3 in [8]. □

The theorems above will be used while proving the homogenization result in Theorem 10.

4. Homogenization

In this section we study the homogenization of the parabolic equation

$$\begin{aligned}(9)\qquad \partial_t u^\varepsilon(x,t)-\nabla\cdot\Big(a\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\nabla u^\varepsilon(x,t)\Big)&=f(x,t)\quad\text{in }\Omega\times(0,T),\\ u^\varepsilon(x,0)&=u_0(x)\quad\text{in }\Omega,\\ u^\varepsilon(x,t)&=0\quad\text{on }\partial\Omega\times(0,T),\end{aligned}$$

introduced in Section 1, where $f\in L^2(\Omega_T)$ and $u_0\in L^2(\Omega)$. This problem allows a unique solution $u^\varepsilon\in H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ for any fixed $\varepsilon>0$ if a is assumed to belong to $L^\infty_\sharp(\mathcal{Y}_{1,2})$ and to satisfy the coercivity condition

$$a(y,s_1,s_2)\xi\cdot\xi\geqslant\alpha|\xi|^2,\quad \alpha>0$$

for any $(y,s_1,s_2)\in\mathbb{R}^{N+2}$ and all $\xi\in\mathbb{R}^N$. Under these assumptions, $a(x/\varepsilon,t/\varepsilon,t/\varepsilon^r)$ satisfies the standard conditions for G-convergence for linear parabolic problems, see [6] and [14]. This means that, at least for a suitable subsequence, there exists a well posed limit problem of the same type as (9) governed by a coefficient matrix b. Moreover,

$$(10)\qquad \|u^\varepsilon\|_{L^\infty(0,T;L^2(\Omega))}\leqslant C$$

and

$$\|u^\varepsilon\|_{H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))} \leqslant C \tag{11}$$

for some positive constant C. Our aim is to characterize the G-limit b further by means of homogenization procedures.

Theorem 10. *Let $\{u^\varepsilon\}$ be a sequence of solutions in $H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ to (9). Then*

$$u^\varepsilon(x,t) \to u(x,t) \quad \text{in } L^2(\Omega_T)$$

and

$$\nabla u^\varepsilon(x,t) \overset{2,3}{\rightharpoonup} \nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2),$$

where $u \in H^1(0,T;H_0^1(\Omega),H^{-1}(\Omega))$ and $u_1 \in L^2(\Omega_T \times (0,1)^2;H_\sharp^1(Y)/\mathbb{R})$.

Furthermore, u is the unique solution to the homogenized problem

$$\begin{aligned} \partial_t u(x,t) - \nabla\cdot\big(b\nabla u(x,t)\big) &= f(x,t) && \text{in } \Omega_T,\\ u(x,0) &= u_0(x) && \text{in } \Omega,\\ u(x,t) &= 0 && \text{on } \partial\Omega\times(0,T) \end{aligned} \tag{12}$$

where

$$b\nabla u(x,t) = \int_{\mathcal{Y}_{1,2}} a(y,s_1,s_2)\big(\nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2)\big)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1.$$

For $0<r<2$, $r\neq 1$, the function u_1 is determined by the local problem

$$-\nabla_y\cdot\big(a(y,s_1,s_2)\big(\nabla u(x,t)+\nabla_y u_1(x,t,y,s_1,s_2)\big)\big) = 0, \tag{13}$$

for $r=2$ by

$$\partial_{s_2} u_1 - \nabla_y\cdot\big(a(y,s_1,s_2)(\nabla u(x,t)+\nabla_y u_1(x,t,y,s_1,s_2))\big) = 0, \tag{14}$$

and for $r>2$ by the system of local problems

$$-\nabla_y\cdot\left(\left(\int_0^1 a(y,s_1,s_2)\,\mathrm{d}s_2\right)\big(\nabla u(x,t)+\nabla_y u_1(x,t,y,s_1)\big)\right) = 0, \tag{15}$$

$$\partial_{s_2} u_1(x,t,y,s_1,s_2) = 0. \tag{16}$$

P r o o f. Throughout the proof we use the Einstein tensor summation convention. We study the weak form of (9), i.e. we require that

$$\begin{aligned}(17)\qquad \int_{\Omega_T} -u^{\varepsilon}(x,t)v(x)\partial_t c(t) + a_{ij}\Bigl(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\frac{t}{\varepsilon^r}\Bigr)\partial_{x_j}u^{\varepsilon}(x,t)\partial_{x_i}v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t \\ = \int_{\Omega_T} f(x,t)v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

