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SOME SPECTRAL PROPERTIES OF THE STREAMING OPERATOR WITH GENERAL BOUNDARY CONDITIONS

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Abstract. This paper deals with the spectral study of the streaming operator with general boundary conditions defined by means of a boundary operator K. We study the positivity and the irreducibility of the generated semigroup proved in [M. Boulanouar, L'opérateur d'Advection: existence d'un C_0-semi-groupe (I), Transp. Theory Stat. Phys. *31*, 2002, 153–167], in the case $\|K\| \geqslant 1$. We also give some spectral properties of the streaming operator and we characterize the type of the generated semigroup in terms of the solution of a characteristic equation.

Keywords: compactness, essential type, positivity and irreducibility, spectral properties, streaming operator, strongly continuous semigroups

MSC 2000: 47D06

1. Introduction

In this paper, we are concerned with the study of some properties of the streaming operator with general boundary conditions. Let $X \subset \mathbb{R}^n$ be a smoothly bounded open subset and $\mathrm{d}\mu$ be a Radon measure on $\mathbb{R}^n$ with bounded support V. In this context, we introduce the streaming operator by

$$T_K\varphi(x,v) = -v\cdot\nabla_x\varphi(x,v), \quad (x,v)\in X\times V, \tag{1.1}$$

supplemented with the following boundary conditions

$$\varphi_{|\Gamma_-} = K(\varphi_{|\Gamma_+}), \tag{1.2}$$

where $\varphi_{|\Gamma_+}$ and $\varphi_{|\Gamma_-}$ present respectively the outgoing and the incoming fluxes and K is a bounded linear operator. These boundary conditions generalize in a natural way all well-known boundary conditions such as vacuum, specularly reflective, diffusely reflective and periodic boundary conditions. For the convenience of the reader and more explanations, we refer for instance to [2], [7, Chapter XI and XII], [8, Chapter 21], [13] and [14].

The existence of a strongly continuous semigroup generated by the streaming operator has been investigated by several authors and several important results have been clarified. Beals and Protopopescu [2] and Greenberg, van der Mee and Protopopescu [7] have given two approaches (method of characteristics and semigroup method) to discuss the case $\|K\| < 1$. For the same case, the semigroup approach has been also used by Dautray and Lions [8, Chapter 21], Sentis [12], Ukai [13] and Voigt [14]. The case $\|K\| = 1$ with $K \geqslant 0$ has been treated in [2], [7], [13] by approximating from the case $\|K\| < 1$. The case $\|K\| > 1$ has been studied by Borgioli and Totaro [4] for two-dimensional spatial domains and in [3] for three-dimensional spatial domains. We have also studied the case $\|K\| \geqslant 1$ in [5] by using some geometrical restrictions on X and V.

In the case $\|K\| < 1$, the positivity and the irreducibility of the generated semigroup and the spectral properties of the streaming operator have been developed for instance in [7, Chapter XII]. In [1], we have also found that the spectral bound of the streaming operator is negative.

The motivation of the present work is to deal with the positivity and the irreducibility of the generated semigroup when $\|K\| \geqslant 1$. In this work we also study in this case the spectrum of the streaming operator. In the next section we recall some facts concerning the generation of the semigroup of the streaming operator we are concerned with. In the third section we study the positivity and the irreducibility of the generated semigroup. In the last section we give the characterization of the type of the generated semigroup by means of a relation we call a characteristic equation. We note that all of these results are obtained under suitable hypotheses on the boundary operator K.

2. Statement of the problem

We consider the Banach space $L^p(X \times V)$ $(1 \leqslant p < \infty)$ with its natural norm

$$\|\varphi\|_p = \left[\int_{X\times V} |\varphi(x,v)|^p \, \mathrm{d}x \, \mathrm{d}\mu(v)\right]^{1/p}, \tag{2.1}$$

and the partial Sobolev space

$$W^p(X \times V) = \{\varphi \in L^p(X \times V),\ v \cdot \nabla_x \varphi \in L^p(X \times V)\},$$

with the norm $\|\varphi\|_{W^p(X\times V)} = [\|\varphi\|_p^p + \|v\cdot\nabla_x\varphi\|_p^p]^{1/p}$. We let $n(x)$ be the outer unit normal at $x \in \partial X$, where ∂X is the boundary of X equipped with the measure $\mathrm{d}\gamma$. We denote

