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THE EXISTENCE OF PERIODIC SOLUTIONS FOR A CLASS OF
NONLINEAR FUNCTIONAL DIFFERENTIAL EQUATIONS*

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Abstract. This paper is concerned with periodic solutions of first-order nonlinear functional differential equations with deviating arguments. Some new sufficient conditions for the existence of periodic solutions are obtained. The paper extends and improves some well-known results.

Keywords: nonlinear functional differential equation, differential equation with deviating arguments, periodic solutions, coincidence degree theory

MSC 2000: 34B15, 34K13

1. INTRODUCTION

Recently, periodic solutions of functional differential equations have been extensively studied (see, e.g., [1]–[6]). In [1], the functional differential equation

$$(1) \quad \dot{x}(t) = b(t, x(t + \cdot)) + G(t, x(t + \cdot))$$

is considered where $x(t) \in \mathbb{R}^n$, $x(t + \cdot) \in BC(\mathbb{R}, \mathbb{R}^n)$ is given by $x(t + \cdot)(s) = x(t + s)$, b and G are continuous and boundary operators from $\mathbb{R} \times BC(\mathbb{R}, \mathbb{R}^n)$ to \mathbb{R}^n for any fixed $t \in \mathbb{R}$, $b(t, \varphi)$ is linear with respect to $\varphi \in BC(\mathbb{R}, \mathbb{R}^n)$, there exists a constant $T > 0$ such that $b(t + T, \varphi) = b(t, \varphi)$, $G(t + T, \varphi) = G(t, \varphi)$ for any $(t, \varphi) \in \mathbb{R} \times BC(\mathbb{R}, \mathbb{R}^n)$. Moreover,

$$\lim_{\|\varphi\| \rightarrow \infty} \frac{|G(t, \varphi)|}{\|\varphi\|} = 0$$

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uniformly for $t \in \mathbb{R}$, where $|\cdot|$ and $\|\cdot\|$ denote the norms in \mathbb{R}^n and $BC(\mathbb{R}, \mathbb{R}^n)$, respectively. In the theory of coincidence degree it is proved that if the linear equation $\dot{x}(t) = b(t, x(t + \cdot))$ has only the trivial T -periodic solution, then equation (1) has at least one T -periodic solution. Particularly, if $G(t, \varphi) = G(t)$, it is easy to show that equation (1) has a unique T -periodic solution. A specific example is given in [2] on how periodic solutions can be obtained for the functional differential equation

$$(2) \quad \dot{u}(t) = l(u(t)) + g(t)$$

where $l: C_\omega(\mathbb{R}) \rightarrow L(\mathbb{R})$ is a linear bounded operator and $g \in L_\omega(\mathbb{R})$, and an important particular case

$$(3) \quad \dot{u}(t) = \sum_{k=1}^n p_k(t)u(\tau_k(t)) + g(t)$$

is also studied.

In the present paper, we first consider the nonlinear equation with deviating argument

$$(4) \quad \dot{x}(t) = p(t)f(x(t - \tau(t))).$$

Some new optimal sufficient conditions are established for the existence of the trivial T -periodic solution of equation (4). Next, we consider the functional differential equation

$$(5) \quad \dot{x}(t) = p(t)f(x(t - \tau(t))) + g(t).$$

By the theory of coincidence degree, we obtain sufficient conditions that equation (5) has at least one T -periodic solution.

2. MAIN RESULTS AND PROOFS

Theorem 1. *Assume that*

- (a) $p, \tau \in C(\mathbb{R}, \mathbb{R})$, $p(t) \geq 0$, $p(t)$ is not identically equal to zero for $t \in \mathbb{R}$,
- (b) there exists a constant $T > 0$ such that $p(t + T) = p(t)$, $\tau(t + T) = \tau(t)$ for $t \in \mathbb{R}$,
- (c) f is continuous and $x(t)f(x(t)) > 0$, ($x(t) \neq 0$). If $f(x(t))/x(t) \leq 1$ ($x(t) \neq 0$) and

$$(6) \quad 0 < \int_0^T p(t) < 4$$

then equation (4) has only the trivial T -periodic solution.

Proof. Assume the contrary. Let there exist a nontrivial T -periodic solution x of equation (4); then $\max_{0 \leq t \leq T} |x(t)| > 0$. There are two cases:

Case 1. $M = \max_{0 \leq t \leq T} x(t) \leq 0$, then $x(t) \leq 0$, $f(x) \leq 0$ since $p(t) \geq 0$, so $\dot{x}(t) \leq 0$, which is a contradiction with the assumption.

Case 2. $M = \max_{0 \leq t \leq T} x(t) > 0$, then there are two cases:

(i) $\min_{0 \leq t \leq T} x(t) \geq 0$, then $x(t) \geq 0$, $f(x) \geq 0$ since $p(t) \geq 0$, so $\dot{x}(t) \geq 0$, which is a contradiction with the assumption.

(ii) $\min_{0 \leq t \leq T} x(t) = -m < 0$; choose $t_* \in [0, T]$, $t^* \in [t_*, t_* + T]$ such that $x(t_*) = -m$, $x(t^*) = M$, then $-m \leq f(x(t - \tau(t))) \leq M$.

