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ON CONVERGENCE OF GRADIENT-DEPENDENT INTEGRANDS*

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Dedicated to Jiří V. Outrata on the occasion of his 60th birthday

Abstract. We study convergence properties of $\{v(\nabla u_k)\}_{k \in \mathbb{N}}$ if $v \in C(\mathbb{R}^{m \times n})$, $|v(s)| \leq C(1 + |s|^p)$, $1 < p < +\infty$, has a finite quasiconvex envelope, $u_k \rightarrow u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$ and for some $g \in C(\Omega)$ it holds that $\int_{\Omega} g(x)v(\nabla u_k(x)) \, dx \rightarrow \int_{\Omega} g(x)Qv(\nabla u(x)) \, dx$ as $k \rightarrow \infty$. In particular, we give necessary and sufficient conditions for L^1 -weak convergence of $\{\det \nabla u_k\}_{k \in \mathbb{N}}$ to $\det \nabla u$ if $m = n = p$.

Keywords: bounded sequences of gradients, concentrations, oscillations, quasiconvexity, weak convergence

MSC 2000: 49J45, 35B05

1. INTRODUCTION

Oscillations and/or concentrations appear in many problems in the calculus of variations, partial differential equations, or optimal control theory, which admit only L^p but not L^∞ a priori estimates. While Young measures [31] successfully capture oscillatory behavior (see e.g. [17], [23]) of sequences they completely miss concentrations. There are several tools how to deal with concentrations. They can be considered as generalization of Young measures, see for example Alibert's and Bouchitté's approach [1], DiPerna's and Majda's treatment of concentrations [7], or Fonseca's method described in [10]. An overview can be found in [25], [28]. Moreover, in many cases, we are interested in oscillation/concentration effects generated by sequences of gradients. A characterization of Young measures generated by gradients was completely given by Kinderlehrer and Pedregal [14], [16], cf. also [23], [24]. The first attempt to characterize both oscillations and concentrations in sequences

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of gradients is due to Fonseca, Müller, and Pedregal [11]. They dealt with a special situation of $\{gv(\nabla u_k)\}_{k \in \mathbb{N}}$ where v is positively p -homogeneous, $u_k \in W^{1,p}(\Omega; \mathbb{R}^m)$, and g continuous and vanishing on $\partial\Omega$. Later on, a characterization of oscillation/concentration effects in terms of DiPerna's and Majda's generalization of Young measures was given in [13] for arbitrary integrands.

The aim of our paper is to point out a few consequences of this characterization. This leads to a slight generalization of Kinderlehrer's and Pedregal's results on weak convergence of integrands [15]. They proved that if $0 \leq v \leq C(1 + |\cdot|^p)$ is quasiconvex, $u_k \rightarrow u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, and $\int_{\Omega} v(\nabla u_k) dx \rightarrow \int_{\Omega} v(\nabla u) dx$ then possibly for a subsequence $v(\nabla u_k) \rightarrow v(\nabla u)$ weakly in $L^1(\Omega)$. Here we show that if we instead assume that $|v| \leq C(1 + |\cdot|^p)$ and a condition on $\{u_k\}$ which is too involved to be stated here but which is fulfilled e.g. if $u_k = u$ on the boundary we get $v(\nabla u_k) \rightarrow v(\nabla u)$ weakly* in measures on $\bar{\Omega}$; cf. Theorem 2.3 and Corollary 2.4. We also give necessary and sufficient conditions under which a nonnegative sequence of $\{\det \nabla u_k\}$ converges weakly to $\{\det \nabla u\}$ in L^1 if $u_k \rightarrow u$ in $W^{1,n}(\Omega; \mathbb{R}^n)$ for a smooth bounded domain in \mathbb{R}^n . Proposition 2.7 generalizes some results by Müller [22] and Hogan et al. [12]. Finally, we show that while $u \mapsto \int_{\Omega} v(\nabla u) dx$ does not have to be sequentially weakly lower semicontinuous on $W^{1,p}(\Omega; \mathbb{R}^m)$ if $|v| \leq C(1 + |\cdot|^p)$ is quasiconvex, the weak lower semicontinuity can be recovered by removing an arbitrarily thin "boundary layer" of Ω ; cf. Theorem 2.11. The main tool of our analysis is a recently proved characterization of generalized Young measures generated by gradients [13].

1.1. Basic notation

Let us start with a few definitions and with the explanation of our notation. Having a bounded domain $\Omega \subset \mathbb{R}^n$ we denote by $C(\Omega)$ the space of continuous functions: $\Omega \rightarrow \mathbb{R}$. Then $C_0(\Omega)$ consists of functions from $C(\Omega)$ whose support is contained in Ω . In what follows "rca(S)" denotes the set of regular countably additive set functions on the Borel σ -algebra on a metrizable set S (cf. [8]), its subset, $\text{rca}_1^+(S)$, denotes regular probability measures on a set S . We write " γ -almost all" or " γ -a.e." if we mean "up to a set with the γ -measure zero". If γ is the n -dimensional Lebesgue measure and $M \subset \mathbb{R}^n$ we omit writing γ in the notation. Further, $W^{1,p}(\Omega; \mathbb{R}^m)$, $1 \leq p < +\infty$ denotes the usual space of measurable mappings which are together with their first (distributional) derivatives integrable in the p th power. The support of a measure $\sigma \in \text{rca}(\Omega)$ is a smallest closed set S such that $\sigma(A) = 0$ if $S \cap A = \emptyset$. Finally, if $\sigma \in \text{rca}(S)$ we write σ_s and d_σ for the singular part and density of σ defined by the Lebesgue decomposition, respectively. Finally, we denote by "w-lim" the weak limit.