$$\begin{aligned}
\Gamma &= \partial X \times V,\\
\Gamma_0 &= \{(x,v)\in\Gamma,\ v\cdot n(x) = 0\},\\
\Gamma_+ &= \{(x,v)\in\Gamma,\ v\cdot n(x) > 0\},\\
\Gamma_- &= \{(x,v)\in\Gamma,\ v\cdot n(x) < 0\},
\end{aligned}$$

and suppose that $\mathrm{d}\gamma(\Gamma_0) = 0$. For $(x,v)\in X\times V$, the time which a particle starting at x with velocity $-v$ needs until it reaches the boundary ∂X of X is denoted by

$$t(x,v) = \inf\{t,\ x - tv \notin X\}.$$

Similarly, if $(x,v)\in\Gamma_+$ we set

$$\tau(x,v) = \inf\{t,\ x - tv \notin X\}.$$

In the sequel we need

Definition 2.1. A pair (X,V) is called regular if and only if

$$\tau(X,V) \stackrel{\mathrm{def}}{=} \inf_{(x,v)\in\Gamma_+} \tau(x,v) > 0.$$

We also consider the trace spaces $L^p(\Gamma_\pm)$ equipped with the norm

$$\|\varphi\|_{L^p(\Gamma_\pm)} = \left[\int_{\Gamma_\pm} |\varphi(x,v)|^p\,\mathrm{d}\xi\right]^{1/p},$$

where $\mathrm{d}\xi = |v\cdot n(x)|\,\mathrm{d}\gamma\,\mathrm{d}\mu$. In this case we have

Lemma 2.1 (see [5]). *If (X,V) is a regular pair then the trace mappings*

$$\gamma_+\colon\ W^p(X\times V) \longrightarrow L^p(\Gamma_+) \quad \text{and} \quad \gamma_-\colon\ W^p(X\times V)\longrightarrow L^p(\Gamma_-)$$

are continuous.

In this context we consider the boundary operator

$$K \in \mathcal{L}(L^p(\Gamma_+), L^p(\Gamma_-)), \tag{2.2}$$

and by the previous lemma we can give sense to the streaming operator T_K as follows

$$\begin{cases} T_K\varphi = -v\cdot\nabla_x\varphi \ \text{ on the domain},\\ D(T_K) = \{\varphi\in W^p(X\times V),\ \gamma_-\varphi = K\gamma_+\varphi\}.\end{cases}$$

If $K = 0$ the operator T_0 has some properties which we summarize as follows.

Lemma 2.2 (see [8, Chapter 21]). *The operator T_0 generates, on $L^p(X\times V)$, a positive strongly continuous semigroup $\{U_0(t)\}_{t\geqslant 0}$. Moreover, for all $\lambda>0$ and all $\varphi\in L^p(X\times V)$, the operator $(\lambda-T_0)^{-1}\in\mathcal{L}(L^p(X\times V))$ given by*

$$(\lambda-T_0)^{-1}\varphi(x,v)=\int_0^{t(x,v)}\mathrm{e}^{-\lambda s}g(x-sv,v)\,\mathrm{d}s,\quad (x,v)\in X\times V$$

and $\gamma_+(\lambda-T_0)^{-1}\in\mathcal{L}(L^p(X\times V),L^p(\Gamma_+))$ given by

$$\gamma_+(\lambda-T_0)^{-1}\varphi(x,v)=\int_0^{\tau(x,v)}\mathrm{e}^{-\lambda s}g(x-sv,v)\,\mathrm{d}s,\quad (x,v)\in\Gamma_+$$

are positive operators. Furthermore, we have

$$\gamma_+(\lambda-T_0)^{-1}\big((L^p(X\times V))_+\setminus\{0\}\big)\subset(L^p(\Gamma_+))_+\setminus\{0\}. \tag{2.3}$$

Remark 2.1. In the sequel until the end of this paper we suppose that (X,V) is a regular pair and we denote by $A\psi(x,v)=\psi(x-\tau(x,v),v)$ the Albedo operator associated to the following problem

$$\begin{cases} v\cdot\nabla_x u=0,\\ \gamma_-u=\psi\in L^p(\Gamma_-).\end{cases}$$

Note the positivity of A and $\|A\|_{\mathcal{L}(L^p(\Gamma_-),L^p(\Gamma_+))}=1$. We also suppose that $\|K\|=\|K\|_{\mathcal{L}(L^p(\Gamma_+),L^p(\Gamma_-))}\geqslant 1$.