for all $v \in H_0^1(\Omega)$ and $c \in D(0,T)$. The a priori estimates (10) and (11) allow us to apply Theorem 8. Hence, up to a subsequence,

$$u^{\varepsilon}(x,t) \to u(x,t) \quad \text{in } L^2(\Omega_T)$$

and

$$\nabla u^{\varepsilon}(x,t) \overset{2,3}{\rightharpoonup} \nabla u(x,t) + \nabla_y u_1(x,t,y,s_1,s_2),$$

where $u \in L^2(0,T;H_0^1(\Omega))$ and $u_1 \in L^2(\Omega_T \times (0,1)^2; H_\sharp^1(Y)/\mathbb{R})$. Choosing $v \in H_0^1(\Omega)$ and $c \in D(0,T)$ independent of ε in (17) and letting ε tend to zero we get, according to Theorem 8,

$$\begin{aligned}(18)\quad \int_{\Omega_T} & -u(x,t)v(x)\partial_t c(t) \\ & + \Bigl(\int_{\mathcal{Y}_{1,2}} a_{ij}(y,s_1,s_2)\bigl(\partial_{x_j}u(x,t) + \partial_{y_j}u_1(x,t,y,s_1,s_2)\bigr)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\Bigr) \\ & \qquad\qquad\qquad\qquad \times \partial_{x_i}v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t \\ = & \int_{\Omega_T} f(x,t)v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t.\end{aligned}$$

In order to find the local problems we study the difference between (17) and (18), i.e.

$$\begin{aligned}(19)\quad \int_{\Omega_T} & \bigl(u^{\varepsilon}(x,t) - u(x,t)\bigr)v(x)\partial_t c(t) \\ & + \biggl(\Bigl(\int_{\mathcal{Y}_{1,2}} a_{ij}(y,s_1,s_2)\bigl(\partial_{x_j}u(x,t) + \partial_{y_j}u_1(x,t,y,s_1,s_2)\bigr)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\Bigr) \\ & - a_{ij}\Bigl(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\frac{t}{\varepsilon^r}\Bigr)\partial_{x_j}u^{\varepsilon}(x,t)\biggr)\partial_{x_i}v(x)c(t)\,\mathrm{d}x\,\mathrm{d}t = 0.\end{aligned}$$

We choose the test functions

$$v(x) = \varepsilon v_1(x)v_2\Bigl(\frac{x}{\varepsilon}\Bigr), \quad c(t) = c_1(t)c_2\Bigl(\frac{t}{\varepsilon}\Bigr)c_3\Bigl(\frac{t}{\varepsilon^r}\Bigr),$$

where $v_1 \in D(\Omega)$, $v_2 \in C^\infty_\sharp(Y)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_2, c_3 \in C^\infty_\sharp(0,1)$ in (19), and get

$$\begin{aligned}
(20)\quad \int_{\Omega_T} &\frac{1}{\varepsilon}\big(u^\varepsilon(x,t) - u(x,t)\big)v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)\Bigg(\varepsilon^2\partial_t c_1(t)c_2\Big(\frac{t}{\varepsilon}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big) \\
&+ \varepsilon c_1(t)\partial_{s_1}c_2\Big(\frac{t}{\varepsilon}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big) + \varepsilon^{2-r}c_1(t)c_2\Big(\frac{t}{\varepsilon}\Big)\partial_{s_2}c_3\Big(\frac{t}{\varepsilon^r}\Big)\Bigg) \\
&+ \Bigg(\Bigg(\int_{\mathcal{Y}_{1,2}} a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t) + \partial_{y_j}u_1(x,t,y,s_1,s_2)\big)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\Bigg) \\
&\qquad - a_{ij}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\frac{t}{\varepsilon^r}\Big)\partial_{x_j}u^\varepsilon(x,t)\Bigg) \\
&\times \Bigg(\varepsilon\partial_{x_i}v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big) + v_1(x)\partial_{y_i}v_2\Big(\frac{x}{\varepsilon}\Big)\Bigg)c_1(t)c_2\Big(\frac{t}{\varepsilon}\Big)c_3\Big(\frac{t}{\varepsilon^r}\Big)\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned}$$

