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RICHARDSON EXTRAPOLATION AND DEFECT CORRECTION OF MIXED FINITE ELEMENT METHODS FOR INTEGRO-DIFFERENTIAL EQUATIONS IN POROUS MEDIA*

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Abstract. Asymptotic error expansions in the sense of L^{∞}-norm for the Raviart-Thomas mixed finite element approximation by the lowest-order rectangular element associated with a class of parabolic integro-differential equations on a rectangular domain are derived, such that the Richardson extrapolation of two different schemes and an interpolation defect correction can be applied to increase the accuracy of the approximations for both the vector field and the scalar field by the aid of an interpolation postprocessing technique, and the key point in deriving them is the establishment of the error estimates for the mixed regularized Green's functions with memory terms presented in R. Ewing at al., Int. J. Numer. Anal. Model *2* (2005), 301–328. As a result of all these higher order numerical approximations, they can be used to generate a posteriori error estimators for this mixed finite element approximation.

Keywords: integro-differential equations, mixed finite element methods, mixed regularized Green's functions, asymptotic expansions, interpolation defect correction, interpolation postprocessing, a posteriori error estimators

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1. INTRODUCTION

The aim of this paper is to discuss the asymptotic behavior of the mixed finite ele-

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ment approximation for a parabolic integro-differential equation with the Neumann boundary condition: Find $u = u(x,t)$ such that

$$
\begin{aligned}
u_t &= \nabla \cdot \boldsymbol{\sigma} + cu + f && \text{in } \Omega \times J,\\
\boldsymbol{\sigma} &= A(t) \cdot \nabla u - \int_0^t B(t,s) \cdot \nabla u(s)\,\mathrm{d}s && \text{in } \Omega \times J,\\
\frac{\partial u}{\partial \mathbf{n}} &= 0 && \text{on } \partial\Omega \times J,\\
u &= u_0(x) && x \in \Omega,\ t = 0,
\end{aligned}
\tag{1.1}
$$

where Ω is an open bounded domain in $\mathbb{R}^2$ with Lipschitz boundary $\partial\Omega$, and $\mathbf{n}$ is the outward unit normal vector along $\partial\Omega$, $J = (0,T)$ with $T > 0$, $A(t) = A(x,t)$ and $B(t,s) = B(x,t,s)$ are two 2×2 matrices with A being positive definite, c, f and u_0 are known smooth functions.

At present there is an extensive literature available for numerical approximations of the problem (1.1). See, for instance, [6], [7], [17], [19], [20], [22]–[24], [27], [28], [30]–[33] for the finite element method and the finite difference method. Recently, mixed finite element methods have been investigated for (1.1) in [10]–[13], and [16], in which optimal and superconvergent estimates in the L^2-norm and the L^∞-norm as well as Richardson extrapolation in L^2-norm have been obtained by means of a mixed Ritz-Volterra projection, regularized Green's functions and an interpolation postprocessing method.

In the present paper we study two numerical approaches of higher accuracy–the Richardson extrapolation method of two different forms in the L^∞-norm and an interpolation defect correction method in the L^2-norm and the L^∞-norm. As an efficient numerical method to increase the accuracy of approximations, Richardson extrapolation has been demonstrated in [29] for the difference method, in [3], [5], [8], [15], [19], [21]–[25], [26], [34], [35], [37], [38] for the (Galerkin and Petrov-Galerkin) finite element method and the mixed finite element method, in [18] and [36] for the collocation method and the boundary element method, respectively.

The defect correction of (Galerkin and Petrov-Galerkin) finite elements by means of an interpolation postprocessing technique is another numerical method to obtain approximations of higher accuracy, which has been probed for a wide variety of models. See, for example, [2], [5], [19], [21], [24], and the references cited therein.

As we have done in [14], we employ the analysis for the "short side" in the FE-right triangle plus the sharp integral estimates of the "hypotenuse" (see, for example, [19] and [20]) to present an immediate analysis for the asymptotic expansion of the error between the mixed finite element solution and the corresponding interpolation function of the exact solution to (1.1) in the sense of L^∞-norm on the basis of the estimates for the mixed regularized Green's functions with memory terms introduced in [13]. Thus, the asymptotic expansion of the error in the mixed finite element solution is derived, by which the Richardson extrapolation of two different types and the interpolation defect correction can be applied to generate mixed finite element approximations of higher accuracy. In addition, by means of these approximations with higher precision, a class of a posteriori error estimators are constructed for this mixed finite element method.

This paper is organized as follows. In Section 2, the approximate subspace and the variational formula of (1.1) are provided. Also, the asymptotic expansion of the Raviart-Thomas projection is presented in this section for the future need. Section 3 is devoted to investigating the asymptotic expansion of the error between the mixed finite element solution and the Raviart-Thomas projection of the exact solution to (1.1) in the L^∞-norm. Finally, Section 4 deals with an interpolation defect correction approximation in the L^2- and the L^∞-norm based on the results given in Section 3. Furthermore, at each end of Sections 3 and 4, a posteriori error estimators are furnished as by-products of these numerical solutions with higher convergence rates.

2. The asymptotic expansion

In this section we first give the formula of the mixed finite element method for the parabolic integro-differential equation (1.1). For the sake of simplicity of analysis, we take the domain Ω to be a rectangle in this paper.

Let

$$W := L^2(\Omega) \quad \text{and} \quad \mathbf{V} := H(\mathrm{div}, \Omega) = \{\boldsymbol{\sigma} \in (L^2(\Omega))^2 \colon \nabla \cdot \boldsymbol{\sigma} \in L^2(\Omega)\}$$

be the standard L^2-space on Ω with norm $\|\cdot\|_0$ and the Hilbert space equipped with the norm

$$\|\boldsymbol{\sigma}\|_{\mathbf{V}} := (\|\boldsymbol{\sigma}\|_0^2 + \|\nabla \cdot \boldsymbol{\sigma}\|_0^2)^{1/2},$$

respectively. Moreover, set

$$\mathbf{V}_0 := \{\boldsymbol{\sigma} \in \mathbf{V} \colon \boldsymbol{\sigma} \cdot \mathbf{n} = 0 \text{ on } x \in \partial\Omega\}.$$

Then, from [10] we recall that the weak mixed formulation of (1.1) is given by finding $(u, \boldsymbol{\sigma}) \in W \times \mathbf{V}_0$ such that

$$\begin{gathered} (u_t, w) - (\nabla \cdot \boldsymbol{\sigma}, w) - (cu, w) = (f, w), \quad w \in W, \\ (\alpha\boldsymbol{\sigma}, \mathbf{v}) + \int_0^t (M(t,s)\boldsymbol{\sigma}(s), \mathbf{v})\, \mathrm{d}s + (\nabla \cdot \mathbf{v}, u) = 0, \quad \mathbf{v} \in \mathbf{V}_0, \\ u(0,x) = u_0(x) \quad \text{in } L^2(\Omega), \end{gathered} \tag{2.1}$$

where $\alpha = A^{-1}(t)$, $M(t,s) = R(t,s)A^{-1}(s)$, and $R(t,s)$ is the resolvent of the matrix $A^{-1}(t)B(t,s)$ and is presented by

$$R(t,s) = A^{-1}(t)B(t,s) + \int_s^t A^{-1}(t)B(t,\tau)R(\tau,s)\,\mathrm{d}\tau, \quad t > s \geqslant 0.$$

Let $T_{h,k}$ be a finite element partition of $\overline{\Omega}$ into uniform rectangles and $\mathbf{V}_{h,k} \times W_{h,k} \subset \mathbf{V} \times W$ denote a pair of finite element spaces associated with this partition which satisfy the inf-sup stability condition of Babuška and Brezzi (see, for instance, [1] and [4]), where h and k are the mesh sizes in x- and y-axis, respectively. Some spaces like that have been constructed and analyzed for rectangular elements. However, here our analysis is concentrated on the Raviart-Thomas space of the lowest order; i.e.,

$$\begin{aligned}
\mathbf{V}_{h,k} &:= \{\mathbf{v}_{h,k} \in \mathbf{V}\colon\ \mathbf{v}_{h,k}|_e \in Q_{1,0}(e) \times Q_{0,1}(e),\ e \in T_{h,k}\},\\
W_{h,k} &:= \{w_{h,k} \in W\colon\ w_{h,k}|_e \in Q_0(e),\ e \in T_{h,k}\},
\end{aligned} \tag{2.2}$$

where $Q_{m,n}(e)$ stands for the space of polynomials of degree no more than m and n in x and y on e, respectively. The extension to other stable rectangular element spaces can also be made.

The corresponding semi-discrete version of (2.1) seeks a pair $(u_{h,k}, \boldsymbol{\sigma}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k} \subset W \times \mathbf{V}_0$ to satisfy

$$\begin{gathered}
(u_{h,k,t}, w_{h,k}) - (\nabla \cdot \boldsymbol{\sigma}_{h,k}, w_{h,k}) - (cu_{h,k}, w_{h,k}) = (f, w_{h,k}), \quad w_{h,k} \in W_{h,k},\\
(\alpha\boldsymbol{\sigma}_{h,k}, \mathbf{v}_{h,k}) + \int_0^t (M(t,s)\boldsymbol{\sigma}_{h,k}(s), \mathbf{v}_{h,k})\,\mathrm{d}s + (u_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) = 0,\\
\mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \quad u_{h,k}(0,x) = u_{0,h,k},
\end{gathered} \tag{2.3}$$

where $u_{0,h,k} \in W_{h,k}$ is some appropriately chosen approximation of the initial data $u_0(x)$. Furthermore, $\boldsymbol{\sigma}_{h,k}(0,x)$ is also chosen to satisfy (2.3) at $t = 0$; i.e.,

$$(\alpha(0)\boldsymbol{\sigma}_{h,k}(0), \mathbf{v}_{h,k}) + (u_{0,h,k}, \nabla \cdot \mathbf{v}_{h,k}) = 0, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}. \tag{2.4}$$

In addition, from (2.1) and (2.3) one derives the following mixed finite element error equation:

$$\begin{gathered}
(u_t - u_{h,k,t}, w_{h,k}) - (\nabla \cdot (\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,k}), w_{h,k}) - (c(u - u_{h,k}), w_{h,k}) = 0,\\
w_{h,k} \in W_{h,k},\\
(\alpha(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,k}), \mathbf{v}_{h,k}) + \int_0^t (M(t,s)(\boldsymbol{\sigma} - \boldsymbol{\sigma}_{h,k})(s), \mathbf{v}_{h,k})\,\mathrm{d}s + (u - u_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) = 0,\\
\mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}.
\end{gathered} \tag{2.5}$$