Integrating equation (4) from t_* to t^* and from t^* to $t_* + T$, respectively, we have

$$(7) \quad M + m = \int_{t_*}^{t^*} p(t)f(x(t - \tau(t))) dt \leq M \int_{t_*}^{t^*} p(t) dt$$

and

$$(8) \quad m + M = - \int_{t^*}^{t_* + T} p(t)f(x(t - \tau(t))) dt \leq m \int_{t^*}^{t_* + T} p(t) dt.$$

Therefore, summing the last two inequalities,

$$4 \leq 2 + \frac{M}{m} + \frac{m}{M} \leq \int_{t_*}^{t_* + T} p(t) dt = \int_0^T p(t) dt,$$

which is a contradiction with the condition (6). The proof is complete. \square

Remark. If equation (4) has only the trivial T -periodic solution, we cannot conclude that equation (5) has at least one T -periodic solution.

Theorem 2. Assume that $p, \tau, g \in C(\mathbb{R}, \mathbb{R})$, $p(t) \geq 0$ and $p(t)$ is not identically equal to zero for $t \in \mathbb{R}$, there exists a constant $T > 0$ such that $p(t + T) = p(t)$, $\tau(t + T) = \tau(t)$, $g(t + T) = g(t)$ for $t \in \mathbb{R}$, f is continuous and $x(t)f(x(t)) > 0$ ($x(t) \neq 0$), $f(0) = 0$. Let the following conditions hold:

- (i) $|f(x(t))| \leq |x(t)|$ for all $x(t) \in \mathbb{R}$;
- (ii) $0 < \int_0^T p(t) dt < 4$;
- (iii) $\int_0^T g(t) dt = 0$.

Then equation (5) has at least one T -periodic solution.

To prove the theorem, we first consider the auxiliary equation

$$(9) \quad \dot{x}(t) = \lambda p(t)f(x(t - \tau(t))) + \lambda g(t), \quad \lambda \in (0, 1).$$

Lemma 1. For each possible T -periodic solution x_λ of equation (9), if the conditions of Theorem 2 hold, then there exists a constant D which is independent of λ such that

$$(10) \quad |x_\lambda(t)| \leq D, \quad t \in \mathbb{R}.$$

Proof. Let x denote x_λ . There are two possible cases:

Case 1. $x(t)$ is of a constant sign, i.e., either $x(t) \geq 0$ or $x(t) \leq 0$ for $t \in \mathbb{R}$. Integrating both sides of equation (9) from 0 to T , note that x is T -periodic and (iii) yields

$$(11) \quad \int_0^T p(t)f(x(t - \tau(t))) = dt = 0.$$

From (11), in view of the fact that $p(t)$ is not identically equal to zero, we obtain that there exists $t_0 \in [0, T]$ such that $x(t_0) = 0$. Moreover, there exists $t_1 < t_0$ such that $|x(t_1)| = \|x\|_C$ with $\|x\|_C = \max_{0 \leq t \leq T} x(t)$. Now integration of (9) on $[t_1, t_0]$ yields

$$(12) \quad \|x\|_C \leq \|g\|_L$$

where $\|g\|_L = \int_0^\tau |g(t)| dt$.

Case 2. The function x assumes both positive and negative values. Let $I = [t_2, t_3]$, $J = [t_3, t_2 + T]$, where t_2 and t_3 are such that $t_2 < t_3 < t_2 + T$ and $x(t_2) = -\min_{0 \leq t \leq T} x(t)$ and $x(t_3) = \max_{0 \leq t \leq T} x(t)$. Then integration of (9) on I and J , in view of $-m \leq f(x(t - \tau(t))) \leq M$ and $\lambda \in (0, 1)$, yields

$$(13) \quad m \leq M \left(\int_I p(t) dt - 1 \right) + \|g\|_L$$

and

$$(14) \quad M \leq m \left(\int_J p(t) dt - 1 \right) + \|g\|_L$$

where $M = \max_{0 \leq t \leq T} x(t) > 0$, $m = -\min_{0 \leq t \leq T} x(t) > 0$.

There are four cases:

Case a) $\int_I p(t) dt \leq 1$ and $\int_J p(t) dt \leq 1$. Then from (13) and (14) we get $\|x\|_C \leq \|g\|_L$.

Case b) $\int_I p(t) dt \leq 1$ and $\int_J p(t) dt > 1$. Then from (13) we have $m \leq \|g\|_L$, which together with (14) implies $M \leq \|p\|_L \|g\|_L$, i.e., $\|x\|_C \leq (\|p\|_L + 1)\|g\|_L$.

Case c) $\int_I p(t) dt > 1$ and $\int_J p(t) dt \leq 1$. Analogously to Case b, we obtain $\|x\|_C \leq (\|p\|_L + 1)\|g\|_L$.

