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Pseudo-amenability of Brandt semigroup algebras

MAYSAM MAYSAMI SADR

Abstract. In this paper it is shown that for a Brandt semigroup S over a group G with an arbitrary index set I , if G is amenable, then the Banach semigroup algebra $\ell^1(S)$ is pseudo-amenable.

Keywords: pseudo-amenability, Brandt semigroup algebra, amenable group

Classification: 43A20, 43A07, 46H20

1. Introduction

The concept of amenability for Banach algebras was introduced by Johnson in 1972 [6]. Several modifications of this notion, such as approximate amenability and pseudo-amenability, were introduced in [2] and [4]. In the current paper we investigate the pseudo-amenability of Brandt semigroup algebras. It was shown in [2] and [4] that for the group algebra $L^1(G)$, amenability, approximate amenability and pseudo-amenability coincide and are equivalent to the amenability of locally compact group G . In the semigroup case we know that, if S is a discrete semigroup, then amenability of $\ell^1(S)$ implies that S is regular and amenable [1]. Ghahramani et al. [3] have shown that, if $\ell^1(S)$ is approximately amenable, then S is regular and amenable. The present author and Pourabbas in [9] have shown that for a Brandt semigroup S over a group G with an index set I , the following are equivalent.

- (i) $\ell^1(S)$ is amenable.
- (ii) $\ell^1(S)$ is approximately amenable.
- (iii) I is finite and G is amenable.

This result corrects [7, Theorem 1.8]. In the present paper we show that for a Brandt semigroup S over a group G with an arbitrary (finite or infinite) index set I , amenability of G implies pseudo-amenability of $\ell^1(S)$.

2. Preliminaries

Throughout $\hat{\otimes}$ denotes the completed projective tensor product. For an element x of a set X , δ_x is its point mass measure in $\ell^1(X)$. Also, we frequently use the identification $\ell^1(X \times Y) = \ell^1(X) \hat{\otimes} \ell^1(Y)$ for the sets X and Y .

A Banach algebra A is called (approximately) amenable, if for any dual Banach A -bimodule E , every bounded derivation from A to E is (approximately) inner. It is well known that amenability of A is equivalent to existence of a bounded approximate diagonal, that is a bounded net $(m_i) \in A \hat{\otimes} A$ such that for every

$a \in A, a \cdot m_i - m_i \cdot a \rightarrow 0$ and $\pi(m_i)a \rightarrow a$, where $\pi : A \hat{\otimes} A \rightarrow A$ is the continuous bimodule homomorphism defined by $\pi(a \otimes b) := ab$ ($a, b \in A$), and called the *diagonal map*. The famous Johnson Theorem [6], says that, for any locally compact group G , amenability of G and $L^1(G)$ are equivalent. For a modern account on amenability see [8] and for approximate amenability see the original papers [2] and [3].

A Banach algebra A is called pseudo-amenable ([4]) if there is a net $(n_i) \in A \hat{\otimes} A$, called an *approximate diagonal* for A , such that $a \cdot n_i - n_i \cdot a \rightarrow 0$ and $\pi(n_i)a \rightarrow a$ for each $a \in A$.

Let I be a nonempty set and let G be a discrete group. Consider the set $T := I \times G \times I$, add a null element \emptyset to T , and define a semigroup multiplication on $S := T \cup \{\emptyset\}$, as follows. For $i, i', j, j' \in I$ and $g, g' \in G$, let

$$(i, g, j)(i', g', j') = \begin{cases} (i, gg', j') & \text{if } j = i', \\ \emptyset & \text{if } j \neq i', \end{cases}$$

also let $\emptyset(i, g, j) = (i, g, j)\emptyset = \emptyset$ and $\emptyset\emptyset = \emptyset$. Then S becomes a semigroup that is called Brandt semigroup over G with index I , and usually denoted by $B(I, G)$. For more details see [5].