We recall that the Raviart-Thomas projection

$$\mathbf{\Pi}_{h,k}^0 \times P_{h,k}^0 \colon \mathbf{V} \times W \to \mathbf{V}_{0,h,k} \times W_{h,k}$$

is defined by the following conditions:

$$\int_{s_i} (\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma}) \cdot \mathbf{n} \, \mathrm{d}s = 0, \quad i = 1, 2, 3, 4, \tag{2.6}$$
$$\int_e (u - P_{h,k}^0 u) = 0,$$

where s_i $(i = 1, 2, 3, 4)$ are the four edges of the rectangle $e \in T_{h,k}$ and $\mathbf{n}$ is the outward normal direction on the s_i. This projection has the following properties [9]:
(i) $P_{h,k}^0$ is the local $L^2(\Omega)$ projection;
(ii) $\mathbf{\Pi}_{h,k}^0$ and $P_{h,k}^0$ satisfy

$$(\nabla \cdot (\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma}), w_{h,k}) = 0, \quad w_{h,k} \in W_{h,k}, \tag{2.7}$$
$$(\nabla \cdot \mathbf{v}_{h,k}, u - P_{h,k}^0 u) = 0, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{h,k};$$

(iii) there hold the approximation properties,

$$\begin{aligned} \|\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma}\|_{0,p} &\leqslant CH \|\boldsymbol{\sigma}\|_{1,p}, && && 1 \leqslant p \leqslant \infty, \\ \|\nabla \cdot (\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma})\|_{-s,p} &\leqslant CH^{1+s} \|\nabla \cdot \boldsymbol{\sigma}\|_{1,p}, && 0 \leqslant s \leqslant 1, && 1 \leqslant p \leqslant \infty, \\ \|u - P_{h,k}^0 u\|_{-s,p} &\leqslant CH^{1+s} \|u\|_{1,p}, && 0 \leqslant s \leqslant 1, && 1 \leqslant p \leqslant \infty, \end{aligned} \tag{2.8}$$

where $H := \max\{h, k\}$.

Also, from [14] we recall the following two theorems to conclude the section.

Theorem 2.1. *Assume that $\boldsymbol{\sigma} \in \mathbf{V} \cap (W^{4,p}(\Omega))^2$ for some $p \in [1, \infty]$. Then, for sufficiently smooth functions $\alpha_{ij}(x, y)$ $(1 \leqslant i, j \leqslant 2)$ we have*

$$\begin{aligned} (\alpha \cdot (\boldsymbol{\sigma} - \mathbf{\Pi}_{h,k}^0 \boldsymbol{\sigma}), \mathbf{v}_{h,k}) = &- \frac{h^2}{3} \int_\Omega [\alpha_{11} (\sigma_1)_{xx} + \alpha_{12} (\sigma_2)_{xx}] v_{1,h,k} \\ &+ \frac{h^2}{3} \int_\Omega [(\alpha_{22})_x (\sigma_2)_x - \alpha_{21} (\sigma_1)_{xx}] v_{2,h,k} \\ &+ \frac{k^2}{3} \int_\Omega [(\alpha_{11})_y (\sigma_1)_y - \alpha_{12} (\sigma_2)_{yy}] v_{1,h,k} \\ &- \frac{k^2}{3} \int_\Omega [\alpha_{22} (\sigma_2)_{yy} + \alpha_{21} (\sigma_1)_{yy}] v_{2,h,k} \\ &+ O(H^4) \|\boldsymbol{\sigma}\|_{4,p} \|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \end{aligned}$$

where σ_1, σ_2 and $v_{1,h,k}$, $v_{2,h,k}$ are the first components and the second components of the vector-valued functions $\boldsymbol{\sigma}$ and $\mathbf{v}_{h,k}$, respectively, and $q = p/(p-1)$ is the conjugate of p and $\alpha = (\alpha_{ij})_{2\times 2}$.

Theorem 2.2. *Assume that $u \in W^{3,p}(\Omega)$ for some $p \in [1,\infty]$. Then, for a sufficiently smooth function $c(x,y)$ we have the following asymptotic expansion:*

$$(c(u - P^0_{h,k}u), w_{h,k}) = \frac{h^2}{3}\int_\Omega c_x u_x w_{h,k} + \frac{k^2}{3}\int_\Omega c_y u_y w_{h,k} + O(H^4)\|u\|_{3,p}\|w_{h,k}\|_{0,q},$$
$$w_{h,k} \in W_{h,k},$$

where $q = p/(p-1)$ is the conjugate of p.

From Theorems 2.1 and 2.2 we immediately obtain

Corollary 2.1. *If $\boldsymbol{\sigma} \in \mathbf{V} \cap (W^{2,p}(\Omega))^2$ for some $p \in [1,\infty]$, then, we have for sufficiently smooth functions α_{ij} $(1 \leqslant i,\ j \leqslant 2)$ that*

$$|(\alpha \cdot (\boldsymbol{\sigma} - \boldsymbol{\Pi}^0_{h,k}\boldsymbol{\sigma}), \mathbf{v}_{h,k})| \leqslant CH^2\|\boldsymbol{\sigma}\|_{2,p}\|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k},$$

where $q = p/(p-1)$.

Corollary 2.2. *If $u \in W^{1,p}(\Omega)$, we have for a sufficiently smooth function $c(x,y)$ that*

$$|(c(u - P^0_{h,k}u), w_{h,k})| \leqslant CH^2\|u\|_{1,p}\|w_{h,k}\|_{0,q}, \quad w_{h,k} \in W_{h,k},$$

where $q = p/(p-1)$.

3. The Richardson extrapolation

On the basis of Theorems 2.1 and 2.2, we discuss in this section the asymptotic expansion of the error between the mixed finite element solution and the Raviart-Thomas projection of the exact solution of (1.1) in the L^∞-norm in order to establish the asymptotic error expansion of the mixed finite element approximation in maximum norm. First, let us define the following two linear operators $M*$ and $M**$ for any smooth function $f(t)$ defined on $(0,T)$ by

$$(M*f)(t) := \int_0^t M(t,s)f(s)\,\mathrm{d}s \quad \text{and} \quad (M**f)(t) := \int_t^T M(s,t)f(s)\,\mathrm{d}s.$$

Then, we have the following three lemmas according to [13].

Lemma 3.1. *We have*

$$\int_0^T M * f(t)g(t)\,\mathrm{d}t = \int_0^T f(t)M ** g(t)\,\mathrm{d}t.$$

Lemma 3.2. *Assume that $f(t)$, $g(t) \in L^1(0,T^*)$ and there exists $C > 0$ such that for an arbitrary non-negative $\varphi(t) \in C^\infty[0,T]$,*

$$\left|\int_0^T f(t)\varphi(t)\,\mathrm{d}t\right| \leqslant C\left|\int_0^T g(t)(1+\varphi(t))\,\mathrm{d}t\right|, \quad 0 \leqslant T \leqslant T^*.$$

Then, we have

$$|f(t)| \leqslant C\left|g(t) + \int_0^t g(s)\,\mathrm{d}s\right|, \quad \forall\, t \in (0,T), \ \text{a.e.}$$

Especially, if

$$\left|\int_0^T f(t)\varphi(t)\,\mathrm{d}t\right| \leqslant C\left|\int_0^T g(t)\varphi(t)\,\mathrm{d}t\right|,$$

then,

$$|f(t)| \leqslant C|g(t)|, \quad \forall\, t \in (0,T), \ \text{a.e.}$$

Lemma 3.3. *Suppose that the matrix $A(t)$ is positive definite. Then, the norms $\|\boldsymbol{\sigma}\|_0 := (\boldsymbol{\sigma},\boldsymbol{\sigma})^{1/2}$ and $\|\boldsymbol{\sigma}\|_{A^{-1}} := (A^{-1}\boldsymbol{\sigma},\boldsymbol{\sigma})^{1/2}$ are equivalent.*

Next we shall define two regularized Green's functions with memory terms for problem (1.1) in mixed form in the fashion analogous to that employed earlier for finite elements methods of elliptic and integro-differential equations ([39] and [28]) as well as mixed finite element methods of elliptic equations [35], which will play an important role in the analysis of L^∞-norm estimates.

For an arbitrary point $z_0 \in \bar{\Omega}$ the first pair regularized Green's function $(\mathbf{G}_1, \lambda_1) = (\mathbf{G}_1(z,z_0), \lambda_1(z,z_0))$ with memory is defined by

$$\begin{aligned} \alpha\mathbf{G}_1 + M ** \mathbf{G}_1 - \nabla\lambda_1 &= 0 && \text{in } \Omega \times (0,T), \\ \operatorname{div}\mathbf{G}_1 &= \delta_1^h \varphi_1(t) && \text{in } \Omega \times (0,T), \\ \lambda_1 &= 01 && \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{3.1}$$

where $\varphi_1(t) \in C^\infty(0,T)$, and $\delta_1^{h,k} = \delta_1^{h,k}(z,z_0) \in W_h$ is the regularized Dirac δ-function at any fixed point $z_0 \in \bar{\Omega}$. Also, the second pair regularized Green's function $(\mathbf{G}_2, \lambda_2) = (\mathbf{G}_2(z, z_0), \lambda_2(z, z_0))$ is defined such that

$$\begin{aligned} \alpha \mathbf{G}_2 + M ** \mathbf{G}_2 - \nabla \lambda_2 &= \delta_2^h \varphi_2(t) && \text{in } \Omega \times (0,T), \\ \operatorname{div} \mathbf{G}_2 &= 0 && \text{in } \Omega \times (0,T), \\ \lambda_2 &= 0 && \text{on } \partial\Omega \times (0,T), \end{aligned} \tag{3.2}$$

where $\varphi_2(t) \in C^\infty(0,T)$ and $\delta_2^{h,k}$ is either $(\delta_2^{h,k}, 0)$ or $(0, \delta_2^{h,k})$ with $\delta_2^{h,k}$ being a regularized Dirac δ-function at z_0. From [13] we know that

$$\begin{aligned} \|w_{h,k}\|_\infty &\leqslant C|(w_{h,k}, \delta_1^{h,k})|, \quad w_h \in W_h, \\ \|\mathbf{v}_{h,k}\|_\infty &\leqslant C|(\mathbf{v}_{h,k}, \delta_2^{h,k})|, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{h,k}. \end{aligned} \tag{3.3}$$

In addition, we have the following estimates for the regularized Green's functions [13].

Lemma 3.4. *Assume that* $(\mathbf{G}_1, \lambda_1)$, $(\mathbf{G}_2, \lambda_2)$ *and* $(\mathbf{G}_1^{h,k}, \lambda_1^{h,k})$, $(\mathbf{G}_2^{h,k}, \lambda_2^{h,k})$ *are the true solutions and the mixed finite element solutions of* (3.1) *and* (3.2), *respectively. Then, we have*

$$\begin{aligned} \|\mathbf{G}_1^{h,k}\|_0 &\leqslant C|\log H|^{1/2}(1 + \varphi_1(t)), \\ \|\nabla \lambda_1^{h,k}\|_0 &\leqslant C|\log H|^{1/2}(1 + \varphi_1(t)), \\ \|\mathbf{G}_2^{h,k}\|_1 &\leqslant C|\log H|(1 + \varphi_2(t)), \\ \|\lambda_2^{h,k}\|_0 &\leqslant C|\log H|^{1/2}(1 + \varphi_2(t)), \\ \|\nabla \lambda_2^{h,k}\|_0 &\leqslant CH^{-1}(1 + \varphi_2(t)). \end{aligned}$$

3.1. The global Richardson extrapolation in two directions

We first discuss the extrapolation method of mixed finite element approximation for (1.1) in both x and y directions as follows.