Next we let ε pass to zero. For the case when $0 < r < 2$ we have

$$\begin{aligned}
\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} &\Bigg(\Bigg(\int_{\mathcal{Y}_{1,2}} a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t) + \partial_{y_j}u_1(x,t,y,s_1,s_2)\big)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\Bigg) \\
&\quad - a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t) + \partial_{y_j}u_1(x,t,y,s_1,s_2)\big)\Bigg) \\
&\qquad \times v_1(x)\partial_{y_i}v_2(y)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}$$

and due to the periodicity of v_2 we obtain

$$\begin{aligned}
\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} &-a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t) + \partial_{y_j}u_1(x,t,y,s_1,s_2)\big) \\
&\times v_1(x)\partial_{y_i}v_2(y)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}x\,\mathrm{d}t = 0.
\end{aligned}$$

By the variational lemma

$$\int_Y -a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t) + \partial_{y_j}u_1(x,t,y,s_1,s_2)\big)\partial_{y_i}v_2(y)\,\mathrm{d}y = 0$$

for all $v_2 \in C^\infty_\sharp(Y)/\mathbb{R}$, and hence, by density, for all $v_2 \in H^1_\sharp(Y)/\mathbb{R}$, a.e. in $\Omega_T \times (0,1)^2$. This is the weak form of (13). For $r = 2$, according to Theorem 9, (20) approaches

$$\begin{aligned}
\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} &u_1(x,t,y,s_1,s_2)v_1(x)v_2(y)c_1(t)c_2(s_1)\partial_{s_2}c_3(s_2)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}x\,\mathrm{d}t \\
&+ \int_{\Omega_T}\int_{\mathcal{Y}_{1,2}}\Bigg(\int_{\mathcal{Y}_{1,2}} a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t) + \partial_{y_j}u_1(x,t,y,s_1,s_2)\big)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\Bigg) \\
&\qquad \times v_1(x)\partial_{y_i}v_2(y)c_1(t)c_2(s_1)c_3(s_2) \\
&- a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t) + \partial_{y_j}u_1(x,t,y,s_1,s_2)\big) \\
&\qquad \times v_1(x)\partial_{y_i}v_2(y)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}x\,\mathrm{d}t = 0,
\end{aligned}$$

when ε tends to zero. Since v_2 is periodic the middle term vanishes and we have

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} & u_1(x,t,y,s_1,s_2)v_1(x)v_2(y)c_1(t)c_2(s_1)\partial_{s_2}c_3(s_2)\\ & - a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t)+\partial_{y_j}u_1(x,t,y,s_1,s_2)\big)\\ & \times v_1(x)\partial_{y_i}v_2(y)c_1(t)c_2(s_1)c_3(s_2)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

Applying the variational lemma we arrive at

$$\begin{aligned}\int_{\mathcal{Y}_{1,1}} & u_1(x,t,y,s_1,s_2)v_2(y)\partial_{s_2}c_3(s_2)\\ & - a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t)+\partial_{y_j}u_1(x,t,y,s_1,s_2)\big)\partial_{y_i}v_2(y)c_3(s_2)\,\mathrm{d}y\,\mathrm{d}s_2=0,\end{aligned}$$

for all $v_2 \in H^1_\sharp(Y)/\mathbb{R}$ and all $c_3 \in C^\infty_\sharp(0,1)$, a.e. in $\Omega_T \times (0,1)$. We have found the weak form of (14).

For the case when $r > 2$ we choose the test functions

$$v(x)=\varepsilon v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big),\quad c(t)=c_1(t)c_2\Big(\frac{t}{\varepsilon}\Big)$$

in (17), where $v_1 \in D(\Omega)$, $v_2 \in C^\infty_\sharp(Y)$, $c_1 \in D(0,T)$ and $c_2 \in C^\infty_\sharp(0,1)$. We obtain

