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AN INTERPOLATION PROBLEM
FOR MULTIVARIATE STATIONARY SEQUENCES

LUTZ KLOTZ

Let $X$ and $Y$ be stationarily cross-correlated multivariate stationary sequences. Assume that all values of $Y$ and all but one values of $X$ are known. We determine the best linear interpolation of the unknown value on the basis of the known values and derive a formula for the interpolation error matrix. Our assertions generalize a result of Budinský [1].

1. INTRODUCTION

In [1] Budinský studied the following problem. Let $X$ and $Y$ be two univariate stationarily cross-correlated stationary sequences. Assume that all values of $Y$ and all but one values of $X$ are known. Find the linear interpolation error of the unknown value of $X$ on the basis of all known values. In the present paper we generalize Budinský's result to multivariate sequences $X$ and $Y$. The main tool of our investigations is the Hellinger-spectral domain of a stationary sequence. H. Salehi first used Hellinger integrals in the interpolation of multivariate stationary sequences, see [6] and [7]. His method was developed and completed by Makagon and Weron, cf. [2,3], and [8]. Some results of these authors, on which we heavily lean, are summarized in Section 2. Section 3 is devoted to the solution of the interpolation problem mentioned above. We obtain a formula for the interpolation error matrix as well as a recipe for determining the best linear interpolation of the unknown value. Since our formulas are rather difficult to apply in the general situation, in Section 4 we study some special cases and, using some facts on the Moore–Penrose inverse of a non-negative Hermitian block matrix, derive more tractable formulas for the interpolation error matrix.

2. PRELIMINARIES AND NOTATIONS

Let $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{C}$ be the sets of positive integers, integers, and complex numbers, resp. For $r \in \mathbb{N}$, the symbol $\mathcal{M}_r$, stands for the space of $r \times r$-matrices with complex entries. If $A \in \mathcal{M}_r$, then $A^*$, $\mathcal{R}(A)$, $\text{Ker} A$, and $\rho(A)$ denote its adjoint, range, kernel, and rank, resp. Furthermore, $A^+$ is the Moore–Penrose inverse of $A$, cf. formulas (1.2) in [4]. If $A$ is regular, its inverse $A^{-1}$ coincides with $A^+$. 


The symbol $I$ stands for a unit matrix, where its size should become clear from the context.

Let $\mathcal{H}$ be a Hilbert space over $C$ and $\mathcal{H}^r$ the Cartesian product of $r$ copies of $\mathcal{H}$. We will consider $\mathcal{H}^r$ as a left $M_r$-module, i.e., the generic element $u$ of $\mathcal{H}^r$ is written as a column vector so that for each $A \in M_r$ the product $Au$ is defined in a natural way and belongs to $\mathcal{H}^r$. The zero element of $\mathcal{H}^r$ is denoted by $O_r$, whereas the symbol 0 stands for $O_1$ as well as for several zero matrices. For two vectors $u, v$ of $\mathcal{H}^r$ let $\langle u, v \rangle$ be their Grammian matrix. Finally, $e_k$ denotes the $k$th unit vector of $C^r$, i.e. the vector whose $k$th entry is 1 and all its other elements are 0, $k \in \{1, \ldots, r\}$.

An $r$-variate stationary sequence is a map $S: \mathbb{Z} \ni n \rightarrow s_n \in \mathcal{H}^r$ such that $\langle s_m, s_n \rangle$ depends only on $m - n, m, n \in \mathbb{Z}$. By $\overline{M}$ we denote the time domain of $S$, i.e. the closed subspace of $\mathcal{H}^r$ spanned by all $s_n$, $n \in \mathbb{Z}$, with coefficients from $M_r$. Recall that $\overline{M} = \mathcal{M}^r$, where $\mathcal{M}$ is the closed linear subspace of $\mathcal{H}$, spanned by the entries of all $s_n$, $n \in \mathbb{Z}$.