Theorem 3.1. *Suppose that* $(u, \boldsymbol{\sigma})$ *and* $(u_{h,k}, \boldsymbol{\sigma}_{h,k})$ *are the exact solution of* (2.1) *and its mixed finite element solution, respectively, with the chosen initial value* $u_{h,k}(0) = P_{h,k}^0 u_0$. *Then, in the sense of* L^∞*-norm we have the following asymptotic expansions under the conditions that* $(u, \boldsymbol{\sigma})$, c, α *and* M *are sufficiently smooth:*

$$\begin{aligned} u_{h,k} - P_{h,k}^0 u &= H^2 \xi_{h,k} + O(H^4 |\log H|^{1/2}), \\ \boldsymbol{\sigma}_{h,k} - \boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma} &= H^2 \boldsymbol{\eta}_{h,k} + O(H^3), \end{aligned}$$

where $(\xi_{h,k}, \boldsymbol{\eta}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ *and* $P^0_{h,k} \times \mathbf{\Pi}^0_{h,k}\colon\ W \times \mathbf{V}_0 \to W_{h,k} \times \mathbf{V}_{0,h,k}$ *is the Raviart-Thomas projection operator.*

P r o o f. Set

$$\varrho_{h,k} := u_{h,k} - P^0_{h,k} u \quad \text{and} \quad \boldsymbol{\theta}_{h,k} := \boldsymbol{\sigma}_{h,k} - \mathbf{\Pi}^0_{h,k}\boldsymbol{\sigma}.$$

Then, it follows from (2.5) and (2.7) that

$$\begin{aligned}
(3.4)\qquad & (\alpha\boldsymbol{\theta}_{h,k}, \mathbf{v}_{h,k}) + \int_0^t (M(t,s)\boldsymbol{\theta}_{h,k}(s), \mathbf{v}_{h,k})\,\mathrm{d}s + (\varrho_{h,k}, \nabla\cdot\mathbf{v}_{h,k}) \\
&= (\alpha(\boldsymbol{\sigma} - \mathbf{\Pi}^0_{h,k}\boldsymbol{\sigma}), \mathbf{v}_{h,k}) + \int_0^t (M(t,s)(\boldsymbol{\sigma} - \mathbf{\Pi}^0_{h,k}\boldsymbol{\sigma})(s), \mathbf{v}_{h,k})\,\mathrm{d}s, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}, \\
& (\varrho_{h,k,t}, w_{h,k}) - (\nabla\cdot\boldsymbol{\theta}_{h,k}, w_{h,k}) - (c\varrho_{h,k}, w_{h,k}) \\
&= -(c(u - P^0_{h,k}u), w_{h,k}), \quad w_{h,k} \in W_{h,k}.
\end{aligned}$$

Thus, from Theorems 2.1 and 2.2 we know that for $\mathbf{v}_{hk} \in \mathbf{V}_{0,h,k}$ and $w_{hk} \in W_{hk}$

$$\begin{aligned}
(3.5)\quad & (\alpha(\boldsymbol{\sigma} - \mathbf{\Pi}^0_{h,k}\boldsymbol{\sigma}) + M * (\boldsymbol{\sigma} - \mathbf{\Pi}^0_{h,k}\boldsymbol{\sigma}), \mathbf{v}_{h,k}) = H^2\mathbf{L}_{h,k}(\mathbf{v}_{h,k}) + O(H^4)\|\mathbf{v}_{h,k}\|_{0,q}, \\
& -(c(u - P^0_{h,k}u), w_{h,k}) = H^2 G_{h,k}(w_{h,k}) + O(H^4)\|w_{h,k}\|_{0,q},
\end{aligned}$$

where $\alpha = (\alpha_{ij})_{2\times2}$, $M = (m_{ij})_{2\times2}$, and

$$\begin{aligned}
G_{h,k}(\varphi) &= -\frac{1}{3}\Big(\frac{h}{H}\Big)^2 \int_\Omega c_x u_x \varphi - \frac{1}{3}\Big(\frac{k}{H}\Big)^2 \int_\Omega c_y u_y \varphi, \\
\mathbf{L}_{h,k}(\psi) &= \frac{1}{3}\Big(\frac{h}{H}\Big)^2 \int_\Omega \{-[\alpha_{11}(\sigma_1)_{xx} + \alpha_{12}(\sigma_2)_{xx}]\psi_1 + [(\alpha_{22})_x(\sigma_2)_x - \alpha_{21}(\sigma_1)_{xx}]\psi_2\} \\
&\quad + \frac{1}{3}\Big(\frac{h}{H}\Big)^2 \int_\Omega\int_0^t \{-[m_{11}(\sigma_1)_{xx} + m_{12}(\sigma_2)_{xx}]\psi_1 \\
&\qquad\qquad + [(m_{22})_x(\sigma_2)_x - m_{21}(\sigma_1)_{xx}]\psi_2\}\,\mathrm{d}s \\
&\quad + \frac{1}{3}\Big(\frac{k}{H}\Big)^2 \int_\Omega \{[(\alpha_{11})_y(\sigma_1)_y - \alpha_{12}(\sigma_2)_{yy}]\psi_1 - [\alpha_{22}(\sigma_2)_{yy} + \alpha_{21}(\sigma_1)_{yy}]\psi_2\} \\
&\quad + \frac{1}{3}\Big(\frac{k}{H}\Big)^2 \int_\Omega\int_0^t \{[(m_{11})_y(\sigma_1)_y - m_{12}(\sigma_2)_{yy}]\psi_1 \\
&\qquad\qquad - [m_{22}(\sigma_2)_{yy} + m_{21}(\sigma_1)_{yy}]\psi_2\}\,\mathrm{d}s
\end{aligned}$$

with $\psi = (\psi_1, \psi_2)$ being a vector-valued function. Evidently,

$$(3.6)\qquad \mathbf{L}_{h/2,k/2}(\boldsymbol{\psi}) = \mathbf{L}_{h,k}(\psi) \quad \text{and} \quad G_{h/2,k/2}(\varphi) = G_{h,k}(\varphi).$$

Let $(\xi, \boldsymbol{\eta}) \in W \times \mathbf{V}_0$ and $(\xi_{h,k}, \boldsymbol{\eta}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ be the variational solution and the mixed finite element solution of the auxiliary problem, respectively,

$$
\begin{aligned}
(3.7) \qquad & (\alpha\boldsymbol{\eta}, \mathbf{v}) + (M * \boldsymbol{\eta}, \mathbf{v})\,\mathrm{d}s + (\xi, \nabla\cdot\mathbf{v}) = \mathbf{L}_{h,k}(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_0,\\
& (\xi_t, w) - (\nabla\cdot\boldsymbol{\eta}, w) - (c\xi, w) = G_{h,k}(w), \quad w \in W,\\
& \xi(0) = 0.
\end{aligned}
$$

Then, from (3.4), (3.5), and (3.7) one finds that

$$
\begin{aligned}
&(\alpha(\boldsymbol{\theta}_{h,k} - H^2\boldsymbol{\eta}_{h,k}) + M * \boldsymbol{\theta}_{h,k}(s) - H^2\boldsymbol{\eta}_{h,k}(s), \mathbf{v}_{h,k}) + (\varrho_{h,k} - H^2\xi_{h,k}, \nabla\cdot\mathbf{v}_{h,k})\\
&\qquad = O(H^4)\|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k},\\
&(\varrho_{h,k,t} - H^2\xi_{h,k,t}, w_{h,k}) - (\nabla\cdot(\boldsymbol{\theta}_{h,k} - H^2\boldsymbol{\eta}_{h,k}), w_{h,k}) - (c(\varrho_{h,k} - H^2\xi_{h,k}), w_{h,k})\\
&\qquad = O(H^4)\|w_{h,k}\|_0, \quad w_{h,k} \in W_{h,k}.
\end{aligned}
$$

Let

$$
\boldsymbol{\theta}^*_{h,k} := \boldsymbol{\theta}_{h,k} - H^2\boldsymbol{\eta}_{h,k} \quad \text{and} \quad \varrho^*_{h,k} := \varrho_{h,k} - H^2\xi_{h,k}.
$$

Thus, we have

$$
(3.8) \quad (\alpha\boldsymbol{\theta}^*_{h,k} + M * \boldsymbol{\theta}^*_{h,k}, \mathbf{v}_{h,k}) + (\varrho^*_{h,k}, \nabla\cdot\mathbf{v}_{h,k}) = O(H^4)\|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k}.
$$

Moreover, in [16] we have proved that

$$
(3.9) \qquad \|\boldsymbol{\theta}^*_{h,k}\|_0 + \|\varrho^*_{h,k}\|_0 \leqslant CH^4.
$$

Set $\mathbf{v}_{h,k} = \mathbf{G}_1^{h,k}$ in (3.8). Then, it follows from Lemma 3.4 for $q = 2$ that

$$
(\alpha\boldsymbol{\theta}^*_{h,k} + M * \boldsymbol{\theta}^*_{h,k}, \mathbf{G}_1^{h,k}) + (\varrho^*_{h,k}, \nabla\cdot\mathbf{G}_1^{h,k}) = O(H^4|\log H|^{1/2})(1 + \varphi_1(t)),
$$

and from Lemma 3.1 and (3.1) that

$$
\begin{aligned}
\left|\int_0^T (\varrho^*_{h,k}, \delta_1^{h,k})\varphi_1(t)\,\mathrm{d}t\right| &\leqslant \left|\int_0^T (\alpha\mathbf{G}_1^{h,k} + M ** \mathbf{G}_1^{h,k}, \boldsymbol{\theta}^*_{h,k})\,\mathrm{d}t\right|\\
&\quad + CH^4|\log H|^{1/2}\int_0^T (1 + \varphi_1(t))\,\mathrm{d}t\\
&= \left|\int_0^T (\nabla\lambda_1^{h,k}, \boldsymbol{\theta}^*_{h,k})\,\mathrm{d}t\right| + CH^4|\log H|^{1/2}\int_0^T (1 + \varphi_1(t))\,\mathrm{d}t.
\end{aligned}
$$