Now we recall the generation result for the streaming operator under consideration.

Lemma 2.3 (see [5]). *The operator T_K generates, on $L^p(X\times V)$, a C_0-semigroup $\{U_K(t)\}_{t\geqslant 0}$. Moreover, for all $\lambda>\tau(X,V)^{-1}\ln\|K\|$ we have*

$$\begin{aligned}(\lambda-T_K)^{-1}\varphi(x,v)&=\varepsilon_\lambda(x,v)(I-K_\lambda)^{-1}K\gamma_+(\lambda-T_0)^{-1}\varphi(x-t(x,v)v,v)\\&\quad+(\lambda-T_0)^{-1}\varphi(x,v),\end{aligned} \tag{2.4}$$

where $K_\lambda\in\mathcal{L}(L^p(\Gamma_-))$ is given by $K_\lambda\psi=K(\theta_\lambda A\psi)$, $\varepsilon_\lambda(x,v)=\mathrm{e}^{-\lambda t(x,v)}$ and $\theta_\lambda(x,v)=(\gamma_+\varepsilon_\lambda)(x,v)=\mathrm{e}^{-\lambda\tau(x,v)}$.

3. Positivity and irreducibility of the generated semigroup

In this section, we study the positivity and the irreducibility of the generated semigroup under some suitable hypothesis on the boundary operator K. We begin this section by discussing the positivity of the generated semigroup.

Theorem 3.1. *If K is a positive operator, the semigroup $\{U_K(t)\}_{t\geqslant 0}$ is also positive.*

Proof. Let $\lambda > 1/(\tau(X,V))\ln\|K\|$ and $\varphi \in (L^p(X\times V))_+$. Note that $(I - K_\lambda)^{-1}$ is invertible because

$$\|K_\lambda\|_{\mathcal{L}(L^p(\Gamma_-))} \leqslant \mathrm{e}^{-\lambda\tau(X,V)}\|K\|\|A\| \leqslant \mathrm{e}^{-\lambda\tau(X,V)}\|K\| < 1.$$

As K is a positive operator, K_λ is also positive and therefore the positivity announced in Lemma 2.2 yields that

$$\begin{aligned}(\lambda - T_K)^{-1}\varphi(x,v) &\geqslant \varepsilon_\lambda(x,v)(I-K_\lambda)^{-1}K\gamma_+(\lambda - T_0)^{-1}\varphi(x-t(x,v)v,v)\\ &= \varepsilon_\lambda(x,v)\sum_{n\geqslant 0}K_\lambda^n K\gamma_+(\lambda - T_0)^{-1}\varphi(x-t(x,v)v,v) \geqslant 0\end{aligned}$$

for a.e. $(x,v)\in X\times V$. Thus $(\lambda - T_K)^{-1}$ is a positive operator and hence [6, Proposition 7.1] implies the result. □

Remark 3.1. Under the hypothesis of the previous theorem and the relation (2.4) we get that $(\lambda - T_K)^{-1} \geqslant (\lambda - T_0)^{-1}$ for $\lambda > 1/(\tau(X,V))\ln\|K\|$, which implies by the Exponential formula that $U_K(t)\geqslant U_0(t)$, for all $t\geqslant 0$.