If not said otherwise, we will suppose in the sequel that $\Omega \subset \mathbb{R}^n$ is a bounded domain with a Lipschitz boundary. Some generalizations to less regular domains are possible, however they seem to be technically much more involved.

1.2. Quasiconvex functions

Let $\Omega \subset \mathbb{R}^n$ be a bounded regular domain. We say that a function $v: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is quasiconvex [21] if for any $s_0 \in \mathbb{R}^{m \times n}$ and any $\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)$

$$v(s_0)|\Omega| \leq \int_{\Omega} v(s_0 + \nabla\varphi(x)) \, dx.$$

If $v: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ is not quasiconvex we define its quasiconvex envelope $Qv: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ as

$$Qv = \sup\{h \leq v; h: \mathbb{R}^{m \times n} \rightarrow \mathbb{R} \text{ quasiconvex}\}$$

and if the set on the right-hand side is empty we put $Qv = -\infty$. If v is locally bounded and Borel measurable then for any $s_0 \in \mathbb{R}^{m \times n}$ (see [6])

$$(1.1) \quad Qv(s_0) = \inf_{\varphi \in W_0^{1,\infty}(\Omega; \mathbb{R}^m)} \frac{1}{|\Omega|} \int_{\Omega} v(s_0 + \nabla\varphi(x)) \, dx.$$

1.3. Young measures

For $p \geq 0$ we define the following subspace of the space $C(\mathbb{R}^{m \times n})$ of all continuous functions on $\mathbb{R}^{m \times n}$:

$$C_p(\mathbb{R}^{m \times n}) = \{v \in C(\mathbb{R}^{m \times n}); v(s) = o(|s|^p) \text{ for } |s| \rightarrow \infty\}.$$

The Young measures on a bounded domain $\Omega \subset \mathbb{R}^n$ are weakly* measurable mappings $x \mapsto \nu_x: \Omega \rightarrow \text{rca}(\mathbb{R}^{m \times n})$ with values in probability measures; and the adjective “weakly* measurable” means that, for any $v \in C_0(\mathbb{R}^{m \times n})$, the mapping $\Omega \rightarrow \mathbb{R}: x \mapsto \langle \nu_x, v \rangle = \int_{\mathbb{R}^{m \times n}} v(\lambda) \nu_x(d\lambda)$ is measurable in the usual sense. Let us remind that, by the Riesz theorem, $\text{rca}(\mathbb{R}^{m \times n})$, normed by the total variation, is a Banach space which is isometrically isomorphic with $C_0(\mathbb{R}^{m \times n})^*$, where $C_0(\mathbb{R}^{m \times n})$ stands for the space of all continuous functions $\mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ vanishing at infinity. Let us denote the set of all Young measures by $\mathcal{Y}(\Omega; \mathbb{R}^{m \times n})$. It is known that $\mathcal{Y}(\Omega; \mathbb{R}^{m \times n})$ is a convex subset of $L_w^\infty(\Omega; \text{rca}(\mathbb{R}^{m \times n})) \cong L^1(\Omega; C_0(\mathbb{R}^{m \times n}))^*$, where the subscript “w” indicates the property “weakly* measurable”. A classical result [27], [30] is that, for every sequence $\{y_k\}_{k \in \mathbb{N}}$ bounded in $L^\infty(\Omega; \mathbb{R}^{m \times n})$, there exists its subsequence (denoted by the same indices for notational simplicity) and a Young measure $\nu = \{\nu_x\}_{x \in \Omega} \in \mathcal{Y}(\Omega; \mathbb{R}^{m \times n})$ such that

$$(1.2) \quad \forall v \in C_0(\mathbb{R}^{m \times n}): \lim_{k \rightarrow \infty} v \circ y_k = v_\nu \text{ weakly* in } L^\infty(\Omega),$$

where $[v \circ y_k](x) = v(y_k(x))$ and

$$(1.3) \quad v_\nu(x) = \int_{\mathbb{R}^{m \times n}} v(\lambda) \nu_x(d\lambda).$$

Let us denote by $\mathcal{Y}^\infty(\Omega; \mathbb{R}^{m \times n})$ the set of all Young measures which are created in this way, i.e. by taking all bounded sequences in $L^\infty(\Omega; \mathbb{R}^{m \times n})$. Note that (1.2) actually holds for any $v: \mathbb{R}^{m \times n} \rightarrow \mathbb{R}$ continuous.

A generalization of this result was formulated by Schonbek [26] (cf. also [2]): if $1 \leq p < +\infty$, for every sequence $\{y_k\}_{k \in \mathbb{N}}$ bounded in $L^p(\Omega; \mathbb{R}^{m \times n})$ there exists its subsequence (denoted by the same indices) and a Young measure $\nu = \{\nu_x\}_{x \in \Omega} \in \mathcal{Y}(\Omega; \mathbb{R}^{m \times n})$ such that

$$(1.4) \quad \forall v \in C_p(\mathbb{R}^{m \times n}): \lim_{k \rightarrow \infty} v \circ y_k = v_\nu \text{ weakly in } L^1(\Omega).$$

We say that $\{y_k\}$ generates ν if (1.4) holds.