Hence, from Lemma 3.4 and (3.9) we get

$$
\left|\int_0^T (\varrho^*_{h,k}, \delta_1^{h,k})\varphi_1(t)\,\mathrm{d}t\right| \leqslant C|\log H|^{1/2}H^4\int_0^T (1 + \varphi_1(t))\,\mathrm{d}t
$$

which, together with Lemma 3.2 and (3.3), leads to

$$\|\varrho_{h,k}^*\|_\infty \leqslant CH^4|\log H|^{1/2}. \tag{3.10}$$

Letting $\mathbf{v}_{h,k} = \mathbf{G}_2^{h,k}$ in (3.8), we have according to Lemma 3.4 that

$$(\alpha\boldsymbol{\theta}_{h,k}^* + M * \boldsymbol{\theta}_{h,k}^*, \mathbf{G}_2^{h,k}) + (\varrho_{h,k}^*, \nabla \cdot \mathbf{G}_2^{h,k}) = O(H^4|\log H|)(1 + \varphi_2(t)).$$

Since $\nabla \cdot \mathbf{G}_2^{h,k} = 0$ by the mixed finite element approximation of (3.2), we further obtain by means of Lemmas 3.1, 3.4, and (3.9) that

$$\begin{aligned}\left|\int_0^T (\boldsymbol{\theta}_{h,k}^*, \delta_2^{h,k})\varphi_2(t)\,\mathrm{d}t\right| &\leqslant \left|\int_0^T (\nabla\lambda_2^{h,k}, \boldsymbol{\theta}_{h,k}^*)\,\mathrm{d}t\right| + CH^4|\log H| \int_0^T (1 + \varphi_2(t))\,\mathrm{d}t \\ &\leqslant CH^3 \int_0^T (1 + \varphi_2(t))\,\mathrm{d}t.\end{aligned}$$

Finally, we get via Lemma 3.2 and (3.3) that

$$\|\boldsymbol{\theta}_{h,k}^*\|_\infty \leqslant CH^3$$

which, together with (3.10), completes the proof of Theorem 3.1. □

For the future need we shall next present a superconvergent estimate for the mixed finite element solution of (3.7). For this purpose we first introduce the so-called mixed Ritz-Volterra projection.

Definition 3.1. For $(u, \boldsymbol{\sigma}) \in W \times \mathbf{V}$ its mixed Ritz-Volterra projection

$$(\bar{u}_{h,k}, \bar{\boldsymbol{\sigma}}_{h,k})\colon\ [0,T] \to W_{h,k} \times \mathbf{V}_{0,h,k}$$

is defined by

$$\begin{aligned}(\alpha(\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}_{h,k}) + M * (\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}_{h,k}), \mathbf{v}_{h,k}) + (u - \bar{u}_{h,k}, \nabla \cdot \mathbf{v}_{h,k}) = 0, &\quad \mathbf{v}_{h,k} \in \mathbf{V}_{h,k},\\ (\nabla \cdot (\boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}_{h,k}), w_{h,k}) + (c(u - \bar{u}_{h,k}), w_{h,k}) = 0, &\quad w_{h,k} \in W_{h,k}.\end{aligned}$$

In [10] it has been shown that the mixed Ritz-Volterra projection $(\bar{u}_{h,k}, \bar{\boldsymbol{\sigma}}_{h,k}) \in W_{h,k} \times \mathbf{V}_{h,k}$ for an arbitrary pair $(u, \boldsymbol{\sigma}) \in W \times \mathbf{V}$ exists uniquely. In addition, we have [10]

$$\|P_{h,k}^0 u - \bar{u}_{h,k}\|_W + \|\boldsymbol{\Pi}_{h,k}^0 \boldsymbol{\sigma} - \bar{\boldsymbol{\sigma}}_{h,k}\|_{\mathbf{V}} = O(H^2), \tag{3.11}$$

where

$$\|u\|_W := \|u\|_0 \quad \text{and} \quad \|\boldsymbol{\sigma}\|_{\mathbf{V}} := (\|\boldsymbol{\sigma}\|_0^2 + \|\nabla \cdot \boldsymbol{\sigma}\|_0^2)^{1/2}.$$

In order to obtain superconvergence of the mixed finite element approximation for the variational equation (3.7), we choose the initial data approximation as the mixed elliptic projection; i.e.,

$$\begin{aligned}
(3.12)\quad & (\alpha(0)(\boldsymbol{\eta}_{h,k}-\boldsymbol{\eta}(0)),\mathbf{v}_{h,k})+(\xi_{h,k}(0)-\xi(0),\nabla\cdot\mathbf{v}_{h,k})=0,\quad \mathbf{v}_{h,k}\in\mathbf{V}_{0,h,k},\\
& (\nabla\cdot(\boldsymbol{\eta}_{h,k}(0)-\boldsymbol{\eta}(0)),w_{h,k})+(c(0)(\xi_{h,k}(0)-\xi(0)),w_{h,k})=0,\quad w_{h,k}\in W_{h,k}.
\end{aligned}$$

Theorem 3.2. *Assume that* $(\xi,\boldsymbol{\eta})$ *and* $(\xi_{h,k},\boldsymbol{\eta}_{h,k})$ *are the exact solution of* (3.7) *and its mixed finite element solution, respectively, and* $(\xi_{h,k}(0),\boldsymbol{\eta}_{h,k}(0))$ *is chosen to satisfy* (3.12). *Then, we have*

$$\|\xi_{h,k}-P^0_{h,k}\xi\|_0+\|(\xi_{h,k}-P^0_{h,k}\xi)_t\|_0+\|\boldsymbol{\eta}_{h,k}-\boldsymbol{\Pi}^0_{h,k}\boldsymbol{\eta}\|_0\leqslant CH^2.$$

Proof. Let $\tau_{h,k}:=\xi_{h,k}-P^0_{h,k}\xi$ and $\gamma_{h,k}:=\boldsymbol{\eta}_{h,k}-\boldsymbol{\Pi}^0_{h,k}\boldsymbol{\eta}$. Then, from (2.7) and the mixed finite element error equation of (3.7) one derives that

$$\begin{aligned}
(3.13)\qquad & (\alpha\gamma_{h,k}+M*\gamma_{h,k},\mathbf{v}_{h,k})+(\tau_{h,k},\nabla\cdot\mathbf{v}_{h,k})\\
&=(\alpha(\boldsymbol{\eta}-\boldsymbol{\Pi}^0_{h,k}\boldsymbol{\eta})+M*(\boldsymbol{\eta}-\boldsymbol{\Pi}^0_{h,k}\eta),\mathbf{v}_{h,k}),\quad \mathbf{v}_{h,k}\in\mathbf{V}_{0,h,k},\\
& (\tau_{h,k,t},w_{h,k})-(\nabla\cdot\gamma_{h,k},w_{h,k})-(c\tau_{h,k},w_{h,k})\\
&=-(c(\xi-P^0_{h,k}\xi),w_{h,k}),\quad w_{h,k}\in W_{h,k},
\end{aligned}$$

and by Corollaries 2.1 and 2.2 that

$$\begin{aligned}
&(\alpha\gamma_{h,k}+M*\gamma_{h,k},\mathbf{v}_{h,k})+(\tau_{h,k},\nabla\cdot\mathbf{v}_{h,k})=O(H^2)\|\mathbf{v}_{h,k}\|_{0,q},\quad \mathbf{v}_{h,k}\in\mathbf{V}_{0,h,k},\\
&(\tau_{h,k,t},w_{h,k})-(\nabla\cdot\gamma_{h,k},w_{h,k})-(c\tau_{h,k},w_{h,k})=O(H^2)\|w_{h,k}\|_{0,q},\quad w_{h,k}\in W_{h,k}.
\end{aligned}$$

Therefore, setting $w_{h,k}=\tau_{h,k}$ and $\mathbf{v}_{h,k}=\gamma_{h,k}$ in (3.13), we obtain from their sum that

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\tau_{h,k}\|_0^2-(c\tau_{h,k},\tau_{h,k})+\|\gamma_{h,k}\|^2_{A^{-1}}\\
&\qquad\leqslant-\int_0^t(M(t,s)\gamma_{h,k}(s),\gamma_{h,k})\,\mathrm{d}s+CH^2(\|\gamma_{h,k}\|_0+\|\tau_{h,k}\|_0),
\end{aligned}$$

and by means of Lemma 3.3 and the ε-type inequality that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tau_{h,k}\|_0^2+\|\gamma_{h,k}\|_0^2\leqslant C\Big\{\int_0^t\|\gamma_{h,k}(s)\|_0^2\,\mathrm{d}s+\|\tau_{h,k}\|_0^2+H^4\Big\}.$$

Integrating the above inequality from 0 to t and noticing $\tau_{h,k}(0) = 0$ leads to

$$\|\tau_{h,k}\|_0^2 + \int_0^t \|\gamma_{h,k}(s)\|_0^2 \,\mathrm{d}s \leqslant C \int_0^t \left\{ \|\tau_{h,k}(s)\|_0^2 + \int_0^s \|\gamma_{h,k}(\tau)\|_0^2 \,\mathrm{d}\tau \right\} \mathrm{d}s + CH^4$$

which, together with Gronwall's lemma, implies

$$\|\tau_{h,k}\|_0^2 + \int_0^t \|\gamma_{h,k}(s)\|_0^2 \,\mathrm{d}s \leqslant CH^4.$$

Thus,

$$\|\tau_{h,k}\|_0 \leqslant CH^2. \tag{3.14}$$

In order to get the estimate for $\gamma_{h,k}$ we first differentiate (3.13) to obtain

$$(\alpha_t \gamma_{h,k} + \alpha \gamma_{h,k,t} + M\gamma_{h,k} + M_t * \gamma_{h,k}, \mathbf{v}_{h,k}) + (\tau_{h,k,t}, \nabla \cdot \mathbf{v}_{h,k}) = O(H^2)\|\mathbf{v}_{h,k}\|_0,$$
$$\mathbf{v}_{h,k} \in \mathbf{V}_{0,h,k},$$

and then by setting $\mathbf{v}_{h,k} = \gamma_{h,k}$ in the above equation and $w_{h,k} = \tau_{h,k,t}$ in (3.13), we have from their sum that

$$\begin{aligned} \|\tau_{h,k,t}\|_0^2 &+ (\alpha\gamma_{h,k,t}, \gamma_{h,k}) + (\alpha_t \gamma_{h,k}, \gamma_{h,k}) \\ &= -(M\gamma_{h,k} + M_t * \gamma_{h,k}, \gamma_{h,k}) + (c\tau_{h,k}, \tau_{h,k,t}). \end{aligned} \tag{3.15}$$