Theorem 3.2. *Suppose that K is a positive operator and there exists $\lambda > \tau(X,V)^{-1}\ln\|K\|$ such that $(I-K_\lambda)^{-1}K$ is strongly positive. Then the semigroup $\{U_K(t)\}_{\geqslant 0}$ is irreducible.*

Proof. Let $\lambda > 1/(\tau(X,V))\ln\|K\|$ and $\varphi\in (L^p(X\times V))_+$ with $\varphi\neq 0$. From Lemma 2.2 we get that $\gamma_+(\lambda - T_0)^{-1}\varphi \geqslant 0$ with $\gamma_+(\lambda - T_0)^{-1}\varphi\neq 0$ and by the strong positivity of $(I-K_\lambda)^{-1}K$ we obtain

$$(I-K_\lambda)^{-1}K\gamma_+(\lambda - T_0)^{-1}\varphi(x-t(x,v)v,v) > 0, \tag{3.1}$$

a.e. $(x,v)\in X\times V$ which implies

$$\varepsilon_\lambda(x,v)(I-K_\lambda)^{-1}K\gamma_+(\lambda - T_0)^{-1}\varphi(x-t(x,v)v,v) > 0, \tag{3.2}$$

a.e. $(x,v)\in X\times V$. By Lemmas 2.2 and 2.3 we get $(\lambda - T_K)^{-1}\varphi(x,v) > 0$, a.e. $(x,v)\in X\times V$ and we achieve the proof by applying [6, Proposition 7.6]. □

Remark 3.2. Note that the strong positivity of the boundary operator K is sufficient to guarantee the irreducibility of the generated semigroup. In fact, this follows from the previous theorem and the inequality

$$(I-K_\lambda)^{-1}K = \sum_{n\geqslant 0}K_\lambda^n K \geqslant K.$$

R e m a r k 3.3. As the geometry of X is arbitrary, it is difficult to obtain the irreducibility of the generated semigroup without imposing some suitable conditions on the boundary operator K. Nevertheless, in some particular cases we can overcome this difficulty. For instance, if $X \times V =]0,1[\times]a,b[$ $(0 \leqslant a < b < \infty)$, then $\tau(X,V) = 1/b$ and the Albedo operator is reduced to the identity operator on the trace spaces $L^p(]a,b[, v\,\mathrm{d}v)$. Thus $K_\lambda \psi = K[\theta_\lambda \psi]$ where $\psi \in L^p(]a,b[v\,\mathrm{d}v)$ and $\theta_\lambda(v) = \mathrm{e}^{-1/v}$. In this case, the positivity and the irreducibility of K are sufficient to guarantee the irreducibility of the generated semigroup. In fact, let $\lambda > b \ln \|K\|$. If $a > 0$ (resp. $a = 0$), by Lemma 3.1 (resp. Remark 3.4) we obtain the irreducibility of K_λ and therefore the strong positivity of $(I - K_\lambda)^{-1}K$ by

$$(I - K_\lambda)^{-1}K = \sum_{n \geqslant 0} K_\lambda K \geqslant K_\lambda^n, \quad n \geqslant 0. \tag{3.3}$$

Now we finish this head by the following lemma we will need in the next section.

Lemma 3.1. *If K is positive and such that KA is irreducible, then K_λ is irreducible for all $\lambda \geqslant 0$.*

P r o o f. The case $\lambda = 0$ is clear. Let $\lambda > 0$ and M be a closed ideal of the trace space $L^p(\Gamma_+)$ such that $K_\lambda(M) \subset M$. Using the characterization of a closed ideal in L^p-like spaces (see [9, p. 309]), there exists a measurable subset $\omega \subset \Gamma_+$ such that

$$M = \{\psi \in L^p(\Gamma_+) \colon\ \psi(x,v) = 0, \text{ for a.e. } (x,v) \in \omega\}.$$

Note that $M \subset \theta_\lambda M$ and if $g \in KA(M) \subset KA(\theta_\lambda M)$, then there exists $\psi \in M$ such that $g = KA\theta_\lambda \psi = K_\lambda \psi \in K_\lambda(M) \subset M$, which implies that $KA(M) \subset M$. Thus the irreducibility of KA implies the irreducibility of K_λ. $\square$

R e m a r k 3.4. Note that, if $0 \notin V$, then $\tau(x,v) \leqslant d/v_0 < \infty$ (d is the diameter of X and v_0 is the minimum speed), thus $K_\lambda \geqslant \mathrm{e}^{-\lambda d/v_0} KA$, and therefore the irreducibility of KA implies that of K_λ.

4. Spectral properties

In this section, we give a characterization of the spectral bound $s(T_K)$ of the streaming operator. This characterization is possible under some hypothesis on the boundary operator K. We first begin with the localization of the essential spectrum of the streaming operator. We set $\mathbb{C}_- = \{\lambda,\ \Re(\lambda) \leqslant 0\}$ and $\mathbb{C}_+ = \{\lambda,\ \Re(\lambda) \geqslant 0\}$.