Let us denote by $\mathcal{Y}^p(\Omega; \mathbb{R}^{m \times n})$ the set of all Young measures which are created in this way, i.e. by taking all bounded sequences in $L^p(\Omega; \mathbb{R}^{m \times n})$.

1.4. DiPerna-Majda measures

Let us take a complete (i.e. containing constants, separating points from closed subsets and closed with respect to the Chebyshev norm) separable ring \mathcal{R} of continuous bounded functions $\mathbb{R}^{m \times n} \rightarrow \mathbb{R}$. It is known [9, Sect. 3.12.21] that there is a one-to-one correspondence $\mathcal{R} \mapsto \beta_{\mathcal{R}} \mathbb{R}^{m \times n}$ between such rings and metrizable compactifications of $\mathbb{R}^{m \times n}$; by a compactification we mean here a compact set, denoted by $\beta_{\mathcal{R}} \mathbb{R}^{m \times n}$, into which $\mathbb{R}^{m \times n}$ is embedded homeomorphically and densely. For simplicity, we will not distinguish between $\mathbb{R}^{m \times n}$ and its image in $\beta_{\mathcal{R}} \mathbb{R}^{m \times n}$. Similarly, we will not distinguish between elements of \mathcal{R} and their unique continuous extensions to $\beta_{\mathcal{R}} \mathbb{R}^{m \times n}$.

Let $\sigma \in \text{rca}(\bar{\Omega})$ be a positive Radon measure on a bounded domain $\Omega \subset \mathbb{R}^n$. A mapping $\hat{\nu}: x \mapsto \hat{\nu}_x$ belongs to the space $L_w^\infty(\bar{\Omega}, \sigma; \text{rca}(\beta_{\mathcal{R}} \mathbb{R}^{m \times n}))$ if it is weakly* σ -measurable (i.e., for any $v_0 \in C_0(\mathbb{R}^{m \times n})$, the mapping $\bar{\Omega} \rightarrow \mathbb{R}: x \mapsto \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n}} v_0(s) \hat{\nu}_x(ds)$ is σ -measurable in the usual sense). If additionally $\hat{\nu}_x \in \text{rca}_1^+(\beta_{\mathcal{R}} \mathbb{R}^{m \times n})$ for σ -a.a. $x \in \bar{\Omega}$ the collection $\{\hat{\nu}_x\}_{x \in \bar{\Omega}}$ is the so-called Young measure on $(\bar{\Omega}, \sigma)$ [31], see also [2], [25], [27], [29], [30].

DiPerna and Majda [7] showed that having a bounded sequence in $L^p(\Omega; \mathbb{R}^{m \times n})$ with $1 \leq p < +\infty$ and Ω an open domain in \mathbb{R}^n , there exists its subsequence (denoted by the same indices) a positive Radon measure $\sigma \in \text{rca}(\bar{\Omega})$ and a Young measure $\hat{\nu}: x \mapsto \hat{\nu}_x$ on $(\bar{\Omega}, \sigma)$ such that $(\sigma, \hat{\nu})$ is attainable by a sequence $\{y_k\}_{k \in \mathbb{N}} \subset$

$L^p(\Omega; \mathbb{R}^{m \times n})$ in the sense that $\forall g \in C(\bar{\Omega}) \forall v_0 \in \mathcal{R}$:

$$(1.5) \quad \lim_{k \rightarrow \infty} \int_{\Omega} g(x)v(y_k(x)) \, dx = \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n}} g(x)v_0(s)\hat{\nu}_x(ds)\sigma(dx),$$

where

$$v \in \Upsilon_{\mathcal{R}}^p(\mathbb{R}^{m \times n}) := \{v_0(1 + |\cdot|^p); v_0 \in \mathcal{R}\}.$$

In particular, putting $v_0 = 1 \in \mathcal{R}$ in (1.5) we can see that

$$(1.6) \quad \lim_{k \rightarrow \infty} (1 + |y_k|^p) = \sigma \quad \text{weakly}^* \text{ in } \text{rca}(\bar{\Omega}).$$

If (1.5) holds, we say that $\{y_k\}_{k \in \mathbb{N}}$ generates $(\sigma, \hat{\nu})$. Let us denote by $\mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ the set of all pairs $(\sigma, \hat{\nu}) \in \text{rca}(\bar{\Omega}) \times L_{\mathbb{w}}^{\infty}(\bar{\Omega}, \sigma; \text{rca}(\beta_{\mathcal{R}} \mathbb{R}^{m \times n}))$ attainable by sequences from $L^p(\Omega; \mathbb{R}^{m \times n})$; note that, taking $v_0 = 1$ in (1.5), one can see that these sequences must inevitably be bounded in $L^p(\Omega; \mathbb{R}^{m \times n})$. The explicit description of the elements from $\mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$, called DiPerna-Majda measures, for unconstrained sequences was done in [19, Theorem 2].