The Banach space $\ell^1(T)$, with the convolution product,

$$(ab)(i, g, j) = \sum_{k \in I, h \in G} a(i, gh^{-1}, k)b(k, h, j),$$

for $a, b \in \ell^1(T), i, j \in I, g \in G$, becomes a Banach algebra. (Note that if G is the one point group, and I is finite, then $\ell^1(T)$ is an ordinary matrix algebra.) We have a closed relation between the Banach algebra $\ell^1(T)$ and the Banach semigroup algebra $\ell^1(S)$:

Lemma 1. *There exists a homeomorphic isomorphism $\ell^1(S) \cong \ell^1(T) \oplus \mathbb{C}$ of Banach algebras, where the multiplication of $\ell^1(T) \oplus \mathbb{C}$ is coordinatewise.*

PROOF: Consider the following short exact sequence of Banach algebras and continuous algebra homomorphisms:

$$0 \rightarrow \ell^1(T) \rightarrow \ell^1(S) \rightarrow \mathbb{C} \rightarrow 0,$$

where the second arrow $\Psi : \ell^1(T) \rightarrow \ell^1(S)$ is defined by $\Psi(b)(t) := b(t)$ and $\Psi(b)(\emptyset) := -\sum_{s \in T} b(s)$, for $b \in \ell^1(T)$ and $t \in T \subset S$, and the third arrow $\Phi : \ell^1(S) \rightarrow \mathbb{C}$ is the integral functional, $\Phi(a) := \sum_{s \in S} a(s)$ ($a \in \ell^1(S)$). Now, let $\Theta : \ell^1(S) \rightarrow \ell^1(T)$ be the restriction map, $\Theta(a) := a|_T$. Then Θ is a continuous algebra homomorphism and $\Theta\Psi = \text{Id}_{\ell^1(T)}$. Thus the exact sequence splits and we have $\ell^1(S) \cong \ell^1(T) \oplus \mathbb{C}$. □

Lemma 2. *If $\ell^1(T)$ is pseudo-amenable, then so is $\ell^1(S)$.*

PROOF: Suppose that $\ell^1(T)$ is pseudo-amenable. Then by Lemma 1 and [4, Proposition 2.1], $\ell^1(S)$ is pseudo-amenable. □

3. The main result

Let S, T, G and I be as above. We need some other notations and computations:

For $a \in \ell^1(T)$ and every $u, v \in I$, let $a_{(u,v)}$ be an element of $\ell^1(G)$ defined by $a_{(u,v)}(g) := a(u, g, v)$ ($g \in G$). Note that

$$\|a\|_{\ell^1(T)} = \sum_{u,v \in I} \|a_{(u,v)}\|_{\ell^1(G)}.$$

For $b \in \ell^1(G \times G)$, $c \in \ell^1(G)$ and any $i, j, i', j' \in I$, let $E^b_{(i,j,i',j')}$ and $H^c_{(i,j)}$ be elements of $\ell^1(T \times T)$ and $\ell^1(T)$ respectively, defined by

$$E^b_{(i,j,i',j')}(u, g, v, u', g', v') = \begin{cases} b(g, g') & \text{if } u = i, v = j, u' = i', v' = j', \\ 0 & \text{otherwise,} \end{cases}$$

$$H^c_{(i,j)}(u, g, v) = \begin{cases} c(g) & \text{if } u = i, v = j, \\ 0 & \text{otherwise,} \end{cases}$$

where $u, v, u', v' \in I$ and $g, g' \in G$. Also note that

$$(1) \quad \|E^b_{(i,j,i',j')}\|_{\ell^1(T \times T)} = \|b\|_{\ell^1(G \times G)}, \quad \|H^c_{(i,j)}\|_{\ell^1(T)} = \|c\|_{\ell^1(G)}.$$

For $u, v \in I$ and $g \in G$, the module action of $\ell^1(T)$ on $\ell^1(T \times T)$ becomes

$$(2) \quad \delta_{(u,g,v)} \cdot E^b_{(i,j,i',j')} = \begin{cases} E^{\delta_g \cdot b}_{(u,j,i',j')} & \text{if } i = v, \\ 0 & \text{if } i \neq v, \end{cases}$$

$$(3) \quad E^b_{(i,j,i',j')} \cdot \delta_{(u,g,v)} = \begin{cases} E^{b \cdot \delta_g}_{(i,j,i',v)} & \text{if } j' = u, \\ 0 & \text{if } j' \neq u. \end{cases}$$

For the multiplication of $\ell^1(T)$ we have

$$(4) \quad \delta_{(u,g,v)} H^c_{(i,j)} = \begin{cases} H^{\delta_g c}_{(u,j)} & \text{if } i = v, \\ 0 & \text{if } i \neq v, \end{cases} \quad H^c_{(i,j)} \delta_{(u,g,v)} = \begin{cases} H^{c \delta_g}_{(i,v)} & \text{if } j = u, \\ 0 & \text{if } j \neq u. \end{cases}$$