Lemma 4.1. *If K is a compact operator, then we have $\sigma_{\mathrm{ess}}(T_K) \subset \mathbb{C}_-$.*

P r o o f. Let $\lambda > 1/(\tau(X,V)) \ln \|K\|$. As K is compact operator then the operator

$$\varphi \longrightarrow \varepsilon_\lambda (I - K_\lambda)^{-1} K \gamma_+ (\lambda - T_0)^{-1} \varphi(\cdot - t(\cdot,\cdot)\cdot, \cdot)$$

is also compact from $L^p(X \times V)$ into itself which implies, by the relation (2.4), the compactness of the operator $(\lambda - T_K)^{-1} - (\lambda - T_0)^{-1}$ from $L^p(X \times V)$ into itself. Now, [11, Theorem 4.7, p. 17] and the contractiveness of the semigroup $\{U_0(t)\}_{t \geqslant 0}$ (see Lemma 2.2) imply $\sigma_{\mathrm{ess}}(T_K) = \sigma_{\mathrm{ess}}(T_0) \subset \mathbb{C}_-$. $\square$

R e m a r k 4.1. Note that if $0 \notin \overline{V}$ then $\sigma_{\mathrm{ess}}(T_K) = \sigma_{\mathrm{ess}}(T_0) = \emptyset$. This follows from the fact that the semigroup generated by T_0 is compact for all $t > d/v_0$.

Lemma 4.2. *Let $\lambda \in \mathbb{C}_+$. Then we have*

$$\lambda \in \sigma_p(T_K) \Longleftrightarrow 1 \in \sigma_p(K_\lambda). \tag{4.1}$$

P r o o f. Let $\lambda \in \mathbb{C}_+$. If $\lambda \in \sigma_p(T_K)$, then there exists $\varphi \in D(T_K)$ with $\varphi \neq 0$ such that $\lambda\varphi = T_K\varphi$. As φ is given by

$$\varphi(x,v) = \varepsilon_\lambda(x,v)\psi(x - t(x,v)v, v)$$

where $\psi \in L^p(\Gamma_-)$, then the fact that $\varphi \in D(T_K)$ implies that

$$\psi = \gamma_-\varphi = K\gamma_+\varphi = K[\theta_\lambda A\psi] = K_\lambda\psi$$

and therefore $1 \in \sigma_p(K_\lambda)$.

Conversely, if $1 \in \sigma_p(K_\lambda)$, then there exists $\psi \in L^p(\Gamma_+)$ with $\psi \neq 0$ such that $\psi = K_\lambda\psi$. As $\varphi(x,v) = \varepsilon_\lambda(x,v)\psi(x - t(x,v)v, v)$ with $\varphi \neq 0$ solves the equation $\lambda\varphi = T_K\varphi$, then $\lambda \in \sigma_p(T_K)$. $\square$

Lemma 4.3. *Suppose that K is a positive and compact operator such that KA is irreducible, then the following mapping*

$$0 \leqslant \lambda \mapsto r(K_\lambda) \tag{4.2}$$

is continuous and strictly decreasing.

P r o o f. Let $\lambda \geqslant 0$ and $\mu \geqslant 0$. From the hypothesis of the lemma, we get that K_λ is a positive and compact operator and by Lemma 3.1 it is also irreducible. By [10], it follows that $r(K_\lambda) > 0$ and there exists ψ_λ a quasi-interior point of

$(L^p(\Gamma_-))_+$ (resp. strictly positive eigenvector $\psi_\lambda^* \in L^q(\Gamma_-)$) of the operator K_λ (resp. K_λ^* which is the adjoint operator of K_λ) associated with eigenvalue $r(K_\lambda)$, where q is the harmonic conjugate of p, i.e. $p^{-1}+q^{-1}=1$. The same properties hold for $r(K_\mu)$ and we obtain