Let us recall that for any $(\sigma, \hat{\nu}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ there is precisely one $(\sigma^{\circ}, \hat{\nu}^{\circ}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ such that

$$(1.7) \quad \int_{\Omega} \int_{\mathbb{R}^{m \times n}} v_0(s)\hat{\nu}_x(ds)g(x)\sigma(dx) = \int_{\bar{\Omega}} \int_{\mathbb{R}^{m \times n}} v_0(s)\hat{\nu}_x^{\circ}(ds)g(x)\sigma^{\circ}(dx)$$

for any $v_0 \in C_0(\mathbb{R}^{m \times n})$ and any $g \in C(\bar{\Omega})$ and $(\sigma^{\circ}, \hat{\nu}^{\circ})$ is attainable by a sequence $\{y_k\}_{k \in \mathbb{N}}$ such that the set $\{|y_k|^p; k \in \mathbb{N}\}$ is relatively weakly compact in $L^1(\Omega)$; see [19], [25] for details. We call $(\sigma^{\circ}, \hat{\nu}^{\circ})$ the nonconcentrating modification of $(\sigma, \hat{\nu})$. We call $(\sigma, \hat{\nu}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ nonconcentrating if

$$\int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \hat{\nu}_x(ds)\sigma(dx) = 0.$$

There is a one-to-one correspondence between nonconcentrating DiPerna-Majda measures and Young measures; cf. [25].

We wish to emphasize the following fact: if $\{y_k\} \in L^p(\Omega; \mathbb{R}^{m \times n})$ generates $(\sigma, \hat{\nu}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ and σ is absolutely continuous with respect to the Lebesgue measure it generally *does not* mean that $\{|y_k|^p\}$ is weakly relatively compact in $L^1(\Omega)$. Simple examples can be found e.g. in [20], [25].

Having a sequence bounded in $L^p(\Omega; \mathbb{R}^{m \times n})$ generating a DiPerna-Majda measure $(\sigma, \hat{\nu}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ it also generates an L^p -Young measure $\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^{m \times n})$. It easily follows from [25, Theorem 3.2.13] that

$$(1.8) \quad \nu_x(ds) = d_{\sigma^{\circ}}(x) \frac{\hat{\nu}_x^{\circ}(ds)}{1 + |s|^p} \quad \text{for a.a. } x \in \Omega.$$

Note that (1.8) is well-defined as $\hat{\nu}_x^\circ$ is supported on $\mathbb{R}^{m \times n}$. As pointed out in [19, Remark 2] for almost all $x \in \Omega$

$$(1.9) \quad d_\sigma(x) = \left(\int_{\mathbb{R}^{m \times n}} \frac{\hat{\nu}_x(ds)}{1 + |s|^p} \right)^{-1}.$$

In fact, that (1.7) can even be improved to

$$(1.10) \quad \int_{\Omega} \int_{\mathbb{R}^{m \times n}} v_0(s) \hat{\nu}_x(ds) g(x) \sigma(dx) = \int_{\bar{\Omega}} \int_{\mathbb{R}^{m \times n}} v_0(s) \hat{\nu}_x^\circ(ds) g(x) \sigma^\circ(dx)$$

for any $v_0 \in \mathcal{R}$ and any $g \in C(\bar{\Omega})$. The one-to-one correspondence between Young and DiPerna-Majda measures, in particular (see (1.8) and (1.10))

$$\int_{\mathbb{R}^{m \times n}} v(s) \nu_x(ds) = d_\sigma(x) \int_{\mathbb{R}^{m \times n}} v_0(s) \hat{\nu}_x(ds)$$

whenever $v \in \Upsilon_{\mathcal{R}}^p(\mathbb{R}^{m \times n})$, finally yields that $\forall g \in C(\bar{\Omega}) \forall v \in \Upsilon_{\mathcal{R}}^p(\mathbb{R}^{m \times n})$:

$$(1.11) \quad \lim_{k \rightarrow \infty} \int_{\Omega} g(x) v(y_k(x)) dx = \int_{\Omega} \int_{\mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) d_\sigma(x) g(x) dx \\ + \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) g(x) \sigma(dx),$$

where $\nu \in \mathcal{Y}^p(\Omega; \mathbb{R}^{m \times n})$ and $(\sigma, \hat{\nu}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ are Young and DiPerna-Majda measures generated by $\{y_k\}_{k \in \mathbb{N}}$, respectively. We will denote the elements from $\mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ which are generated by $\{\nabla u_k\}_{k \in \mathbb{N}}$ for some bounded $\{u_k\} \subset W^{1,p}(\Omega; \mathbb{R}^m)$ by $\mathcal{GDM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$.

The following proposition from [19] explicitly characterizes elements of $\mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$.

Proposition 1.1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open domain, \mathcal{R} a separable complete subring of the ring of all continuous bounded functions on $\mathbb{R}^{m \times n}$ and $(\sigma, \hat{\nu}) \in \text{rca}(\bar{\Omega}) \times L_w^\infty(\bar{\Omega}, \sigma; \text{rca}(\beta_{\mathcal{R}} \mathbb{R}^{m \times n}))$ and $1 \leq p < +\infty$. Then the following two statements are equivalent to each other:*

- (i) *the pair $(\sigma, \hat{\nu})$ is a DiPerna-Majda measure, i.e. $(\sigma, \hat{\nu}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$,*
- (ii) *the following properties are satisfied simultaneously:*
 1. *σ is positive,*
 2. *$\sigma_{\hat{\nu}} \in \text{rca}(\bar{\Omega})$ defined by $\sigma_{\hat{\nu}}(dx) = (\int_{\mathbb{R}^{m \times n}} \hat{\nu}_x(ds)) \sigma(dx)$ is absolutely continuous with respect to the Lebesgue measure ($d_{\sigma_{\hat{\nu}}}$ will denote its density),*

3. for a.a. $x \in \Omega$ it holds

$$\int_{\mathbb{R}^{m \times n}} \hat{\nu}_x(ds) > 0, \quad d_{\sigma_{\hat{\nu}}}(x) = \left(\int_{\mathbb{R}^{m \times n}} \frac{\hat{\nu}_x(ds)}{1 + |s|^p} \right)^{-1} \int_{\mathbb{R}^{m \times n}} \hat{\nu}_x(ds),$$

4. for σ -a.a. $x \in \bar{\Omega}$ it holds

$$\hat{\nu}_x \geq 0, \quad \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n}} \hat{\nu}_x(ds) = 1.$$

The following two theorems were proved in [13].