Let us assume that the spectral measure $F$ of $S$ is absolutely continuous with respect to the Lebesgue measure $\sigma$ on the $\sigma$-algebra $\mathcal{B}$ of Borel sets of $[-\pi, \pi)$. Let $f$ be the spectral density and $L^2(F)$ the spectral domain of $S$, i.e. the left Hilbert $M_r$-module of (equivalence classes of) $\mathcal{B}$-measurable $M_r$-valued functions $\Phi$ such that $\int_{-\pi}^{\pi} \Phi(\lambda)f(\lambda)\Phi(\lambda)^*\sigma(d\lambda) = \int \Phi f \Phi^* d\sigma$ exists.

In the following we will omit the integration variable and the domain of integration $[-\pi, \pi)$ in the notation. Furthermore, relations between $\mathcal{B}$-measurable functions are to be understood as relations that hold $\sigma$-a.e., although we will not emphasize this each time.

Let $U$ be Kolmogorov's isomorphism between the time domain and the spectral domain of $S$, i.e., $U$ is an isometric $M_r$-linear isomorphism of $\overline{M}$ onto $L^2(F)$ such that

$$Us_n = e^{in}\cdot I, \quad n \in \mathbb{Z}. $$

Let us consider the Hilbert-$M_r$-module $H^2(F)$ of (equivalence classes of) $\mathcal{B}$-measurable $M_r$-valued functions $M$ such that $\ker M \supset \ker f$ and $\int M f^+ M^* d\sigma$ exists. The mapping

$$V: \Phi \rightarrow \Phi f$$

establishes an isometric $M_r$-linear isomorphism of $L^2(F)$ onto $H^2(F)$, cf. [6, Theorem 1] and [3, Theorem 3.3 (b)].

It is not hard to see that

$$V^{-1}M = M f^+, \quad M \in H^2(F).$$

In [3, Theorem 3.4 and Lemma 3.7] and [8, Lemma 4.5 (b)] it was proved the following result.
Lemma 1. A vector $u$ of $\mathcal{M}$ is orthogonal to all $s_n$, $n \in \mathbb{Z} \setminus \{0\}$, if and only if $VUu$ is equal to a constant $\mathcal{M}_r$-valued function, where its value $A$ has the following properties:

\[ \mathcal{R}(A) \subseteq \mathcal{R}(f) \]  
\[ \int Af^+A^*d\sigma \]  
exists. The matrix $A$ can be computed by

\[ A = (u, s_0). \]

Conversely, if $A \in \mathcal{M}_r$ has properties (2) and (3), then there exists a vector $u \in \mathcal{M}$, which is orthogonal to all $s_n$, $n \in \mathbb{Z} \setminus \{0\}$, such that $VUu = A \sigma$-a.e.

3. AN INTERPOLATION PROBLEM

Let $p, q \in \mathbb{N}$ and let $X$ be a $p$-variate and $Y$ a $q$-variate stationary sequence such that $S : s_n := \left( x_n \right)$, $n \in \mathbb{Z}$, is a $(p+q)$-variate stationary sequence. Let $\mathcal{M}_0$ be the closed $\mathcal{M}_{p+q}$-linear hull of all $s_n$, $n \in \mathbb{Z} \setminus \{0\}$, and $\left( O_p \right)$. Denote the vector $\left( x_0 \right)$ by $x_0$. Motivated by a paper of Budinsky [1] we study the following interpolation problem:

Find the orthogonal projection $\hat{x}_0$ of $x_0'$ onto $\mathcal{M}_0$ and the interpolation error matrix

\[ \Delta := (x'_0 - \hat{x}_0, x'_0 - \hat{x}_0). \]

Since $\mathcal{M}_0$ is of the form $\mathcal{M}_0 = \mathcal{M}_{p+q}_0$, where $\mathcal{M}_0$ is the closed subspace of $\mathcal{H}$ spanned by the entries of all $s_n$, $n \in \mathbb{Z} \setminus \{0\}$, and the entries of $y_0$, the problem is equivalent to determining the orthogonal projections of the entries of $x_0$ onto $\mathcal{M}_0$. However, we find it convenient to study the larger space $\mathcal{M}_0$ since this allows us to use the isomorphisms $U$ and $V$.