$$
\begin{aligned}
(4.3)\qquad r(K_\mu) &= \frac{\langle K_\mu^*\psi_\mu^*, \psi_\lambda\rangle}{\langle \psi_\mu^*, \psi_\lambda\rangle} = \frac{\langle \psi_\mu^*, K_\mu\psi_\lambda\rangle}{\langle \psi_\mu^*, \psi_\lambda\rangle} \\
&= \frac{\langle \psi_\mu^*, K_\lambda\psi_\lambda\rangle}{\langle \psi_\mu^*, \psi_\lambda\rangle} - \frac{\langle \psi_\mu^*, (K_\lambda - K_\mu)\psi_\lambda\rangle}{\langle \psi_\mu^*, \psi_\lambda\rangle} = r(K_\lambda) - \alpha(\lambda,\mu),
\end{aligned}
$$

where

$$\alpha(\lambda,\mu) = \frac{\langle \psi_\mu^*, (K_\lambda - K_\mu)\psi_\lambda\rangle}{\langle \psi_\mu^*, \psi_\lambda\rangle}.$$

Thus

$$
\begin{aligned}
(4.4)\qquad |r(K_\lambda) - r(K_\mu)| = |\alpha(\lambda,\mu)| &= \left|\frac{\langle \psi_\mu^*, (K_\lambda - K_\mu)\psi_\lambda\rangle}{\langle \psi_\mu^*, \psi_\lambda\rangle}\right| \\
&\leqslant \frac{\|\psi_\mu^*\|_{L^q(\Gamma_-)}}{\langle \psi_\mu^*, \psi_\lambda\rangle} \|(K_\lambda - K_\mu)\psi_\lambda\|_{L^p(\Gamma_-)}.
\end{aligned}
$$

As we have

$$
\begin{aligned}
(4.5)\qquad \|(K_\lambda - K_\mu)\psi_\lambda\|^p_{L^p(\Gamma_-)} &\leqslant \|K\|^p \int_{\Gamma_-} |\theta_\lambda(x,v) - \theta_\mu(x,v)|^p |\psi_\lambda(x,v)|^p \,\mathrm{d}\xi, \\
|\theta_\lambda - \theta_\mu|^p|\psi_\lambda|^p &\leqslant 2^p|\psi_\lambda|^p \in L^1(\Gamma_+),
\end{aligned}
$$

and

$$\lim_{\lambda\to\mu} |\theta_\lambda(x,v) - \theta_\mu(x,v)|^p |\psi_\lambda(x,v)|^p = 0,$$

a.e. $(x,v) \in \Gamma_+$, then $\lim\limits_{\lambda\to\mu} |\alpha(\lambda,\mu)| = 0$, and therefore the continuity asked for is proved.

Let $0 \leqslant \lambda < \mu$ and $\psi \in (L^p(\Gamma_-))_+$. Writing

$$K_\lambda\psi \geqslant \mathrm{e}^{(\mu-\lambda)\tau(X,V)} K_\mu\psi$$

it follows that

$$r(K_\lambda) \geqslant \mathrm{e}^{(\mu-\lambda)\tau(X,V)} r(K_\mu) > r(K_\mu).$$

The proof is finished. □

Now we state the main theorem of this section as follows.

Theorem 4.1. *Suppose that K is a positive and compact operator such that KA is irreducible. If $r(AK) > 1$, then $\omega(U_K(t))$ is the unique solution of the characteristic equation*

$$r(K_\lambda) = 1. \tag{4.6}$$

Moreover, $\omega(U_K(t)) > 0$.

P r o o f. Note that $r(K_0) = r(KA) > 1$ and

$$\lim_{\lambda\to\infty} r(K_\lambda) \leqslant \lim_{\lambda\to\infty} \|K_\lambda\| \leqslant \lim_{\lambda\to\infty} \mathrm{e}^{-\lambda\tau(X,V)}\|K\|\|A\| = 0.$$

Thus, by the previous lemma, there exists a unique $\lambda_0 > 0$ such that $r(K_{\lambda_0}) = 1$. Moreover, the positivity of K_{λ_0} implies that $1 = r(K_{\lambda_0}) \in \sigma_p(K_{\lambda_0})$ which implies, by Lemma 4.2, that $\lambda_0 \in \sigma_p(T_K)$.