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with Lipschitz boundary, $1 < p < +\infty$ and $(\sigma, \hat{\nu}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$. Then there is $u \in W^{1,p}(\Omega; \mathbb{R}^m)$ and a bounded sequence $\{u_k - u\}_{k \in \mathbb{N}} \subset W_0^{1,p}(\Omega; \mathbb{R}^m)$ such that $\{\nabla u_k\}_{k \in \mathbb{N}}$ generates $(\sigma, \hat{\nu})$ if and only if the following three conditions hold:*

$$(1.12) \quad \text{for a.a. } x \in \Omega : \nabla u(x) = d_{\sigma}(x) \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n}} \frac{s}{1 + |s|^p} \hat{\nu}_x(ds),$$

for almost all $x \in \Omega$ and for all $v \in \Upsilon_{\mathcal{R}}^p(\mathbb{R}^{m \times n})$ the following inequality is fulfilled

$$(1.13) \quad Qv(\nabla u(x)) \leq d_{\sigma}(x) \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds),$$

and for σ -almost all $x \in \bar{\Omega}$ and all $v \in \Upsilon_{\mathcal{R}}^p(\mathbb{R}^{m \times n})$ with $Qv > -\infty$ it holds that

$$(1.14) \quad 0 \leq \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds).$$

The next theorem addresses DiPerna-Majda measures generated by gradients of maps with possibly different traces.

Theorem 1.3. *Let Ω be an arbitrary bounded domain, $1 < p < +\infty$ and $(\sigma, \hat{\nu}) \in \mathcal{GDM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ be generated by $\{\nabla u_k\}_{k \in \mathbb{N}}$ such that $w\text{-}\lim_{k \rightarrow \infty} u_k = u$ in $W^{1,p}(\Omega; \mathbb{R}^m)$. Then the conditions (1.12), (1.13) hold, and (1.14) is satisfied for σ -a.a. $x \in \Omega$.*

Remark 1.4. (i) It can happen that under the assumptions of Theorem 1.3 the formula (1.14) does not hold on $\partial\Omega$. See an example in [3] showing the violation of weak sequential continuity of $W^{1,2}(\Omega; \mathbb{R}^2) \rightarrow L^1(\Omega): u \mapsto \det \nabla u$ if $\Omega = (-1, 1)^2$.

(ii) On the other hand, having $1 < p < +\infty$ we can ask what condition besides quasiconvexity must $v \in C(\mathbb{R}^{2 \times 2})$, $|v| \leq C(1 + |\cdot|^p)$ satisfy so that $W^{1,p}(\Omega; \mathbb{R}^2) \rightarrow \mathbb{R}: u \mapsto I(u) := \int_{\Omega} v(\nabla u(x)) \, dx$ is sequentially weakly lower semicontinuous. Suppose that $v \in C(\mathbb{R}^{2 \times 2})$ is positively p -homogeneous, i.e. $v(\lambda s) = \lambda^p v(s)$ for all $s \in \mathbb{R}^{2 \times 2}$, $\lambda \geq 0$, and take arbitrary $u \in W_0^{1,p}((-1, 1)^2; \mathbb{R}^2)$ and extend it by zero to \mathbb{R}^2 . Define further for all $k \in \mathbb{N}$ $u_k(x) = k^{2/p-1}u(kx)$. Then $\nabla u_k(x) = k^{2/p}\nabla u(kx)$. Clearly, $u_k \rightarrow 0$ weakly in $W^{1,p}((-1, 1)^2; \mathbb{R}^2)$. Take $\Omega := (-1, 0)^2$. A simple calculation shows that if I is weakly lower semicontinuous then for all $u \in W_0^{1,p}((-1, 1)^2; \mathbb{R}^2)$ it holds

$$(1.15) \quad \int_{(-1,0)^2} v(\nabla u(x)) \, dx \geq 0.$$

2. CONVERGENCE OF INTEGRANDS

We start with a remark on (1.14).

Remark 2.1. Condition (1.14) for $x \in \partial\Omega$ is satisfied if $\{u_k - u\} \subset W_0^{1,p}(\Omega; \mathbb{R}^m)$ but the same trace for all terms in the sequence is far from being necessary for (1.14) to hold on $\partial\Omega$. As Ω is supposed to be Lipschitz $\{u_k\}$ can be extended to $\{\tilde{u}_k\}_{k \in \mathbb{N}} \subset W^{1,p}(B; \mathbb{R}^m)$ for some ball $\mathbb{R}^n \supset B \supset \Omega$. Moreover, $\{\tilde{u}_k\}$ is uniformly bounded in $W^{1,p}(B; \mathbb{R}^m)$. Then (1.14) holds if $\{|\nabla \tilde{u}_k|^p\}$ is weakly relatively compact in $L^1(B \setminus \Omega)$. If $\{u_k\}$ satisfies this condition for some ball $B \subset \mathbb{R}^n$ we say that it has a p -equiintegrable extension.