$$\begin{aligned}\int_{\Omega_T} & -u^\varepsilon(x,t)v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)\bigg(\varepsilon\partial_t c_1(t)c_2\Big(\frac{t}{\varepsilon}\Big)+c_1(t)\partial_{s_1}c_2\Big(\frac{t}{\varepsilon}\Big)\bigg)\\ & + a_{ij}\Big(\frac{x}{\varepsilon},\frac{t}{\varepsilon},\frac{t}{\varepsilon^2}\Big)\partial_{x_j}u^\varepsilon(x,t)\bigg(\varepsilon\partial_{x_i}v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)+v_1(x)\partial_{y_i}v_2\Big(\frac{x}{\varepsilon}\Big)\bigg)\\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\times c_1(t)c_2\Big(\frac{t}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}t\\ = & \int_{\Omega_T} f(x,t)\varepsilon v_1(x)v_2\Big(\frac{x}{\varepsilon}\Big)c_1(t)c_2\Big(\frac{t}{\varepsilon}\Big)\,\mathrm{d}x\,\mathrm{d}t\end{aligned}$$

and when ε goes to zero we get

$$\begin{aligned}\int_{\Omega_T}\int_{\mathcal{Y}_{1,2}} & -u(x,t)v_1(x)v_2(y)c_1(t)\partial_{s_1}c_2(s_1)\\ & + a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t)+\partial_{y_j}u_1(x,t,y,s_1,s_2)\big)\\ & \times v_1(x)\partial_{y_i}v_2(y)c_1(t)c_2(s_1)\,\mathrm{d}y\,\mathrm{d}s_2\,\mathrm{d}s_1\,\mathrm{d}x\,\mathrm{d}t=0.\end{aligned}$$

The periodicity of c_2 and the variational lemma imply that

$$\int_{\mathcal{Y}_{1,1}} a_{ij}(y,s_1,s_2)\big(\partial_{x_j}u(x,t)+\partial_{y_j}u_1(x,t,y,s_1,s_2)\big)\partial_{y_i}v_2(y)\,\mathrm{d}y\,\mathrm{d}s_2=0$$

for all $v_2 \in H^1_\sharp(Y)/\mathbb{R}$ a.e. in $\Omega_T \times (0,1)$. This is the weak form of (15).

Next we study the difference (19) for the test functions

$$v(x) = \varepsilon^{r-1} v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big), \quad c(t) = c_1(t) c_2\Big(\frac{t}{\varepsilon}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big),$$

where $v_1 \in D(\Omega)$, $v_2 \in C_\sharp^\infty(Y)/\mathbb{R}$, $c_1 \in D(0,T)$ and $c_2, c_3 \in C_\sharp^\infty(0,1)$, and we obtain

$$\begin{aligned}
&\int_{\Omega_T} \frac{1}{\varepsilon}\big(u^\varepsilon(x,t) - u(x,t)\big) v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big)\Big(\varepsilon^r \partial_t c_1(t) c_2\Big(\frac{t}{\varepsilon}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big) \\
&\quad + \varepsilon^{r-1} c_1(t) \partial_{s_1} c_2\Big(\frac{t}{\varepsilon}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big) + c_1(t) c_2\Big(\frac{t}{\varepsilon}\Big) \partial_{s_2} c_3\Big(\frac{t}{\varepsilon^r}\Big)\Big) \\
&\quad + \bigg(\bigg(\int_{\mathcal{Y}_{1,2}} a_{ij}(y, s_1, s_2)\big(\partial_{x_j} u(x,t) + \partial_{y_j} u_1(x,t,y,s_1,s_2)\big) \,\mathrm{d}y \,\mathrm{d}s_2 \,\mathrm{d}s_1\bigg) \\
&\quad - a_{ij}\Big(\frac{x}{\varepsilon}, \frac{t}{\varepsilon}, \frac{t}{\varepsilon^r}\Big) \partial_{x_j} u^\varepsilon(x,t)\bigg) \\
&\quad \times \Big(\varepsilon^{r-1} \partial_{x_i} v_1(x) v_2\Big(\frac{x}{\varepsilon}\Big) + \varepsilon^{r-2} v_1(x) \partial_{y_i} v_2\Big(\frac{x}{\varepsilon}\Big)\Big) c_1(t) c_2\Big(\frac{t}{\varepsilon}\Big) c_3\Big(\frac{t}{\varepsilon^r}\Big) \,\mathrm{d}x \,\mathrm{d}t = 0.
\end{aligned}$$