Let $\lambda \in \mathbb{C}_+$. If $\lambda \in \sigma(T_K)$, then by Lemma 4.2, there exists $\psi \in L^p(\Gamma_-)$ with $\psi \neq 0$ such that $K_\lambda\psi = \psi$. Since K and the Albedo operator A are positive operators (see Remark 2.1) then

$$K_{\Re\lambda}|\psi| = K[\varepsilon_{\Re\lambda}A|\psi|] \geqslant K[|\varepsilon_\lambda A\psi|] \geqslant |K_\lambda\psi| = |\psi|$$

which implies $K^n_{\Re\lambda}|\psi| \geqslant |\psi|$ for all integer n. Consequently $r(K_{\Re\lambda}) \geqslant 1$. Since the mapping (4.2) is strictly decreasing, we conclude that $\Re\lambda \leqslant \lambda_0$, and therefore, $s(T_K) \leqslant \lambda_0$.

Conversely. Since $r(K_{\lambda_0}) = 1$ and by virtue of [10], there exists $\psi \in (L^p(\Gamma_-))_+$ with $\psi \neq 0$ such that $K_{\lambda_0}\psi = \psi$. Defining $\varphi = \varepsilon_{\lambda_0}A\psi$ and noting that $\varphi \neq 0$ because of $\psi > 0$, it is easy to see that $-v\cdot\nabla_x\varphi = \lambda_0\varphi$. Furthermore, as $\theta_\lambda = (\gamma_+\varepsilon_\lambda)$ (see Lemma 2.3), it follows that $\gamma_-\varphi = \psi = K_{\lambda_0}\psi = K[(\gamma_+\varepsilon_{\lambda_0})A\psi] = K\gamma_+[\varepsilon_{\lambda_0}A\psi] = K\gamma_+\varphi$. Thus, $\lambda_0 \in \sigma_p(T_K) \subset \sigma(T_K)$, and therefore, $\lambda_0 \leqslant s(T_K)$. We finish the proof by applying [15]. $\square$

R e m a r k 4.2. Note that the previous theorem holds only for $\|K\| > 1$, because $1 < r(KA) \leqslant \|K\|\|A\| = \|K\|$. For the case $\|K\| = 1$, it is sufficient to consider $r(KA) = 1$. In this case we have $r(K_\lambda) = 1 = r(K_0) = r(KA)$, which implies $\lambda_0 = s(T_K) = \omega(U_K(t)) = 0$.

R e m a r k 4.3. There is another way to characterize the spectral bound $s(T_K)$. In fact, introducing the following boundary operator

$$\overline{K}_\lambda\psi = \theta_\lambda AK\psi,$$

from $L^p(\Gamma_+)$ into itself, it is easy to see that

$$\|K_\lambda^n\| \leqslant \|K\|\|A\|\|\overline{K}_\lambda^{(n-1)}\|,$$

and

$$\|\overline{K}_\lambda^n\| \leqslant \|K\|\|A\|\|K_\lambda^{(n-1)}\|,$$

for all $\lambda \geqslant 0$ and all integers $n \geqslant 0$ and therefore, $r(K_\lambda) = r(\overline{K}_\lambda)$.

The previous theorem is given in a general context. The case $0 \notin \overline{V}$ can be treated in $L^1(X \times V)$ only, under the hypothesis (4.9) and assuming the positivity of the boundary operator K. Before stating this result, we recall some facts developed in [5]. Define on $L^1(X \times V)$ the following norm

$$\|\!|\varphi|\!\|_1 = \int_{X\times V} |(f\varphi)(x,v)| \,\mathrm{d}x \,\mathrm{d}\mu(v), \tag{4.7}$$

where, $f(x,v) = \|K\|^{t(x,v)/(\tau(x+t(x,-v)v,v))}$ which satisfies $1 \leqslant f(x,v) \leqslant \|K\|$, for all $(x,v) \in \overline{X} \times V$. The norms (2.1) and (4.7) are equivalent on $L^1(X \times V)$ because

$$\|\varphi\|_1 \leqslant \|\!|\varphi|\!\|_1 \leqslant \|K\|\|\varphi\|_1. \tag{4.8}$$

Theorem 4.2. *Suppose that $0 \notin V$. If K is positive operator such that*

$$\|K\psi\|_{L^1(\Gamma_-)} \geqslant \|\psi\|_{L^1(\Gamma_+)} \tag{4.9}$$

for all $\psi \in (L^1(\Gamma_+))_+$, then the type of the generated semigroup on $L^1(X \times V)$ satisfies

$$\omega(U_K(t)) \geqslant \frac{v_0}{d} \ln \|K\|,$$

where d is the diameter of X and v_0 is the minimum speed.