First note that the singular part of the spectral measure $F$ of $S$ does not affect on the interpolation error. So we assume that $S$ has a spectral density $f$.

Let

\[ \vec{x}_0 := x'_0 - \hat{x}_0. \]

In the following we have to consider block partitions $A := \left( \begin{array}{cc} A_{11} & A_{12} \\ A_{21} & A_{22} \end{array} \right)$ of matrices $A$ from $\mathcal{M}_{p+q}$. In all these cases the left upper block $A_{11}$ is assumed to belong to $\mathcal{M}_p$. In particular, the block partition $f = \left( \begin{array}{cc} f_{11} & f_{12} \\ f_{21} & f_{22} \end{array} \right)$ of $f$ corresponds to the partition of $S$ into $X$ and $Y$ and the interpolation error matrix $\Delta$ has the form

\[ \Delta = \left( \begin{array}{cc} \Delta_{11} & 0 \\ 0 & 0 \end{array} \right), \]

where $\Delta_{11}$ is non-negative Hermitian and belongs to $\mathcal{M}_p$.

Consider the subset
$L := \{ (c_{O_q}) \in \mathbb{C}^{p+q} : (c_{O_q}) \in \mathcal{R}(f) \text{ and } \int (c_{O_q})^* f^+ (c_{O_q}) d\sigma \text{ exists } \}$ of $\mathbb{C}^{p+q}$.

Since $(c_{O_q}) \in L$ if and only if $(c_{O_q}) \in \mathcal{R}(f)$ and the $C^p$-valued function $((f^+)_{11})^{\frac{1}{2}} c$ is square-integrable, the set $L$ is a subspace of $\mathbb{C}^{p+q}$. Denote by $P$ the orthogonal projection in $\mathbb{C}^{p+q}$ onto $L$.

Let $E$ be the $M_{p+q}$-valued function

$$E(\lambda) := \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}, \quad \lambda \in [-\pi, \pi).$$

**Theorem 2.** The interpolation error matrix $\Delta$ can be calculated by

$$\Delta = \left( \int P f^+ P d\sigma \right)^+. \quad (6)$$

The orthogonal projection of $x_0'$ onto $\tilde{M}_0$ is equal to

$$\tilde{x}_0 = U^{-1}(E - \Delta f^+). \quad (7)$$

**Proof.** Note that $\tilde{x}_0$ is of the form $\begin{pmatrix} u \\ O_q \end{pmatrix}$ for some $u \in M^p_0$, which implies

$$\langle \tilde{x}_0, s_0 \rangle = \langle \tilde{x}_0, \tilde{x}_0 \rangle = \Delta. \quad (8)$$

Since $\tilde{x}_0$ is orthogonal to all $s_n$, $n \in \mathbb{Z} \setminus \{0\}$, from Lemma 1 and (8) we obtain that $VU \tilde{x}_0$ is a constant function whose value is equal to $\Delta$. Since $VU$ is an isometry of $\tilde{M}$ onto $H^2(F)$, it follows

$$\Delta = \langle \tilde{x}_0, \tilde{x}_0 \rangle = \int \Delta f^+ \Delta d\sigma. \quad (9)$$

Relations (2) and (5) yield $\mathcal{R}(\Delta) \subseteq \mathcal{R}(f)$. Thus $\int \Delta f^+ \Delta d\sigma = \int \Delta f^+ P \Delta d\sigma = \Delta \int P f^+ P d\sigma \Delta$. Comparing this with (9), we get