When ε passes to zero we get according to Theorem 9

$$\int_{\Omega_T} \int_{\mathcal{Y}_{1,2}} u_1(x,t,y,s_1,s_2) v_1(x) v_2(y) c_1(t) c_2(s_1) \partial_{s_2} c_3(s_2) \,\mathrm{d}y \,\mathrm{d}s_2 \,\mathrm{d}s_1 \,\mathrm{d}x \,\mathrm{d}t = 0$$

and by the variational lemma

$$\int_0^1 u_1(x,t,y,s_1,s_2) \partial_{s_2} c_3(s_2) \,\mathrm{d}s_2 = 0$$

for all $c_3 \in C_\sharp^\infty(0,1)$, a.e. in $\Omega_T \times Y \times (0,1)$. This is the weak form of (16) and means that u_1 does not depend on s_2. $\square$

Remark 11. Homogenization of linear parabolic equations with oscillations in both space and time was first investigated by means of asymptotic expansions in [4]. This is also studied in [13], where a different proof is provided. The homogenization of monotone nonlinear problems with two scales in space and time, respectively, is proven by means of G-convergence in [15]. Similar results can also be found in [12]. For corrector results for linear parabolic equations with oscillation only in space we refer to [3]. Such results for problems with oscillations also in time have been proven by means of two-scale convergence methods in [8], and with H-convergence techniques in [7]. The corresponding results for nonlinear problems are obtained in [16]. In [9] reiterated homogenization results for linear parabolic operators are proven by means of multiscale convergence methods.

References

[1] *G. Allaire*: Homogenization and two-scale convergence. SIAM J. Math. Anal. *23* (1992), 1482–1518. zbl

[2] *G. Allaire, M. Briane*: Multiscale convergence and reiterated homogenization. Proc. R. Soc. Edinburgh, Sect. A *126* (1996), 297–342. zbl

[3] *S. Brahim-Otsmane, G. A. Francfort, and F. Murat*: Correctors for the homogenization of the heat and wave equations. J. Math. Pures Appl. *71* (1992), 197–231. zbl

[4] *A. Bensoussan, J.-L. Lions, and G. Papanicolaou*: Asymptotic Analysis for Periodic Structures. Stud. Math. Appl. North-Holland, Amsterdam-New York-Oxford, 1978. zbl

[5] *D. Cioranescu, P. Donato*: An Introduction to Homogenization. Oxford Lecture Ser. Math. Appl. Oxford University Press, Oxford, 1999. zbl

[6] *F. Colombini, S. Spagnolo*: Sur la convergence de solutions d'équations paraboliques. J. Math. Pures Appl. *56* (1977), 263–305. (In French.) zbl

[7] *A. Dall'Aglio, F. Murat*: A corrector result for H-converging parabolic problems with time-dependent coefficients. Ann. Sc. Norm. Super. Pisa Cl. Sci., IV. Ser. *25* (1997), 329–373. zbl

[8] *A. Holmbom*: Homogenization of parabolic equations. An alternative approach and some corrector-type results. Appl. Math. *42* (1997), 321–343. zbl

[9] *A. Holmbom, N. Svanstedt, N. Wellander*: Multiscale convergence and reiterated homogenization for parabolic problems. Appl. Math. *50* (2005), 131–151. zbl

[10] *D. Lukkassen, G. Nguetseng, P. Wall*: Two-scale convergence. Int. J. Pure Appl. Math. *2* (2002), 35–86. zbl

[11] *G. Nguetseng*: A general convergence result for a functional related to the theory of homogenization. SIAM J. Math. Anal. *20* (1989), 608–623. zbl

[12] *A. Pankov*: G-convergence and Homogenization of Nonlinear Partial Differential Operators. Mathematics and its Applications. Kluwer, Dordrecht, 1997.

[13] *A. Profeti, B. Terreni*: Uniformità per una convergenza di operatori parabolichi nel caso dell'omogenizzazione. Boll. Unione Math. Ital. Ser. B *16* (1979), 826–841. (In Italian.) zbl

[14] *S. Spagnolo*: Convergence of parabolic equations. Boll. Unione Math. Ital. Ser. B *14* (1977), 547–568. zbl

[15] *N. Svanstedt*: G-convergence of parabolic operators. Nonlinear Anal. Theory Methods Appl. *36* (1999), 807–843. zbl

[16] *N. Svanstedt*: Correctors for the homogenization of monotone parabolic operators. J. Nonlinear Math. Phys. *7* (2000), 268–283. zbl

Authors' address: *L. Flodén*, *M. Olsson*, Department of Engineering, Physics and Mathematics, Mid Sweden University, SE-831 25 Östersund, Sweden, e-mails: `lotta.floden@miun.se`, `marianne.olsson@miun.se`.