It follows from

$$\alpha(\gamma_{h,k}^2)_t = (\alpha\gamma_{h,k}^2)_t - \alpha_t \gamma_{h,k}^2$$

that

$$\begin{aligned} (\alpha\gamma_{h,k,t}, \gamma_{h,k}) = \int_\Omega \alpha\gamma_{h,k,t}\gamma_{h,k} &= \frac{1}{2}\int_\Omega \alpha \frac{\mathrm{d}}{\mathrm{d}t}(\gamma_{h,k}^2) \\ &= \frac{1}{2}\int_\Omega \frac{\mathrm{d}}{\mathrm{d}t}(\alpha\gamma_{h,k}^2) - \frac{1}{2}\int_\Omega \alpha_t\gamma_{h,k}^2 \\ &= \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\gamma_{h,k}\|_{A^{-1}}^2 - \frac{1}{2}(\alpha_t\gamma_{h,k}, \gamma_{h,k}) \end{aligned}$$

which, together with (3.15) and the ε-type inequality, produces that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\gamma_{h,k}\|_{A^{-1}}^2 + \frac{1}{2}(\alpha_t\gamma_{h,k}, \gamma_{h,k}) \leqslant C\left\{ \|\gamma_{h,k}\|_0^2 + \int_0^t \|\gamma_{h,k}\|_0^2 \,\mathrm{d}s \right\} + C\|\tau_{h,k}\|_0^2,$$

and then via (3.14), integrating from 0 to t and using Lemma 3.3,

$$\|\gamma_{h,k}\|_0^2 \leqslant \|\gamma_{h,k}(0)\|_0^2 + C\Big\{H^4 + \int_0^t \|\gamma_{h,k}(s)\|_0^2\,\mathrm{d}s\Big\}.$$

To estimate $\|\gamma_{h,k}(0)\|_0$ we set $t=0$ and $\mathbf{v}_{h,k} = \gamma_{h,k}(0)$ in (3.13) to gain that

$$(\alpha(0)\gamma_{h,k}(0), \gamma_{h,k}(0)) \leqslant CH^2\|\gamma_{h,k}(0)\|_0, \quad \text{where} \quad \tau_{h,k}(0) = 0 \text{ is used,}$$

and by Lemma 3.3 that

$$\|\gamma_{h,k}(0)\|_0 \leqslant CH^2.$$

Thus,

$$\|\gamma_{h,k}\|_0^2 \leqslant C\Big\{H^4 + \int_0^t \|\gamma_{h,k}\|_0^2\,\mathrm{d}s\Big\}$$

which, together with Gronwall's lemma, demonstrates

$$\|\gamma_{h,k}\|_0 \leqslant CH^2. \tag{3.16}$$

Hence, only $\|\tau_{h,k,t}\|_0$ remains to be estimated to finish the proof of the theorem.

We differentiate (3.13) to obtain

$$\begin{gathered}(\alpha_t\gamma_{h,k} + \alpha\gamma_{h,k,t} + M\gamma_{h,k} + M_t * \gamma_{h,k}, \mathbf{v}_{h,k}) + (\tau_{h,k,t}, \nabla\cdot\mathbf{v}_{h,k}) = O(H^2)\|\mathbf{v}_{h,k}\|_0,\\ (\tau_{h,k,t}, w_{h,k}) - (\nabla\cdot\gamma_{h,k,t}, w_{h,k}) - (c_t\tau_{h,k} + c\tau_{h,k,t}, w_{h,k}) = O(H^2)\|w_{h,k}\|_0,\end{gathered}$$

and thus, letting $\mathbf{v}_{h,k} = \gamma_{h,k,t}$ and $w_{h,k} = \tau_{h,k,t}$ yields from their sum, Lemma 3.3, the ε-inequality, (3.14) and (3.16) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\tau_{h,k,t}\|_0^2 \leqslant C\{H^4 + \|\tau_{h,k,t}\|_0^2\},$$

or

$$\|\tau_{h,k,t}\|_0^2 \leqslant \|\tau_{h,k,t}(0)\|_0^2 + C\Big\{H^4 + \int_0^t \|\tau_{h,k,t}(s)\|_0^2\,\mathrm{d}s\Big\}. \tag{3.17}$$

From Definition 3.1 and (3.12) we derive that

$$\begin{gathered}(\alpha(0)(\overline{\boldsymbol{\eta}}_{h,k} - \boldsymbol{\eta}_{h,k})(0), \mathbf{v}_{h,k}) + (\bar{\xi}_{h,k}(0) - \xi_{h,k}(0), \nabla\cdot\mathbf{v}_{h,k}) = 0, \quad \mathbf{v}_{h,k}\in\mathbf{V}_{0,h,k},\\ (\nabla\cdot(\overline{\boldsymbol{\eta}}_{h,k} - \boldsymbol{\eta}_{h,k})(0), w_{h,k}) + (c(0)(\bar{\xi}_{h,k} - \xi_{h,k})(0), w_{h,k}) = 0, \quad w_{h,k}\in W_{h,k},\end{gathered}$$

where $(\bar{\xi}_{h,k}, \overline{\boldsymbol{\eta}}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ is the Ritz-Volterra projection of $(\xi, \boldsymbol{\eta})$. Therefore, the uniqueness of the solution to (3.12) implies

$$\overline{\boldsymbol{\eta}}_{h,k}(0) - \boldsymbol{\eta}_{h,k}(0) = \bar{\xi}_{h,k}(0) - \xi_{h,k}(0) = 0$$

which, together with (3.11), indicates that

$$\|\nabla \cdot \gamma_{h,k}(0)\|_0 = \|\nabla \cdot (\overline{\boldsymbol{\eta}}_{h,k} - \mathbf{\Pi}^0_{h,k}\boldsymbol{\eta})(0)\|_0 \leqslant CH^2.$$

Thus, it follows from (3.14), setting $t = 0$ and $w_{h,k} = \tau_{h,k,t}(0)$ in (3.13) that

$$\|\tau_{h,k,t}(0)\|_0^2 \leqslant CH^2 \|\tau_{h,k,t}(0)\|_0$$

or

$$\|\tau_{h,k,t}(0)\|_0 \leqslant CH^2.$$

This, together with (3.17) and Gronwall's lemma, claims that

$$\|\tau_{h,k,t}\|_0 \leqslant CH^2.$$

□

Theorem 3.3. *We have under the conditions of Theorem* 3.2 *that*

$$|\log H|^{1/2} \|\xi_{h,k} - P^0_{h,k}\xi\|_\infty + \|\boldsymbol{\eta}_{h,k} - \mathbf{\Pi}^0_{h,k}\boldsymbol{\eta}\|_\infty \leqslant CH^2 |\log H|.$$

Proof. Set $\mathbf{v}_{h,k} = \mathbf{G}_1^{h,k}$ in (3.13) and follow the steps for (3.10) to get

$$\|\tau_{h,k}\|_\infty \leqslant CH^2 |\log H|^{1/2}. \tag{3.18}$$

It follows from Theorem 3.2 and (3.13) that

$$\begin{aligned} |(\nabla \cdot \gamma_{h,k}, w_{h,k})| &\leqslant \|\tau_{h,k,t}\|_0 \|w_{h,k}\|_0 + C\|\tau_{h,k}\|_0 \|w_{h,k}\|_0 + CH^2 \|w_{h,k}\|_0 \\ &\leqslant CH^2 \|w_{h,k}\|_0, \end{aligned}$$

which implies

$$\|\nabla \cdot \gamma_{h,k}\|_0 \leqslant CH^2. \tag{3.19}$$

Let $\mathbf{v}_{h,k} = \mathbf{G}_2^{h,k}$ in (3.13) to obtain according to Lemmas 3.4 and 3.1, Green's formula and (3.19) that

$$\begin{aligned}
&\left|\int_0^T (\gamma_{h,k}, \delta_2^{h,k})\varphi_2(t)\,\mathrm{d}t\right| \\
&\qquad \leqslant \left|\int_0^T (\lambda_2^{h,k}, \nabla\cdot\gamma_{h,k})\,\mathrm{d}t\right| + CH^2|\log H|\int_0^T (1+\varphi_2(t))\,\mathrm{d}t \\
&\qquad \leqslant CH^2|\log H|^{1/2}\int_0^T (1+\varphi_2(t))\,\mathrm{d}t + CH^2|\log H|\int_0^T (1+\varphi_2(t))\,\mathrm{d}t.
\end{aligned}$$

Thus, Lemma 3.2 and (3.3) lead to

$$\|\gamma_{h,k}\|_\infty \leqslant CH^2|\log H|, \quad \text{where} \quad \nabla\cdot\mathbf{G}_2^{h,k} = 0 \text{ has been used.}$$

This, together with (3.18), completes the proof of the theorem. □

As we have done in our previous work [10], we now utilize an interpolation postprocessing method to gain the global extrapolation approximations of higher accuracy for the pressure field and the velocity field. For this end, we need to define two postprocessing interpolation operator $\mathbf{\Pi}^2_{3h,3k}$ and $P^3_{4h,4k}$ to satisfy

$$\begin{aligned}
&\mathbf{\Pi}^2_{3h,3k}\mathbf{\Pi}^0_{h,k} = \mathbf{\Pi}^3_{3h,3k}, && \\
&\|\mathbf{\Pi}^2_{3h,3k}\mathbf{v}_{h,k}\|_{0,p} \leqslant C\|\mathbf{v}_{h,k}\|_{0,p} \quad \forall\,\mathbf{v}_{h,k}\in\mathbf{V}_{0,h,k}, && 1\leqslant p\leqslant\infty, \\
&\|\mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}-\boldsymbol{\sigma}\|_{0,p} \leqslant CH^3\|\boldsymbol{\sigma}\|_{3,p} \quad \forall\,\boldsymbol{\sigma}\in(W^{3,p}(\Omega))^2, && 1\leqslant p\leqslant\infty, \\
&P^3_{4h,4k}P^0_{h,k} = P^3_{4h,4k}, && \\
&\|P^3_{4h,4k}w_{h,k}\|_{0,p} \leqslant C\|w_{h,k}\|_{0,p} \quad \forall\,w_{h,k}\in W_{h,k}, && 1\leqslant p\leqslant\infty, \\
&\|P^3_{4h,4k}u-u\|_{0,p} \leqslant CH^4\|u\|_{4,p} \quad \forall\,u\in W^{4,p}(\Omega), && 1\leqslant p\leqslant\infty.
\end{aligned} \tag{3.20}$$

We assume that the rectangular partition $T_{h,k}$ has been obtained from $T_{3h,3k}$ with mesh size $3H$ and $T_{4h,4k}$ with mesh size $4H$ by subdividing each element of $T_{3h,3k}$ and $T_{4h,4k}$ into nine and sixteen small congruent rectangles, respectively. Let $\tau := \bigcup_{i=1}^{9} e_i$ and $\hat\tau := \bigcup_{i=1}^{16} \hat e_i$ with e_i, $\hat e_i \in T_{h,k}$. Then, we can define two projection interpolation operators $\mathbf{\Pi}^2_{3h,3k}$ and $P^3_{4h,4k}$ associated with $T_{3h,3k}$ and $T_{4h,4k}$ of degree at most 2 and 3 in x and y on τ and $\hat\tau$, respectively, according to the following conditions:

$$\begin{aligned}
&\mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}|_\tau \in Q_{3,2}(\tau)\times Q_{2,3}(\tau), \quad P^3_{4h,4k}u|_{\hat\tau}\in Q_{3,3}(\hat\tau), \\
&\int_{s_i}(\boldsymbol{\sigma}-\mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma})\cdot\mathbf{n}\,\mathrm{d}s = 0, \quad i=1,2,\ldots,24, \quad \text{and} \\
&\int_{\hat e_i}(u-P^3_{4h,4k}u) = 0, \quad i=1,2,\ldots,16, \ \text{respectively},
\end{aligned} \tag{3.21}$$

where s_i $(i = 1, 2, \ldots, 24)$ is one of the twenty-four sides of the nine small elements e_i $(i = 1, 2, \ldots, 9)$. It is easy to check that the two operators $\mathbf{\Pi}^2_{3h,3k}$ and $P^3_{4h,4k}$ defined by (3.21) possess the properties described in (3.20).