We put

$$(2.1) \quad V_p = \{v \in C(\mathbb{R}^{m \times n}); \exists C > 0: |v| \leq C(1 + |\cdot|^p), Qv > -\infty\}.$$

As shown in [18] if $v \in V_p$ then $Qv \in V_p$ as well.

Theorem 2.2. *Let $v \in V_p$ and $u_k \rightarrow u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $1 < p < +\infty$. Suppose that there is $g_0 \in C(\bar{\Omega})$, $g_0(x) = 0$ if $x \in \partial\Omega$ and $g_0(x) > 0$ if $x \in \Omega$ such that for $k \rightarrow \infty$*

$$(2.2) \quad \int_{\Omega} g_0(x)v(\nabla u_k(x)) \, dx \rightarrow \int_{\Omega} g_0(x)Qv(\nabla u(x)) \, dx.$$

Then there is a subsequence of $\{u_k\}_{k \in \mathbb{N}}$ (not relabeled) such that for $k \rightarrow \infty$

$$(2.3) \quad v(\nabla u_k) \rightarrow Qv(\nabla u) \quad \text{weakly}^* \text{ in } \text{rca}(\Omega),$$

i.e., we can replace g_0 in (2.2) by all $g \in C_0(\Omega)$.

Proof. We take a separable subring \mathcal{R} such that $v/(1 + |\cdot|^p) \in \mathcal{R}$. As noted in [13], this is always possible. Taking a subsequence of $\{\nabla u_k\}$ generating $(\sigma, \hat{\nu}) \in \mathcal{GDM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ and exploiting Theorem 1.3 we have by (2.2) and (1.11) that for g_0

$$\begin{aligned} \int_{\Omega} g_0(x) Qv(\nabla u(x)) \, dx &= \lim_{k \rightarrow \infty} \int_{\Omega} g_0(x) v(\nabla u_k(x)) \, dx \\ &= \int_{\Omega} \int_{\mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) d_{\sigma}(x) g_0(x) \, dx \\ &\quad + \int_{\Omega} \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) g_0(x) \sigma(dx) \\ &\geq \int_{\Omega} Qv(\nabla u(x)) g_0(x) \, dx \\ &\quad + \int_{\Omega} \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) g_0(x) \sigma(dx) \\ &\geq \int_{\Omega} g_0(x) Qv(\nabla u(x)) \, dx. \end{aligned}$$

Hence,

$$\int_{\Omega} \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) g_0(x) \sigma(dx) = 0$$

and because $g_0 > 0$ in Ω and for σ -a.a. $x \in \Omega$

$$\int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) \geq 0$$

by (1.14) we get that for σ -a.a. $x \in \Omega$

$$(2.4) \quad \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) = 0.$$

Moreover, as $g_0 > 0$ in Ω we get that for a.a. $x \in \Omega$

$$(2.5) \quad d_{\sigma}(x) \int_{\mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) = Qv(\nabla u(x)).$$

The assertion (2.3) follows by (2.4), (2.5), and (1.11). □

Theorem 2.3. Let $v \in V_p$ and $u_k \rightarrow u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $1 < p < +\infty$. Suppose that there is $g_0 \in C(\bar{\Omega})$, $g_0 > 0$ such that for $k \rightarrow \infty$

$$(2.6) \quad \int_{\Omega} g_0(x)v(\nabla u_k(x)) \, dx \rightarrow \int_{\Omega} g_0(x)Qv(\nabla u(x)) \, dx.$$

Suppose further that $\{\nabla u_k\}$ generates a DiPerna-Majda measure $(\sigma, \hat{\nu}) \in \mathcal{GDM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$ such that for σ -a.a. $x \in \partial\Omega$

$$(2.7) \quad \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{v(s)}{1 + |s|^p} \hat{\nu}_x(ds) \geq 0.$$

Then

$$(2.8) \quad v(\nabla u_k) \rightarrow Qv(\nabla u)$$

weakly* in $\text{rca}(\bar{\Omega})$ if $k \rightarrow \infty$, i.e., (2.6) holds for all $g \in C(\bar{\Omega})$ in place of g_0 .

Proof. The proof is almost the same as the proof of Theorem 2.2. Notice that due to (2.7) and Theorem 1.3 formula (1.4) holds for σ -a.a. $x \in \bar{\Omega}$. \square

In view of Remark 2.1 and Theorem 1.2 we have the following consequence of Theorem 2.3.

Corollary 2.4. Let $v \in V_p$ and $u_k \rightarrow u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $1 < p < +\infty$, $\{u_k - u\}_{k \in \mathbb{N}} \subset W_0^{1,p}(\Omega; \mathbb{R}^m)$. Suppose that

$$(2.9) \quad \int_{\Omega} v(\nabla u_k(x)) \, dx \rightarrow \int_{\Omega} Qv(\nabla u(x)) \, dx.$$

Then there is a subsequence of $\{u_k\}$ (not relabeled) such that

$$(2.10) \quad v(\nabla u_k) \rightarrow Qv(\nabla u) \quad \text{weakly* in } \text{rca}(\bar{\Omega}).$$

The convergence in $\text{rca}(\bar{\Omega})$ can be strengthened if $v(\nabla u_k) \geq 0$ for all $k \in \mathbb{N}$. The following theorem was proved for $v \geq 0$ quasiconvex and $g_0 = 1$ in [15].