And finally, the diagonal maps $\pi : \ell^1(T \times T) \longrightarrow \ell^1(T)$ and $\pi : \ell^1(G \times G) \longrightarrow \ell^1(G)$ have the relation

$$(5) \quad \pi(E^b_{(i,j,i',j')}) = \begin{cases} H^{\pi(b)}_{(i,j')} & \text{if } j = i', \\ 0 & \text{if } j \neq i'. \end{cases}$$

We are now ready to prove our main result:

Theorem 3. *Suppose that G is amenable. Then $\ell^1(S)$ is pseudo-amenable.*

PROOF: Let $(m_\lambda)_{\lambda \in \Lambda} \in \ell^1(G \times G)$ be a bounded approximate diagonal for the amenable Banach algebra $\ell^1(G)$. For any finite nonempty subset F of I and $\lambda \in \Lambda$, let

$$W_{F,\lambda} := \frac{1}{\#F} \sum_{i,j \in F} E_{(i,j,j,i)}^{m_\lambda},$$

where $\#F$ denotes the cardinal of F . We show that the net $(W_{F,\lambda}) \in \ell^1(T \times T)$ over the directed set $\Gamma \times \Lambda$, where Γ is the directed set of finite subsets of I ordered by inclusion, is an approximate diagonal for $\ell^1(T)$.

For any $u, v \in I$ and $g \in G$, by equations (2) and (3), we have,

$$\delta_{(u,g,v)} \cdot W_{F,\lambda} = \begin{cases} \frac{1}{\#F} \sum_{j \in F} E_{(u,j,j,v)}^{\delta_g \cdot m_\lambda} & \text{if } v \in F, \\ 0 & \text{if } v \notin F, \end{cases}$$

$$W_{F,\lambda} \cdot \delta_{(u,g,v)} = \begin{cases} \frac{1}{\#F} \sum_{j \in F} E_{(u,j,j,v)}^{m_\lambda \cdot \delta_g} & \text{if } u \in F, \\ 0 & \text{if } u \notin F, \end{cases}$$

and thus,

$$\delta_{(u,g,v)} \cdot W_{F,\lambda} - W_{F,\lambda} \cdot \delta_{(u,g,v)} = \begin{cases} \frac{1}{\#F} \sum_{j \in F} E_{(u,j,j,v)}^{\delta_g \cdot m_\lambda - m_\lambda \cdot \delta_g} & \text{if } u \in F, v \in F, \\ \frac{1}{\#F} \sum_{j \in F} E_{(u,j,j,v)}^{\delta_g \cdot m_\lambda} & \text{if } v \in F, u \notin F, \\ -\frac{1}{\#F} \sum_{j \in F} E_{(u,j,j,v)}^{m_\lambda \cdot \delta_g} & \text{if } u \in F, v \notin F, \\ 0 & \text{if } v \notin F, u \notin F. \end{cases}$$

Then, for $a = \sum_{u,v \in I, g \in G} a(u, g, v) \delta_{(u,g,v)}$ in $\ell^1(T)$ we have

$$\begin{aligned} a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a &= \frac{1}{\#F} \sum_{j,u,v \in F} E_{(u,j,j,v)}^{a(u,v) \cdot m_\lambda - m_\lambda \cdot a(u,v)} \\ &+ \frac{1}{\#F} \sum_{j,v \in F, u \in I-F} E_{(u,j,j,v)}^{a(u,v) \cdot m_\lambda} \\ &- \frac{1}{\#F} \sum_{j,u \in F, v \in I-F} E_{(u,j,j,v)}^{m_\lambda \cdot a(u,v)}, \end{aligned}$$

and thus, by (1),

$$\begin{aligned} \|a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a\| &\leq \sum_{u,v \in F} \|a(u,v) \cdot m_\lambda - m_\lambda \cdot a(u,v)\| \\ (6) \quad &+ \sum_{v \in F, u \in I-F} \|a(u,v) \cdot m_\lambda\| \\ &+ \sum_{u \in F, v \in I-F} \|m_\lambda \cdot a(u,v)\|. \end{aligned}$$