P r o o f. Let λ be a large enough real and let $g \in (L^p(X \times V))_+$. By Lemma 2.3, there exists a unique solution of the equation

$$\lambda\varphi = T_K\varphi + g, \tag{4.10}$$

given by $\varphi = (\lambda - T_K)^{-1}g$. By virtue of Theorem 3.1 and [6, Proposition 7.1], we get that $\varphi = (\lambda - T_K)^{-1}g \in (L^p(X \times V))_+$. Now, multiplying the equation (4.10) by f and integrating on $X \times V$ we obtain

$$\begin{aligned}\lambda\|\!|\varphi|\!\|_1 = &\int_{X\times V} (fv \cdot \nabla_x\varphi)(x,v) \,\mathrm{d}x \,\mathrm{d}\mu(v) \\ &+ \int_{X\times V} (fg)(xv) \,\mathrm{d}x \,\mathrm{d}\mu(v) = I + \|\!|g|\!\|_1.\end{aligned} \tag{4.11}$$

Using Green's Formula, the boundary conditions and the relation (4.9), the term I becomes

$$\begin{aligned}
I &= -\int_{X\times V} v\cdot\nabla_x(f\varphi)(x,v)\,\mathrm{d}x\,\mathrm{d}\mu(v) + \int_{X\times V}[(v\cdot\nabla_x f)\varphi](x,v)\,\mathrm{d}x\,\mathrm{d}\mu(v)\\
&= -\int_{\Gamma_+}\gamma_+(f\varphi)(x,v)\,\mathrm{d}\xi + \int_{\Gamma_-}\gamma_-(f\varphi)(x,v)\,\mathrm{d}\xi\\
&\quad + \ln\|K\|\int_{X\times V}\frac{1}{\tau(x+t(x,-v)v,v)}(f\varphi)(x,v)\,\mathrm{d}x\,\mathrm{d}\mu(v)\\
&\geqslant \|K\gamma_+\varphi\|_{L^1(\Gamma_-)} - \|\gamma_+\varphi\|_{L^1(\Gamma_+)} + \ln\|K\|\frac{v_0}{d}\int_{X\times V}(f\varphi)\,\mathrm{d}x\,\mathrm{d}\mu(v)\\
&\geqslant \ln\|K\|\frac{v_0}{d}|\!|\!|\varphi|\!|\!|_1,
\end{aligned}$$

because $\tau(x+t(x,-v)v,v)\leqslant d/v_0$. Now the relation (4.11) becomes

$$\Big(\lambda-\ln\|K\|\frac{v_0}{d}\Big)|\!|\!|(\lambda-T_K)^{-1}g|\!|\!|_1\geqslant|\!|\!|g|\!|\!|_1,$$

which implies

$$\Big(\lambda-\ln\|K\|\frac{v_0}{d}\Big)^n|\!|\!|[(\lambda-T_K)^{-1}]^n g|\!|\!|_1\geqslant|\!|\!|g|\!|\!|_1,$$

for all large integers n and if $\lambda=n/t$ with $t>0$ then

$$\Big[\Big(1-\frac{t}{n}\ln\|K\|\frac{v_0}{d}\Big)\Big]^n\Big|\!\Big|\!\Big|\Big[\frac{n}{t}\Big(\frac{n}{t}-T_K\Big)^{-1}\Big]^n g\Big|\!\Big|\!\Big|_1\geqslant|\!|\!|g|\!|\!|_1.$$

Now the Exponential formula and the relation (4.8) imply that

$$\|K\|\|U_K(t)g\|_1\geqslant \mathrm{e}^{v_0t/d}\ln\|K\|\|g\|_1$$

which implies $\omega(U_K(t))\geqslant v_0/d\ln\|K\|>0$ and achieves the proof. □

R e m a r k 4.4. Note that the previous Theorem holds for $\|K\|\geqslant 1$ because of the relation (4.9).

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