Theorem 2.5. Let $v \in V_p$, $u_k \rightarrow u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$, $1 < p < +\infty$, and for all $k \in \mathbb{N}$ let $v(\nabla u_k) \geq 0$ almost everywhere in Ω . Suppose that there is $g_0 \in C(\bar{\Omega})$, $g_0 > 0$ such that for $k \rightarrow \infty$

$$(2.11) \quad \int_{\Omega} g_0(x)v(\nabla u_k(x)) \, dx \rightarrow \int_{\Omega} g_0(x)Qv(\nabla u(x)) \, dx.$$

Then there is a subsequence of $\{\nabla u_k\}$ (not relabeled) such that

$$(2.12) \quad v(\nabla u_k) \rightarrow Qv(\nabla u)$$

weakly in $L^1(\Omega)$ if $k \rightarrow \infty$, i.e., (2.11) holds for all $g \in L^\infty(\Omega)$ in place of g_0 .

We will need the following lemma.

Lemma 2.6. Let $f \in V_p$, $f \geq 0$, and $f/(1 + |\cdot|^p) \in \mathcal{R}$. Let further $\{y_k\}_{k \in \mathbb{N}} \subset L^p(\Omega; \mathbb{R}^{m \times n})$ generate $(\sigma, \hat{\nu}) \in \mathcal{DM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$. Then $\{f(y_k)\}_{k \in \mathbb{N}}$ is weakly relatively compact in $L^1(\Omega)$ if and only if

$$(2.13) \quad \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{f(s)}{1 + |s|^p} \hat{\nu}_x(ds) \sigma(dx) = 0.$$

Proof. This lemma was proved in [25, Lemma 3.2.14 (i)] for $f = 1 + |\cdot|^p$. The proof for general f is analogous. \square

Proof of Theorem 2.5. As $v(\nabla u_k) \geq 0$ we can replace v by $|v|$ and take \mathcal{R} such that $|v|/(1 + |\cdot|^p) \in \mathcal{R}$. Moreover, $Q|v| \geq Qv$ because $|v| \geq v$.

We have for a subsequence of $\{\nabla u_k\}$ (not relabeled) generating $(\sigma, \hat{\nu})$:

$$\begin{aligned} \int_{\Omega} g_0(x)Qv(\nabla u(x)) \, dx &= \lim_{k \rightarrow \infty} \int_{\Omega} g_0(x)|v|(\nabla u_k(x)) \, dx \\ &= \int_{\Omega} \int_{\mathbb{R}^{m \times n}} \frac{|v|(s)}{1 + |s|^p} \hat{\nu}_x(ds) d_{\sigma}(x) g_0(x) \, dx \\ &\quad + \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{|v|(s)}{1 + |s|^p} \hat{\nu}_x(ds) g_0(x) \sigma(dx) \\ &\geq \int_{\Omega} Qv(\nabla u(x)) g_0(x) \, dx \\ &\quad + \int_{\bar{\Omega}} \int_{\beta_{\mathcal{R}} \mathbb{R}^{m \times n} \setminus \mathbb{R}^{m \times n}} \frac{|v|(s)}{1 + |s|^p} \hat{\nu}_x(ds) g_0(x) \sigma(dx) \\ &\geq \int_{\Omega} g_0(x)Qv(\nabla u(x)) \, dx. \end{aligned}$$

Taking into account that $g_0 > 0$ and Lemma 2.6 we get that $\{|v|(\nabla u_k)\} = \{v(\nabla u_k)\}$ is weakly relatively compact in $L^1(\Omega)$. Applying [24, Theorem 6.2] we conclude that a subsequence has the weak limit $Qv(\nabla u)$. \square

2.1. Applications to determinants

Suppose now that $p = n > 1$ and consider $v = \det$. Clearly, $v \in V_n$ and because the determinant is quasilinear (i.e. \det as well as $-\det$ are both quasiconvex) (1.13) as well as (1.14) hold as equalities.

Combining (1.14) with Lemma 2.6 and Theorem 2.5 we have the following proposition.

Proposition 2.7. *Let $u_k \rightarrow u$ weakly in $W^{1,n}(\Omega; \mathbb{R}^n)$, let $\{\nabla u_k\}_{k \in \mathbb{N}}$ generate $(\sigma, \hat{\nu}) \in \mathcal{GDM}^n(\Omega; \mathbb{R}^{n \times n})$, and let $\det / (1 + |\cdot|^n) \in \mathcal{R}$. Let for all $k \in \mathbb{N}$ $\det \nabla u_k \geq 0$ almost everywhere in Ω . Then $\{\det \nabla u_k\}_{k \in \mathbb{N}}$ is weakly relatively compact in $L^1(\Omega)$ if and only if*

$$(2.14) \quad \int_{\partial\Omega} \int_{\beta_{\mathcal{R}} \mathbb{R}^{n \times n} \setminus \mathbb{R}^{n \times n}} \frac{\det s}{1 + |s|^n} \hat{\nu}_x(ds) \sigma(dx) = 0.$$

If (2.14) holds then

$$\det \nabla u_k \rightarrow \det \nabla u \quad \text{weakly in } L^1(\Omega).$$

Notice that (2.14) is satisfied if $\sigma(\partial\Omega) = 0$, if $u_k = u$ on $\partial\Omega$, or if $\{\nabla u_k\}$ has a p -equiintegrable extension; cf. Theorem 1.2. Hence, Proposition 2.7 generalizes [15, Theorem 4.1] and [22, Corollary 1.2].