Now, suppose that $M > 0$ is a bound for the norms of m_λ 's. Let $\epsilon > 0$ be arbitrary, and let F_0 be an element of Γ such that

$$\sum_{(u,v) \in J_0, g \in G} |a(u, g, v)| = \sum_{(u,v) \in J_0} \|a_{(u,v)}\| < \epsilon,$$

where $J_0 = (I \times (I - F_0)) \cup ((I - F_0) \times I)$. And choose a $\lambda_0 \in \Lambda$ such that for every $\lambda \geq \lambda_0$,

$$\sum_{u,v \in F_0} \|a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)}\| < \epsilon.$$

Now, if $(F, \lambda) \in \Gamma \times \Lambda$ such that $F_0 \subseteq F$, $\lambda \geq \lambda_0$, then we have,

$$\begin{aligned} \sum_{u,v \in F} \|a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)}\| &\leq \sum_{u,v \in F_0} \|a_{(u,v)} \cdot m_\lambda - m_\lambda \cdot a_{(u,v)}\| \\ &\quad + \sum_{(u,v) \in J_0} \|a_{(u,v)} \cdot m_\lambda\| \\ &\quad + \sum_{(u,v) \in J_0} \|m_\lambda \cdot a_{(u,v)}\| \\ &< \epsilon + \epsilon M + \epsilon M, \end{aligned}$$

and analogously,

$$\sum_{v \in F, u \in I - F} \|a_{(u,v)} \cdot m_\lambda\| < \epsilon M$$

and

$$\sum_{u \in F, v \in I - F} \|m_\lambda \cdot a_{(u,v)}\| < \epsilon M.$$

Thus by (6), we have $\|a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a\| < \epsilon + 4\epsilon M$.

Therefore, we proved that $a \cdot W_{F,\lambda} - W_{F,\lambda} \cdot a \rightarrow 0$, for every $a \in \ell^1(T)$.

Now, we prove that $\pi(W_{F,\lambda})a \rightarrow a$ for any $a \in \ell^1(T)$.

By (5), we have

$$\pi(W_{F,\lambda}) = \frac{1}{\#F} \sum_{i,j \in F} H_{(i,i)}^{\pi(m_\lambda)} = \sum_{i \in F} H_{(i,i)}^{\pi(m_\lambda)}.$$

Thus, (4) implies that

$$\pi(W_{F,\lambda})a = \sum_{i \in F, v \in I} H_{(i,v)}^{\pi(m_\lambda)a_{(i,v)}},$$

since $a = \sum_{u,v \in I} H_{(u,v)}^{a(u,v)}$. Then we have,

$$(7) \quad \begin{aligned} \|\pi(W_{F,\lambda})a - a\| &\leq \sum_{i \in F, v \in I} \|H_{(i,v)}^{\pi(m_\lambda)a(i,v) - a(i,v)}\| \\ &+ \sum_{v \in I, u \in I-F} \|H_{(u,v)}^{a(u,v)}\|. \end{aligned}$$

Let $\epsilon > 0$ be arbitrary, and let F_0 and J_0 be as above. Choose a $\lambda_1 \in \Lambda$ such that for every $\lambda \geq \lambda_1$,

$$\sum_{i,j \in F_0} \|\pi(m_\lambda)a(i,j) - a(i,j)\| < \epsilon.$$

Now, if $(F, \lambda) \in \Gamma \times \Lambda$ is such that $F_0 \subseteq F$, $\lambda \geq \lambda_1$, then by (1) we have,

$$\begin{aligned} \sum_{i \in F, v \in I} \|H_{(i,v)}^{\pi(m_\lambda)a(i,v) - a(i,v)}\| &\leq \sum_{i,j \in F_0} \|\pi(m_\lambda)a(i,j) - a(i,j)\| \\ &+ \sum_{(u,v) \in J_0} \|\pi(m_\lambda)a(u,v)\| + \sum_{(u,v) \in J_0} \|a(u,v)\| \\ &< \epsilon + \epsilon M + \epsilon, \end{aligned}$$

and

$$\sum_{v \in I, u \in I-F} \|H_{(u,v)}^{a(u,v)}\| = \sum_{v \in I, u \in I-F} \|a(u,v)\| < \epsilon.$$

Thus, by (7) we have

$$\|\pi(W_{F,\lambda})a - a\| < 3\epsilon + \epsilon M.$$

This completes the proof. □

We end with a natural question:

Question 4. Does pseudo-amenability of $\ell^1(B(I, G))$ imply amenability of G ?

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