$$\Delta = \Delta \int P f^+ P d\sigma \Delta. \quad (10)$$

If we can show that the range of the matrix $B := \int P f^+ P d\sigma$ is included in $\mathcal{R}(\Delta)$, the result immediately follows from (10). But $\mathcal{R}(B) \subseteq \mathcal{R}(P) \subseteq \mathcal{R}(f)$ and the integral $\int B f^+ B d\sigma = B \int f^+ P d\sigma B$ exists. According to Lemma 1 there exists a vector $u$ of $\tilde{M}$, which is orthogonal to all $s_n$, $n \in \mathbb{Z} \setminus \{0\}$, such that $VU u = B = \langle u, s_0 \rangle \sigma$-a.e. Moreover, since $\int VU u f^+ VU \begin{pmatrix} 0_p \\ u_0 \end{pmatrix} d\sigma = \int B f^+ \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} d\sigma = \int B \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} d\sigma = 0$,

the vector $u$ even belongs to the orthogonal complement of $\tilde{M}_0$. This means that it has the form $u = D \tilde{x}_0$, for some $D \in M_{p+q}$. Then $B = \langle u, s_0 \rangle = \langle D \tilde{x}_0, s_0 \rangle = D (\tilde{x}_0, \tilde{x}_0) = D \Delta$, which implies $\text{Ker} \Delta \subseteq \text{Ker} B$ and, hence, $\mathcal{R}(B) \subseteq \mathcal{R}(\Delta)$.

To prove (7) note that $U \tilde{x}_0 = U x_0' - U \tilde{x}_0, U x_0' = E$, and $VU \tilde{x}_0 = \Delta \sigma$-a.e., thus $U \tilde{x}_0 = V^{-1} \Delta = \Delta f^+$ by (1).
Corollary 3. The range of $\Delta$ is equal to the range of $P$.

Proof. It was shown in the proof of Theorem 2 that $\mathcal{R}(\Delta) \subseteq \mathcal{R}(P)$. Thus, if $P_\Delta$ denotes the orthogonal projector onto $\mathcal{R}(\Delta)$, we get $\int P f^+ P d\sigma = \Delta^+ = P_\Delta \Delta^+ P_\Delta = P_\Delta \int P f^+ P d\sigma P_\Delta = \int P_\Delta P f^+ P P_\Delta d\sigma = \int P_\Delta f^+ P P_\Delta d\sigma$. From this equality it is easy to conclude that $\mathcal{R}(P_\Delta) = \mathcal{R}(P)$. □

4. SPECIAL CASES

Under additional assumptions formula (6) can be brought into a more explicit form. Because of (5) it is enough to give expressions for $\Delta_{11}$.

Corollary 4. If the values of $f$ are regular matrices and

$$\int (f_{11} - f_{12} f_{22}^{-1} f_{12}^*)^{-1} d\sigma$$

exists, then

$$\Delta_{11} = \left( \int (f_{11} - f_{12} f_{22}^{-1} f_{12}^*)^{-1} d\sigma \right)^{-1}. \quad (12)$$

Proof. If the matrix $f(\lambda)$ is regular, then the left upper block of $f(\lambda)^{-1}$ is equal to $(f_{11}(\lambda) - f_{12}(\lambda)f_{22}(\lambda)^{-1} f_{12}(\lambda)^*)^{-1}$ by the well-known Frobenius formula, $\lambda \in [-\pi, \pi)$. Now the result immediately follows from (6). □

The following corollary generalizes Theorem 1 of [1].

Corollary 5. Let $p = 1$ and the values of $f$ be regular matrices. Then $\Delta_{11}$ can be computed by (12), where the right-hand side of (12) is to be interpreted as 0, if the integral (11) does not exist.

Proof. If (11) exists, the result is a special case of Corollary 4. If (11) does not exist, the projection $P$ is equal to 0. □

In the statement and the proof of our next corollary we make use of the following result on matrices, which can be easily obtained from formula (3.24) in [4]. If $A \in \mathcal{M}_{p+q}$ and $A$ is non-negative Hermitian, then $\rho(A) = \rho(A_{22}) + \rho(A_{11} - A_{12} A_{22}^+ A_{21})$. In particular, $\rho(A) = \rho(A_{22})$ if and only if $A_{11} - A_{12} A_{22}^+ A_{21} = 0$.