Dropping the requirement $\det \nabla u_k \geq 0$ we have the following consequence of Theorem 2.3.

Proposition 2.8. *Let $u_k \rightarrow u$ weakly in $W^{1,n}(\Omega; \mathbb{R}^n)$, let $\{\nabla u_k\}_{k \in \mathbb{N}}$ generate $(\sigma, \hat{\nu}) \in \mathcal{GDM}^n(\Omega; \mathbb{R}^{n \times n})$, and let $\det / (1 + |\cdot|^n) \in \mathcal{R}$. If (2.14) holds then*

$$(2.15) \quad \det \nabla u_k \rightarrow \det \nabla u \quad \text{weakly}^* \text{ in } \text{rca}(\bar{\Omega}).$$

In particular, (2.15) is satisfied if $u_k = u$ on $\partial\Omega$, if $\sigma(\partial\Omega) = 0$, or if $\{\nabla u_k\}$ has a p -equiintegrable extension.

Finally, we may even give up (2.14) to hold and by Theorem 1.6 with $g_0 = \text{dist}(\cdot, \partial\Omega)$ we obtain the following fact mentioned already in [3].

Proposition 2.9. *Let $u_k \rightarrow u$ weakly in $W^{1,n}(\Omega; \mathbb{R}^n)$. Then possibly for a subsequence it holds*

$$(2.16) \quad \det \nabla u_k \rightarrow \det \nabla u \quad \text{weakly}^* \text{ in } \text{rca}(\Omega).$$

2.2. Biting lemma for quasiconvex integrands

It is well known that boundedness of a sequence in $L^1(\Omega; \mathbb{R}^m)$ is not sufficient for the existence of a weakly converging subsequence. Nevertheless, Brooks and Chacon showed that removing nested sets of vanishing Lebesgue measure we can recover weak L^1 convergence.

Lemma 2.10 ([5]). *Let $\{y_k\} \subset L^1(\Omega; \mathbb{R}^{m \times n})$ be bounded. Then there is $y \in L^1(\Omega; \mathbb{R}^{m \times n})$ and measurable sets $\{\Omega_j\}_{j \in \mathbb{N}}$, $\Omega_{j+1} \subset \Omega_j \subset \Omega$, $j \in \mathbb{N}$, $|\Omega_j| \rightarrow 0$ such that for $k \rightarrow \infty$*

$$y_k \rightarrow y \quad \text{weakly in } L^1(\Omega \setminus \Omega_j) \text{ for every fixed } j \in \mathbb{N}.$$

Ball and Zhang [4] showed that if $v \in C(\mathbb{R}^{m \times n})$, $|v| \leq C(1 + |\cdot|^p)$ is quasiconvex and $u_k \rightarrow u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$ then there are measurable sets $\{\Omega_j\}_{j \in \mathbb{N}}$, $\Omega_{j+1} \subset \Omega_j \subset \Omega$, $j \in \mathbb{N}$, $|\Omega_j| \rightarrow 0$ such that for any $j \in \mathbb{N}$

$$\liminf_{k \rightarrow \infty} \int_{\Omega_j} v(\nabla u_k(x)) \, dx \geq \int_{\Omega_j} v(\nabla u(x)) \, dx.$$

The next theorem shows that Ω_j can be chosen as arbitrarily thin “boundary layers” of Ω .

Theorem 2.11. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with a Lipschitz boundary such that $0 \in \Omega$. Let $u_k \rightarrow u$ weakly in $W^{1,p}(\Omega; \mathbb{R}^m)$ and let $v \in V_p$. Then for every $0 < \varepsilon < 1$ there is a subsequence of $\{u_k\}$ which we do not relabel and $\varepsilon < \delta < 1$ such that*

$$(2.17) \quad \lim_{k \rightarrow \infty} \int_{\delta\Omega} v(\nabla u_k(x)) \, dx \geq \int_{\delta\Omega} Qv(\nabla u(x)) \, dx,$$

where $\delta\Omega = \{y \in \mathbb{R}^n; y/\delta \in \Omega\}$.

Proof. Take $\varepsilon \in]0, 1[$. We will assume that $\{\nabla u_k\}$ generates $(\sigma, \hat{\nu}) \in \mathcal{GDM}_{\mathcal{R}}^p(\Omega; \mathbb{R}^{m \times n})$. Notice that $\delta\Omega \subset \Omega$. Using [13, Lemma 3.6] we get that $\sigma(\partial\delta\Omega) > 0$ only for at most countably many values of δ . Thus we take $\delta > \varepsilon$ such that $\sigma(\partial\delta\Omega) = 0$. Then using [13, Lemma 3.2] we have that the restriction of $\{u_k\}$ on $\delta\Omega$ has the property that $\{\nabla u_k|_{\delta\Omega}\}$ generates $(\sigma, \hat{\nu})|_{\delta\Omega}$. Then (2.17) follows by (1.11) in view of Theorem 1.2. \square

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