We are now in a position to assert our main result in this section. In fact, using Theorems 3.1, 3.3, (3.20) and following the procedure for Theorem 4.2 in [16] we can establish:

Theorem 3.4. *Under the conditions of Theorem* 3.1 *we have in the sense of L^∞-norm that*

$$P^3_{4h,4k}u_{h,k} - u = H^2\xi + O(H^4|\log H|^{1/2}),$$
$$\mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k} - \boldsymbol{\sigma} = H^2\boldsymbol{\eta} + O(H^3),$$

where $(\xi, \boldsymbol{\eta}) \in W \times \mathbf{V}_0$ is the variational solution of (3.7).

On the basis of Theorem 3.4 we can generate the approximations of higher precision for problem (1.1) by Richardson extrapolation. For this end, in addition to $W_{h,k} \times \mathbf{V}_{0,h,k}$, we employ another Raviart-Thomas mixed finite element space $W_{h/2,k/2} \times \mathbf{V}_{0,h/2,k/2}$ of the lowest order gained by subdividing each element $e_i \in T_{h,k}$ into four small congruent elements $\hat{e}_{i,j} \in T_{h/2,k/2}$ $(j = 1, 2, 3, 4)$. Let $(u_{h/2,k/2}, \boldsymbol{\sigma}_{h/2,k/2}) \in W_{h/2,k/2} \times \mathbf{V}_{0,h/2,k/2}$ and $P^3_{2h,2k} \times \mathbf{\Pi}^2_{3h/2,3k/2}$ stand for the mixed finite element solution and the Raviart-Thomas projection operator with respect to this new partition, respectively. Then, we know from Theorem 3.4 that

$$P^3_{2h,2k}u_{h/2,k/2} - u = \left(\frac{H}{2}\right)^2\xi + O(H^4|\log H|^{1/2}),$$

which gives by applying once the Richardson extrapolation that

$$\frac{4P^3_{2h,2k}u_{h/2,k/2} - P^3_{4h,4k}u_{h,k}}{3} = u + O(H^4|\log H|^{1/2}). \tag{3.22}$$

Similarly, we have

$$\frac{4\mathbf{\Pi}^2_{3h/2,3k/2}\boldsymbol{\sigma}_{h/2,k/2} - \mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k}}{3} = \boldsymbol{\sigma} + O(H^3). \tag{3.23}$$

An important application of the approximations of higher accuracy given by (3.22) and (3.23) is that they can provide efficient a posteriori error estimators to assess the accuracy of the approximate solutions in applications. In fact, following the same way as that for Theorem 5.3 in [16] we are led to

Theorem 3.5. *Under the assumptions of Theorem* 3.1, *we have*

$$\|u - P^3_{2h,2k}u_{h/2,k/2}\|_\infty = \frac{1}{3}\|P^3_{2h,2k}u_{h/2,k/2} - P^3_{4h,4k}u_{h,k}\|_\infty + O(H^4|\log H|^{1/2}),$$
$$\|\boldsymbol{\sigma} - \boldsymbol{\Pi}^2_{3h/2,3k/2}\boldsymbol{\sigma}_{h/2,k/2}\|_\infty = \frac{1}{3}\|\boldsymbol{\Pi}^2_{3h/2,3k/2}\boldsymbol{\sigma}_{h/2,k/2} - \boldsymbol{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k}\|_\infty + O(H^3).$$

Moreover, if there exist positive constants C_1, C_2 *and sufficiently small* $\varepsilon_1, \varepsilon_2 \in (0,1)$ *such that*

$$\|u - P^3_{2h,2k}u_{h/2,k/2}\|_\infty \geqslant C_1 H^{4-\varepsilon_1}|\log H|^{1/2}, \tag{3.24}$$
$$\|\boldsymbol{\sigma} - \boldsymbol{\Pi}^2_{3h/2,3k/2}\boldsymbol{\sigma}_{h/2,k/2}\|_\infty \geqslant C_2 H^{3-\varepsilon_2}, \tag{3.25}$$

then we have

$$\lim_{H\to 0}\frac{3\|u - P^3_{2h,2k}u_{h/2,k/2}\|_\infty}{\|P^3_{2h,2k}u_{h/2,k/2} - P^3_{4h,4k}u_{h,k}\|_\infty} = 1,$$
$$\lim_{H\to 0}\frac{3\|\boldsymbol{\sigma} - \boldsymbol{\Pi}^2_{3h/2,3k/2}\boldsymbol{\sigma}_{h/2,k/2}\|_\infty}{\|\boldsymbol{\Pi}^2_{3h/2,3k/2}\boldsymbol{\sigma}_{h/2,k/2} - \boldsymbol{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k}\|_0} = 1.$$

From Theorem 3.4 we know that the optimal convergence rates of

$$\|u - P^3_{2h,2k}u_{h/2,k/2}\|_\infty \quad \text{and} \quad \|\boldsymbol{\sigma} - \boldsymbol{\Pi}^2_{3h/2,3k/2}\boldsymbol{\sigma}_{h/2,k/2}\|_\infty$$

are $O(H^2)$. Therefore, the assumed conditions (3.24) and (3.25) seem to be reasonable.

3.2. The global Richardson extrapolation in one direction

It is clear according to (3.22) and (3.23) that this extrapolation approach requires a global refinement to generate an approximation of higher accuracy, which wastes computing time and memory. Therefore, it is natural for us to develop a new type of extrapolation to overcome this shortcoming. In fact, here we shall construct a new extrapolation formula of a partial refinement, in which the meshes are refined just in one direction, x- or y-direction, such that it is more efficient and more suitable for parallel computations.

Theorem 3.6. *Under the conditions of Theorem* 3.1 *we have in* L^∞*-norm that*

$$u_{h,k} - P^0_{h,k}u = h^2\xi^1_{h,k} + k^2\xi^2_{h,k} + O(H^4|\log H|^{1/2}),$$
$$\boldsymbol{\sigma}_{h,k} - \mathbf{\Pi}^0_{h,k}\boldsymbol{\sigma} = h^2\boldsymbol{\eta}^1_{h,k} + k^2\boldsymbol{\eta}^2_{h,k} + O(H^3),$$

where $(\xi^1_{h,k}, \boldsymbol{\eta}^1_{h,k}), (\xi^2_{h,k}, \boldsymbol{\eta}^2_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$.

Proof. Let $(\xi^1, \boldsymbol{\eta}^1), (\xi^2, \boldsymbol{\eta}^2) \in W \times \mathbf{V}_0$ and $(\xi^1_{h,k}, \boldsymbol{\eta}^1_{h,k}), (\xi^2_{h,k}, \boldsymbol{\eta}^2_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ be the exact solutions and the mixed finite element solutions of the following two auxiliary variational problems, respectively:

$$\begin{aligned}
(\alpha\boldsymbol{\eta}^1 + M * \boldsymbol{\eta}^1, \mathbf{v}) + (\xi^1, \nabla\cdot\mathbf{v}) &= \mathbf{L}_1(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_0,\\
(\xi^1_t, w) - (\nabla\cdot\boldsymbol{\eta}^1, w) - (c\xi^1, w) &= L_3(w), \quad w \in W,\\
\xi^1(0) &= 0,
\end{aligned} \tag{3.26}$$

and

$$\begin{aligned}
(\alpha\boldsymbol{\eta}^2 + M * \boldsymbol{\eta}^2, \mathbf{v}) + (\xi^2, \nabla\cdot\mathbf{v}) &= \mathbf{L}_2(\mathbf{v}), \quad \mathbf{v} \in \mathbf{V}_0,\\
(\xi^2_t, w) - (\nabla\cdot\boldsymbol{\eta}^2, w) - (c\xi^2, w) &= L_4(w), \quad w \in W,\\
\xi^2(0) &= 0,
\end{aligned} \tag{3.27}$$

where

$$\begin{aligned}
\mathbf{L}_1(\mathbf{v}) &= \frac{1}{3}\int_\Omega \{-[\alpha_{11}(\sigma_1)_{xx} + \alpha_{12}(\sigma_2)_{xx}]v_1 + [(\alpha_{22})_x(\sigma_2)_x - \alpha_{21}(\sigma_1)_{xx}]v_2\}\\
&\quad + \frac{1}{3}\int_\Omega\int_0^t \{-[m_{11}(\sigma_1)_{xx} + m_{12}(\sigma_2)_{xx}](s)v_1\\
&\qquad\qquad + [(m_{22})_x(\sigma_2)_x - m_{21}(\sigma_1)_{xx}](s)v_2\}\,\mathrm{d}s,\\
\mathbf{L}_2(\mathbf{v}) &= \frac{1}{3}\int_\Omega \{[(\alpha_{11})_y(\sigma_1)_y - \alpha_{12}(\sigma_2)_{yy}]v_1 - [\alpha_{22}(\sigma_2)_{yy} + \alpha_{21}(\sigma_1)_{yy}]v_2\}\\
&\quad + \frac{1}{3}\int_\Omega\int_0^t \{[(m_{11})_y(\sigma_1)_y - m_{12}(\sigma_2)_{yy}](s)v_1\\
&\qquad\qquad - [m_{22}(\sigma_2)_{yy} + m_{21}(\sigma_1)_{yy}](s)v_2\}\,\mathrm{d}s,\\
L_3(w) &= -\frac{1}{3}\int_\Omega c_x u_x w,\\
L_4(w) &= -\frac{1}{3}\int_\Omega c_y u_y w.
\end{aligned}$$