Corollary 6. Let $p = 1$. Then $\Delta_{11} = 0$ if one of the following conditions hold:

(i) $\rho(f) = \rho(f_{22})$ or, equivalently, $f_{11} - f_{12} f_{22}^+ f_{12}^* = 0$ on a set of positive measure $\sigma$.

(ii) $\rho(f) > \rho(f_{22})$ $\sigma$-a.e. and the integral

$$\int (f_{11} - f_{12} f_{22}^+ f_{12}^*)^{-1} d\sigma$$

exists.
If \( \rho(f) > \rho(f_{22}) \) \( \sigma \)-a.e. and (13) exists, then \( \Delta_{11} \) is equal to

\[
\Delta_{11} = \left( \int (f_{11} - f_{12}f_{22}^+f_{12}^*)^{-1} d\sigma \right)^{-1}.
\] (14)

Proof. It is not hard to see that the condition \( \rho(f(\lambda)) = \rho(f_{22}(\lambda)) \) is equivalent to the fact that \( e_1 \) does not belong to \( R(f(\lambda)), \lambda \in [-\pi, \pi) \). So, (i) yields \( P = 0 \) and, hence, \( \Delta_{11} = 0 \). If \( \rho(f(\lambda)) > \rho(f_{22}(\lambda)) \), we have

\[
\rho(f_{11}(\lambda) - f_{12}(\lambda)f_{22}(\lambda)^+f_{12}(\lambda)^*) = 1 = \rho(f_{11}(\lambda))
\]

and therefore \( \rho(f(\lambda)) = \rho(f_{11}(\lambda)) + \rho(f_{22}(\lambda)) \). Under this condition the left upper block of \( f(\lambda)^+ \) is equal to \( (f_{11}(\lambda) - f_{12}(\lambda)f_{22}(\lambda)^+f_{12}(\lambda)^*)^{-1} \), cf. formula (3.32) in [4]. Thus, from the non-existence of (13) we again conclude \( P = 0 \) and the existence of (13) yields (14) because of (6).

Corollary 7. Let \( p = 1 \). Then \( \Delta_{11} = 0 \) if and only if \( e_1 \) belongs to \( R(f) \) \( \sigma \)-a.e. and the integral (13) exists.

Proof. In the proof of Corollary 6 it was mentioned that \( e_1 \) belongs to \( R(f(\lambda)) \) if and only if \( \rho(f(\lambda)) > \rho(f_{22}(\lambda)), \lambda \in [-\pi, \pi) \). Hence, Corollary 7 is a consequence of Corollary 6.

Now let us use our results to derive a minimality condition for \( r \)-variate stationary sequences due to Rozanov [5, Theorem 10.2 of Ch. 2].

An \( r \)-variate stationary sequence \( S \) is called minimal in the sense of Rozanov if for each \( k \in \{1, \ldots, r\} \) the \( k \)th entry \( s_{0}^{(k)} \) of \( s_{0} \) does not belong to the space \( H_{k} \) spanned by the entries of all \( s_{n}, n \in Z \setminus \{0\} \), and the elements \( s_{0}^{(j)}, j \neq k \).

Corollary 8. An \( r \)-variate stationary process \( S \) is minimal in the sense of Rozanov if and only if the values of \( f \) are regular matrices and all functions on the principal diagonal of \( f^{-1} \) are integrable.

Proof. From Corollary 7 it follows that \( s_{0}^{(k)} \) does not belong to \( H_{k} \) if and only if \( e_{k} \) belongs to \( R(f) \) and the \( k \)th function on the principal diagonal of \( f^+ \) is integrable. But \( e_{k} \in R(f) \) for all \( k \in \{1, \ldots, r\} \) if and only if \( f^{-1} \) exists.

Remark 9. We conclude with the remark that all results of the present paper can be extended to a multivariate stationary process on a discrete Abelian group in an obvious way.

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