Case d) $\int_I p(t) dt > 1$ and $\int_J p(t) dt > 1$. Then using (14) in (13) or (13) in (14) we have respectively

$$(15) \quad m \leq m \left(\int_I p(t) dt - 1 \right) \left(\int_J p(t) dt - 1 \right) + \left(\int_I p(t) dt - 1 \right) \|g\|_L + \|g\|_L$$

and

$$(16) \quad M \leq M \left(\int_I p(t) dt - 1 \right) \left(\int_J p(t) dt - 1 \right) + \left(\int_J p(t) dt - 1 \right) \|g\|_L + \|g\|_L.$$

Now, in view of the inequality $AB \leq (A + B)^2/4$ we have

$$(17) \quad \left(\int_I p(t) dt - 1 \right) \left(\int_J p(t) dt - 1 \right) \leq \frac{1}{4} \left(\int_{I \cup J} p(t) dt - 2 \right)^2.$$

Consequently, from (15) and (16) we obtain

$$(18) \quad \|x\|_C \leq \frac{1}{4} \left(\int_0^T p(t) dt - 2 \right)^2 \|x\|_C + \|p\|_L \|g\|_L.$$

Then, in view of condition (ii), we have

$$(19) \quad \|x\|_C \leq \left(1 - \frac{1}{4} (\|p\|_L - 2)^2 \right)^{-1} \|p\|_L \|g\|_L.$$

When $m \geq M$, we can obtain the inequality (19) similarly according to (14).

Thus, in both cases, the estimate (10) holds with

$$D = \left(1 + (\|p\|_L + 1) \left(1 - \frac{1}{4} (\|p\|_L - 2)^2 \right)^{-1} \right) \|g\|_L.$$

In order to prove Theorem 2, we also need the continuation theory of coincidence degree developed by Gains and Mawhin in [7]. \square

Lemma 2 (Continuation theorem). *Let X, Z be real Banach spaces, $L: \text{dom } L \subset X \rightarrow Z$ a Fredholm operator with index zero and let $N: \Omega \rightarrow Z$ be L -compact on Ω where Ω is an open subset of X , let $Q: Z \rightarrow Z$ be a continuous projector with $\text{Im } L = \ker Q$ and let $J: \text{Im } Q \rightarrow \ker L$ be an isomorphism. Let*

- (1) $Lx \neq \lambda Nx$ for any $\lambda \in (0, 1)$, $x \in \text{dom } L \cap \partial\Omega$;
- (2) $QNx \neq 0$ for $x \in \ker L \cap \partial\Omega$ and $\deg_B(JQN, \ker L \cap Q, 0) \neq 0$.

Then the operator equation $Lx = Nx$ has at least one solution in $\text{dom } L \cap \bar{\Omega}$.

Proof of Theorem 2. Let $X = Z = \{x \in C(\mathbb{R}, \mathbb{R}) : x(t+T) = x(t)\}$ with the norm $\|x\|_C = \max_{0 \leq t \leq T} |x(t)|$; $\text{dom } L = X \cap C^1(\mathbb{R}, \mathbb{R})$; $\Omega = \{x \in X : |x(t)| < \bar{D}\}$, where \bar{D} is greater than D ; let $L : \text{dom } L \subset X \rightarrow X$ be the differential operator defined by $(Lx)(t) = \dot{x}(t)$, let $N : \bar{\Omega} \rightarrow Z$ be defined by $(Nx)(t) = p(t)f(x(t-\tau(t))) + g(t)$ and $J = \text{id}$. Clearly, $\ker L = \mathbb{R}$. Defining the projectors $P = Q$ as follows:

$$(20) \quad Px(t) = \frac{1}{T} \int_0^T x(s) \, ds.$$

Obviously, $\text{Im } P = \ker L$, $\text{Im } L = \ker Q$, L is a Fredholm operator with index zero and N is L -compact on $\bar{\Omega}$. According to the estimation of the periodic solution of equation (9), we have $Lx \neq \lambda Nx$, for all $x \in \text{dom } L \cap \partial\Omega$, $\lambda \in (0, 1)$. If $x \in \ker L \cap \partial\Omega$, then $x = \pm \bar{D}$, so

$$\begin{aligned} QNx &= \frac{1}{T} \int_0^T [p(t)f(x(t-\tau(t))) + g(t)] \, dt \\ &= \frac{1}{T} \int_0^T p(t)f(\pm \bar{D}) \, dt = f(\pm \bar{D}) \frac{1}{T} \int_0^T p(t) \, dt \neq 0. \end{aligned}$$

Finally, consider the mapping

$$H(x, s) = sx + (1-s)f(x), \quad 0 \leq s \leq 1.$$

Since for every $s \in [0, 1]$ and $x \in \ker L \cap \partial\Omega$, we have

$$xH(x, s) = sx^2 + (1-s)xf(x) > 0,$$

$H(x, s)$ is a homotopy. This shows that

$$\begin{aligned} \deg_B(JQN, \ker L \cap \Omega, 0) &= \deg_B(f, \ker L \cap \Omega, 0) \\ &= \deg_B(\text{id}, \ker L \cap \Omega, 0) \neq 0. \end{aligned}$$

We have thus verified all the assumptions of the continuation theorem. Thus under the conditions of Theorem 2, $Lx = Nx$ has at least one solution in $\text{dom } L \cap \bar{\Omega}$. i.e., equation (5) has at least one T -periodic solution. The proof is complete. \square

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