From (3.4), Theorems 2.1 and 2.2 one finds that

$$(3.28)\qquad \begin{aligned}&(\alpha\boldsymbol{\theta}_{h,k} + M * \boldsymbol{\theta}_{h,k}, \mathbf{v}_{h,k}) + (\varrho_{h,k}, \nabla\cdot\mathbf{v}_{h,k})\\ &= h^2\mathbf{L}_1(\mathbf{v}_{h,k}) + k^2\mathbf{L}_2(\mathbf{v}_{h,k}) + O(H^4)\|\mathbf{v}_{h,k}\|_{0,q}, \quad \mathbf{v}_{h,k}\in\mathbf{V}_{0,h,k},\\ &(\varrho_{h,k,t}, w_{h,k}) - (\nabla\cdot\boldsymbol{\theta}_{h,k}, w_{h,k}) - (c\varrho_{h,k}, w_{h,k})\\ &= h^2L_3(w_{h,k}) + k^2L_4(w_{h,k}) + O(H^4)\|w_{h,k}\|_{0,q}, \quad w_{h,k}\in W_{h,k}.\end{aligned}$$

Let

$$\hat{\boldsymbol{\theta}}_{h,k} := \boldsymbol{\theta}_{h,k} - h^2\boldsymbol{\eta}^1_{h,k} - k^2\boldsymbol{\eta}^2_{h,k}, \quad \hat{\varrho}_{h,k} := \varrho_{h,k} - h^2\xi^1_{h,k} - k^2\xi^2_{h,k}.$$

Then, it follows from (3.26)–(3.28) that

$$(3.29)\qquad \begin{aligned}&(\alpha\hat{\boldsymbol{\theta}}_{h,k} + M * \hat{\boldsymbol{\theta}}_{h,k}, \mathbf{v}_{h,k}) + (\hat{\varrho}_{h,k}, \nabla\cdot\mathbf{v}_{h,k}) = O(H^4)\|\mathbf{v}_{h,k}\|_{0,q}, \ \ \mathbf{v}_{h,k}\in\mathbf{V}_{0,h,k},\\ &(\hat{\varrho}_{h,k,t}, w_{h,k}) - (\nabla\cdot\hat{\boldsymbol{\theta}}_{h,k}, w_{h,k}) - (c\hat{\varrho}_{h,k}, w_{h,k}) = O(H^4)\|w_{h,k}\|_{0,q}, \ \ w_{h,k}\in W_{h,k}.\end{aligned}$$

It has been shown in [16] that

$$(3.30)\qquad \|\hat{\varrho}_{h,k}\|_0 + \|\hat{\boldsymbol{\theta}}_{h,k}\|_0 \leqslant CH^4.$$

Thus, we can also obtain by using (3.29)–(3.30) and following the procedure for the estimates of $\|\varrho^*_{h,k}\|_\infty$ and $\|\boldsymbol{\theta}^*_{h,k}\|_\infty$ in Theorem 3.1 that

$$\|\hat{\varrho}_{h,k}\|_\infty \leqslant CH^4|\log H|^{1/2} \quad \text{and} \quad \|\hat{\boldsymbol{\theta}}_{h,k}\|_\infty \leqslant CH^3.$$

□

Also, we can get the analog of Theorem 3.4 by means of the analog of Theorem 3.2.

Theorem 3.7. *Under the conditions of Theorem* 3.1 *we have in the* L^∞*-norm that*

$$\begin{aligned}P^3_{4h,4k}u_{h,k} - u &= h^2\xi^1 + k^2\xi^2 + O(H^4|\log H|^{1/2}),\\ \boldsymbol{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k} - \boldsymbol{\sigma} &= h^2\boldsymbol{\eta}^1 + k^2\boldsymbol{\eta}^2 + O(H^3),\end{aligned}$$

where $(\xi^1, \boldsymbol{\eta}^1)$, $(\xi^2, \boldsymbol{\eta}^2) \in W\times\mathbf{V}_0$.

Like (3.22) and (3.23), from Theorem 3.7 we can obtain the following unidirectional Richardson extrapolation in the L^∞ norm:

$$(3.31)\qquad \begin{aligned}&\frac{4(\boldsymbol{\Pi}^2_{3h/2,3k}\boldsymbol{\sigma}_{h/2,k} + \boldsymbol{\Pi}^2_{3h,3k/2}\boldsymbol{\sigma}_{h,k/2}) - 5\boldsymbol{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k}}{3} = \boldsymbol{\sigma} + O(H^3),\\ &\frac{4(P^3_{2h,4k}u_{h/2,k} + P^3_{4h,2k}u_{h,k/2}) - 5P^3_{4h,4k}u_{h,k}}{3} = u + O(H^4|\log H|^{1/2}),\end{aligned}$$

where $(u_{h/2,k}, \boldsymbol{\sigma}_{h/2,k})$, $(u_{h,k/2}, \boldsymbol{\sigma}_{h,k/2})$ and $(u_{h,k}, \boldsymbol{\sigma}_{h,k})$ are the mixed finite element solutions corresponding to the meshes $T_{h/2,k}$, $T_{h,k/2}$ and $T_{h,k}$, respectively. Here, $T_{h/2,k}$ and $T_{h,k/2}$ are the meshes gained by subdividing each element of $T_{h,k}$ into two small congruent rectangles in x- and y-direction, respectively.

Similar to Theorem 3.5, a posteriori error indicators can also be constructed by virtue of (3.31). In fact, we have:

Theorem 3.8. *Under the conditions of Theorem* 3.1 *we have*

$$\begin{aligned}
&\|u - P^3_{2h,4k}u_{h/2,k}\|_\infty \\
&= \frac{1}{3}\|P^3_{2h,4k}u_{h/2,k} + 4P^3_{4h,2k}u_{h,k/2} - 5P^3_{4h,4k}u_{h,k}\|_\infty + O(H^4|\log H|^{1/2}),\\
&\|\boldsymbol{\sigma} - \mathbf{\Pi}^2_{3h/2,3k}\boldsymbol{\sigma}_{h/2,k}\|_\infty \\
&= \frac{1}{3}\|\mathbf{\Pi}^2_{3h/2,3k}\boldsymbol{\sigma}_{h/2,k} + 4\mathbf{\Pi}^2_{3h,3k/2}\boldsymbol{\sigma}_{h,k/2} - 5\mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k}\|_\infty + O(H^3),\\
&\|u - P^3_{4h,2k}u_{h,k/2}\|_\infty \\
&= \frac{1}{3}\|4P^3_{2h,4k}u_{h/2,k} + P^3_{4h,2k}u_{h,k/2} - 5P^3_{4h,4k}u_{h,k}\|_\infty + O(H^4|\log H|^{1/2}),\\
&\|\boldsymbol{\sigma} - \mathbf{\Pi}^2_{3h,3k/2}\boldsymbol{\sigma}_{h,k/2}\|_\infty \\
&= \frac{1}{3}\|4\mathbf{\Pi}^2_{3h/2,3k}\boldsymbol{\sigma}_{h/2,k} + \mathbf{\Pi}^2_{3h,3k/2}\boldsymbol{\sigma}_{h,k/2} - 5\mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k}\|_\infty + O(H^3).
\end{aligned}$$

In addition, if there exist positive constants C_1, C_2, C_3, C_4 and sufficiently small ε_1, ε_2, ε_3, $\varepsilon_4 \in (0,1)$ such that

$$\begin{aligned}
\|u - P^3_{2h,4k}u_{h/2,k}\|_\infty &\geqslant C_1 H^{4-\varepsilon_1}|\log H|^{1/2},\\
\|\boldsymbol{\sigma} - \mathbf{\Pi}^2_{3h/2,3k}\boldsymbol{\sigma}_{h/2,k}\|_\infty &\geqslant C_2 H^{3-\varepsilon_2},\\
\|u - P^3_{4h,2k}u_{h,k/2}\|_\infty &\geqslant C_3 H^{4-\varepsilon_3}|\log H|^{1/2},\\
\|\boldsymbol{\sigma} - \mathbf{\Pi}^2_{3h,3k/2}\boldsymbol{\sigma}_{h,k/2}\|_\infty &\geqslant C_4 H^{3-\varepsilon_4},
\end{aligned}$$

then

$$\begin{gathered}
\lim_{H\to 0}\frac{3\|u - P^3_{2h,4k}u_{h/2,k}\|_\infty}{\|P^3_{2h,4k}u_{h/2,k} + 4P^3_{4h,2k}u_{h,k/2} - 5P^3_{4h,4k}u_{h,k}\|_\infty} = 1,\\
\lim_{H\to 0}\frac{3\|\boldsymbol{\sigma} - \mathbf{\Pi}^2_{3h/2,3k}\boldsymbol{\sigma}_{h/2,k}\|_\infty}{\|\mathbf{\Pi}^2_{3h/2,3k}\boldsymbol{\sigma}_{h/2,k} + 4\mathbf{\Pi}^2_{3h,3k/2}\boldsymbol{\sigma}_{h,k/2} - 5\mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k}\|_\infty} = 1,\\
\lim_{H\to 0}\frac{3\|u - P^3_{4h,2k}u_{h,k/2}\|_\infty}{\|4P^3_{2h,4k}u_{h/2,k} + P^3_{4h,2k}u_{h,k/2} - 5P^3_{4h,4k}u_{h,k}\|_\infty} = 1,\\
\lim_{H\to 0}\frac{3\|\boldsymbol{\sigma} - \mathbf{\Pi}^2_{3h,3k/2}\boldsymbol{\sigma}_{h,k/2}\|_\infty}{\|4\mathbf{\Pi}^2_{3h/2,3k}\boldsymbol{\sigma}_{h/2,k} + \mathbf{\Pi}^2_{3h,3k/2}\boldsymbol{\sigma}_{h,k/2} - 5\mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k}\|_0} = 1.
\end{gathered}$$

4. Interpolation defect correction

In this section we propose and investigate an interpolation defect correction scheme (see, for example, [19], [21], [22]) applied to the mixed finite element solution $(u_{h,k}, \boldsymbol{\sigma}_{h,k}) \in W_{h,k} \times \mathbf{V}_{0,h,k}$ to obtain approximations with higher convergence rate. Also, these new approximations are naturally used to form a posteriori error estimators in order to estimate the actual accuracy of the mixed finite element solutions.

First of all, for the future need we construct two projection interpolation operators $P^1_{2h,2k}$ and $\mathbf{\Pi}^1_{2h,2k}$ associated with the mesh $T_{2h,2k}$ to satisfy

$$
\begin{aligned}
&\mathbf{\Pi}^1_{2h,2k}\mathbf{\Pi}^0_{h,k} = \mathbf{\Pi}^1_{2h,2k}, && && \\
&\|\mathbf{\Pi}^1_{2h,2k}\mathbf{v}_{h,k}\|_{0,p} \leqslant C\|\mathbf{v}_{h,k}\|_{0,p} && \forall\, \mathbf{v}_{h,k} \in \mathbf{V}_{h,k}, && 1 \leqslant p \leqslant \infty, \\
&\|\mathbf{\Pi}^1_{2h,2k}\boldsymbol{\sigma} - \boldsymbol{\sigma}\|_{0,p} \leqslant CH^2\|\boldsymbol{\sigma}\|_{2,p} && \forall\, \boldsymbol{\sigma} \in (W^{2,p}(\Omega))^2, && 1 \leqslant p \leqslant \infty, \\
&P^1_{2h,2k}P^0_{h,k} = P^1_{2h,2k}, && && \\
&\|P^1_{2h,2k}w_{h,k}\|_{0,p} \leqslant C\|w_{h,k}\|_{0,p} && \forall\, w_{h,k} \in W_{h,k}, && 1 \leqslant p \leqslant \infty, \\
&\|P^1_{2h,2k}u - u\|_{0,p} \leqslant CH^2\|u\|_{2,p} && \forall\, u \in W^{2,p}(\Omega), && 1 \leqslant p \leqslant \infty.
\end{aligned}
\tag{4.1}
$$

Then, like that seen in the last section, it is assumed that the rectangular partition $T_{h,k}$ has been obtained from $T_{2h,2k}$ with mesh size $2H$ by subdividing each element of $T_{2h,2k}$ into four small congruent rectangles. Set $\hat{e} := \bigcup_{i=1}^{4} e_i$ with $e_i \in T_{h,k}$. And thus, the two interpolation operators $\mathbf{\Pi}^1_{2h,2k}$ and $P^1_{2h,2k}$ of degree at most 1 in x and y on $\hat{e}$, respectively, are defined as follows:

$$
\begin{gathered}
\mathbf{\Pi}^1_{2h,2k}\boldsymbol{\sigma}|_{\hat{e}} \in Q_{2,1}(\hat{e}) \times Q_{1,2}(\hat{e}), \quad P^1_{2h,2k}u|_{\hat{e}} \in Q_{1,1}(\hat{e}), \\
\int_{s_i} (\boldsymbol{\sigma} - \mathbf{\Pi}^1_{2h,2k}\boldsymbol{\sigma}) \cdot \mathbf{n}\,\mathrm{d}s = 0, \quad i = 1, 2, \ldots, 12, \\
\int_{e_i} (u - P^1_{2h,2k}u) = 0, \quad i = 1, 2, 3, 4,
\end{gathered}
$$

where s_i $(i = 1, 2, \ldots, 12)$ are the twelve edges of the four small elements e_i $(i = 1, 2, 3, 4)$. We can also check that $\mathbf{\Pi}^1_{2h,2k}$ and $P^1_{2h,2k}$ defined above possess the properties indicated in (4.1).

In addition, we also need a pair of mixed finite element projection operators $R_{h,k}\times\mathbf{S}_{h,k}\colon W\times\mathbf{V}_0\to W_{h,k}\times\mathbf{V}_{0,h,k}$ defined by

$$((R_{h,k}u)_t-u_t,w_{h,k})-(\nabla\cdot(\mathbf{S}_{h,k}\boldsymbol{\sigma}-\boldsymbol{\sigma}),w_{h,k})-(c(R_{h,k}u-u),w_{h,k})=0,$$
$$w_{h,k}\in W_{h,k},$$
$$(\alpha(\mathbf{S}_{h,k}\boldsymbol{\sigma}-\boldsymbol{\sigma})+M*(\mathbf{S}_{h,k}\boldsymbol{\sigma}-\boldsymbol{\sigma}),\mathbf{v}_{h,k})+(\nabla\cdot\mathbf{v}_{h,k},R_{h,k}u-u)=0,$$
$$\mathbf{v}_{h,k}\in\mathbf{V}_{0,h,k},$$
$$R_{h,k}u(0)=P^0_{h,k}u(0).$$

Then, $(R_{h,k}u,\mathbf{S}_{h,k}\boldsymbol{\sigma})$ is the solution of (2.3) if $(u,\boldsymbol{\sigma})$ is the solution of (2.1).

First of all, we discuss the interpolation defect correction method in the L^2-norm. To this purpose, let us recall the following lemma from [16].

Lemma 4.1. *Assume that $(u,\boldsymbol{\sigma})$ and $(u_{h,k},\boldsymbol{\sigma}_{h,k})$ are the exact solution of* (2.1) *and its mixed finite element solution, respectively, with the chosen initial value $u_{h,k}(0)=P^0_{h,k}u_0$. Then, in the sense of L^2-norm we have under the conditions that $(u,\boldsymbol{\sigma})$, c, α and M are sufficiently smooth that*

$$P^3_{4h,4k}u_{h,k}-u=H^2\xi+O(H^4), \tag{4.2}$$
$$\boldsymbol{\Pi}^3_{4h,4k}\boldsymbol{\sigma}_{h,k}-\boldsymbol{\sigma}=H^2\boldsymbol{\eta}+O(H^4), \tag{4.3}$$

where $(\xi,\boldsymbol{\eta})\in W\times\mathbf{V}_0$ is the variational solution of (3.7).

Theorem 4.1. *Suppose that the conditions of Lemma* 4.1 *are fulfilled. Then, we have*

$$\|u^*_{h,k}-u\|_0+\|\boldsymbol{\sigma}^*_{h,k}-\boldsymbol{\sigma}\|_0\leqslant CH^4,$$

where

$$u^*_{h,k}:=P^3_{4h,4k}u_{h,k}+P^1_{2h,2k}u_{h,k}-P^1_{2h,2k}R_{h,k}P^3_{4h,4k}u_{h,k},$$
$$\boldsymbol{\sigma}^*_{h,k}:=\boldsymbol{\Pi}^3_{4h,4k}\boldsymbol{\sigma}_{h,k}+\boldsymbol{\Pi}^1_{2h,2k}\boldsymbol{\sigma}_{h,k}-\boldsymbol{\Pi}^1_{2h,2k}\mathbf{S}_{h,k}\boldsymbol{\Pi}^3_{4h,4k}\boldsymbol{\sigma}_{h,k}.$$

P r o o f. Multiplying (4.2) by the operator $(I-P^1_{2h,2k}R_{h,k})$, where I is the identity operator, results in

$$\begin{aligned}(I-P^1_{2h,2k}R_{h,k})&(P^3_{4h,4k}u_{h,k}-u)\\&=H^2(I-P^1_{2h,2k}R_{h,k})\xi+O(H^4)\\&=H^2(\xi-P^1_{2h,2k}\xi)+H^2(P^1_{2h,2k}\xi-P^1_{2h,2k}\xi_{h,k})+O(H^4)\\&=H^2P^1_{2h,2k}(P^0_{h,k}\xi-\xi_{h,k})+O(H^4),\end{aligned}$$

since

$$\|\xi - P^1_{2h,2k}\xi\|_0 \leqslant CH^2\|\xi\|_2 \quad \text{and} \quad P^1_{2h,2k}P^0_{h,k} = P^1_{2h,2k}$$

according to the properties of the operator $P^1_{2h,2k}$ described in (4.1). Furthermore, it follows from Theorem 3.2 and the inequality

$$\|P^1_{2h,2k}(P^0_{h,k}\xi - \xi_{h,k})\|_0 \leqslant C\|P^0_{h,k}\xi - \xi_{h,k}\|_0$$

that

$$(I - P^1_{2h,2k}R_{h,k})(P^3_{4h,4k}u_{h,k} - u) = O(H^4),$$

and the left-hand side is nothing but

$$(I - P^1_{2h,2k}R_{h,k})(P^3_{4h,4k}u_{h,k} - u) = u^*_{h,k} - u.$$

Similarly, we can gain in terms of (4.3) that

$$\|\boldsymbol{\sigma}^*_{h,k} - \boldsymbol{\sigma}\|_0 = O(H^4).$$

$\square$

In the same way we can also obtain from Theorems 3.3 and 3.4 the following result.

Theorem 4.2. *We have under the conditions of Theorem* 3.1 *that*

$$\|\hat{u}_{h,k} - u\|_\infty \leqslant CH^4|\log H|^{1/2},$$
$$\|\hat{\boldsymbol{\sigma}}_{h,k} - \boldsymbol{\sigma}\|_\infty \leqslant CH^3,$$

where

$$\hat{u}_{h,k} := P^3_{4h,4k}u_{h,k} + P^1_{2h,2k}u_{h,k} - P^1_{2h,2k}R_{h,k}P^3_{4h,4k}u_{h,k},$$
$$\hat{\boldsymbol{\sigma}}_{h,k} := \mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k} + \boldsymbol{\sigma}_{h,k} - \mathbf{S}_{h,k}\mathbf{\Pi}^2_{3h,3k}\boldsymbol{\sigma}_{h,k}.$$

Analogously to Section 3 we can utilize the superconvergent approximation provided in Theorems 4.1 and 4.2 to establish a posteriori error estimators for the mixed finite element solution of problem (2.1). In fact, we have

Theorem 4.3. *If the conditions of Theorem* 4.1 *are satisfied, then*

$$\|u-u_{h,k}\|_0 = \|u^*_{h,k}-u_{h,k}\|_0 + O(H^4),$$
$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h,k}\|_0 = \|\boldsymbol{\sigma}^*_{h,k}-\boldsymbol{\sigma}_{h,k}\|_0 + O(H^4).$$

Furthermore, if there exist positive constants C_1, C_2 *and sufficiently small* $\varepsilon_1, \varepsilon_2 \in (0,1)$ *such that*

$$\|u-u_{h,k}\|_0 \geqslant C_1 H^{4-\varepsilon_1},$$
$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h,k}\|_0 \geqslant C_2 H^{4-\varepsilon_2},$$

we then have

$$\lim_{H\to 0} \frac{\|u-u_{h,k}\|_0}{\|u^*_{h,k}-u_{h,k}\|_0} = 1,$$
$$\lim_{H\to 0} \frac{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h,k}\|_0}{\|\boldsymbol{\sigma}^*_{h,k}-\boldsymbol{\sigma}_{h,k}\|_0} = 1.$$

Theorem 4.4. *We have under the conditions of Theorem* 4.2 *that*

$$\|u-u_{h,k}\|_\infty = \|\hat{u}_{h,k}-u_{h,k}\|_\infty + O(H^4|\log H|^{1/2}),$$
$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h,k}\|_\infty = \|\hat{\boldsymbol{\sigma}}_{h,k}-\boldsymbol{\sigma}_{h,k}\|_\infty + O(H^3).$$

In addition, if there exist positive constants C_1, C_2 *and sufficiently small* $\varepsilon_1, \varepsilon_2 \in (0,1)$ *such that*

$$\|u-u_{h,k}\|_\infty \geqslant C_1 H^{4-\varepsilon_1}|\log H|^{1/2},$$
$$\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h,k}\|_\infty \geqslant C_2 H^{3-\varepsilon_2},$$

then

$$\lim_{H\to 0} \frac{\|u-u_{h,k}\|_\infty}{\|\hat{u}_{h,k}-u_{h,k}\|_\infty} = 1,$$
$$\lim_{H\to 0} \frac{\|\boldsymbol{\sigma}-\boldsymbol{\sigma}_{h,k}\|_\infty}{\|\hat{\boldsymbol{\sigma}}_{h,k}-\boldsymbol{\sigma}_{h,k}\|_\infty} = 1.$$

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