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COUNTABLE EXTENSION OF TRIANGULAR NORMS AND THEIR APPLICATIONS TO THE FIXED POINT THEORY IN PROBABILISTIC METRIC SPACES

OLGA HADŽIĆ, ENDRE PAP AND MIRKO BUDINČEVIĆ

In this paper a fixed point theorem for a probabilistic $q$-contraction $f : S \to S$, where $(S, \mathcal{F}, T)$ is a complete Menger space, $\mathcal{F}$ satisfies a grow condition, and $T$ is a $g$-convergent $t$-norm (not necessarily $T \geq T_L$) is proved. There is proved also a second fixed point theorem for mappings $f : S \to S$, where $(S, \mathcal{F}, T)$ is a complete Menger space, $\mathcal{F}$ satisfy a weaker condition than in [13], and $T$ belongs to some subclasses of Dombi, Aczél–Alsina, and Sugeno–Weber families of $t$-norms. An application to random operator equations is obtained.

1. INTRODUCTION

The origin of triangular norms was in the theory of probabilistic metric spaces, in the work K. Menger [9], see [4, 7, 14]. It turns out that $t$-norms and related $t$-conorms are crucial operations in several fields, e.g., in fuzzy sets, fuzzy logics (see [7]) and their applications, but also, among other fields, in the theory of generalized measures [7, 11, 17] and in nonlinear differential and difference equations [11].

We present in this paper some results on $t$-norms which are closely related to the fixed point theory in probabilistic metric spaces, see [4]. The first fixed point theorem in probabilistic metric spaces was proved by Sehgal and Bharucha-Reid [15] for mappings $f : S \to S$, where $(S, \mathcal{F}, T_M)$ is a Menger space, where $T_M = \min$. Further development of the fixed point theory in a more general Menger space $(S, \mathcal{F}, T)$ was connected with investigations of the structure of the $t$-norm $T$. Very soon the problem was in some sense completely solved. Namely, if we restrict ourselves to complete Menger spaces $(S, \mathcal{F}, T)$, where $T$ is a continuous $t$-norm, then any probabilistic $q$-contraction $f : S \to S$ has a fixed point if and only if the $t$-norm $T$ is of $H$-type, see [4].

We investigate in this paper the countable extension of $t$-norms and we introduce a new notion: the geometrically convergent (briefly $g$-convergent) $t$-norm, which is closely related to the fixed point property. We prove that $t$-norms of $H$-type and some subclasses of Dombi, Aczél–Alsina, and Sugeno–Weber families of $t$-norms are
geometrically convergent. We prove also some practical criterions for the geometrically convergent t-norms.

A new approach to the fixed point theory in probabilistic metric spaces is given in Tardiff’s paper [16], where some additional growth conditions for the mapping $F : S \times S \rightarrow D^+$ are assumed, and $T \geq T_L$. V. Radu [13] introduced a stronger growth condition for $F$ than in Tardiff’s paper (under the condition $T \geq T_L$), which enables him to define a metric. By metric approach an estimation of the convergence with respect to the solution is obtained, see [4].

We prove in this paper a fixed point theorem for a probabilistic $g$-contraction $f : S \rightarrow S$, where $(S, F, T)$ is a complete Menger space, $F$ satisfies Radu’s condition, and $T$ is a $g$-convergent t-norm (not necessarily $T \geq T_L$). We prove a second fixed point theorem for mappings $f : S \rightarrow S$, where $(S, F, T)$ is a complete Menger space, $F$ satisfy a weaker condition than in [13], and $T$ belongs to some subclasses of Dombi, Aczél–Alsina, and Sugeno–Weber families of t-norms. An application to random operator equations is obtained.

Notions and notations can be found in [4, 7, 11, 14].

2. TRIANGULAR NORMS

A triangular norm (t-norm for short) is a binary operation on the unit interval $[0,1]$, i.e., a function $T : [0,1]^2 \rightarrow [0,1]$ which is commutative, associative, monotone and $T(x,1) = x$. t-conorm $S$ is defined by $S(x,y) = 1 - T(1 - x, 1 - y)$.

If $T$ is a t-norm, $x \in [0,1]$ and $n \in \mathbb{N} \cup \{0\}$ then we shall write

$$x_T^{(n)} = \begin{cases} 
1 & \text{if } n = 0, \\
T\left(x_T^{(n-1)}, x\right) & \text{otherwise}.
\end{cases}$$

**Definition 1.** A t-norm $T$ is of $H$-type if the family $(x_T^{(n)})_{n \in \mathbb{N}}$ is equicontinuous at the point $x = 1$.

A trivial example of a t-norm of $H$-type is $T_M$. There is a nontrivial example of a t-norm $T$ such that $(x_T^{(n)})_{n \in \mathbb{N}}$ is an equicontinuous family at the point $x = 1$.

**Example 2.** Let $\bar{T}$ be a continuous t-norm and let for every $m \in \mathbb{N} \cup \{0\}$:

$$I_m = [1 - 2^{-m}, 1 - 2^{-m-1}].$$

If

$$T(x, y) = 1 - 2^{-m} + 2^{-m-1}\bar{T}(2^{m+1}(x - 1 + 2^{-m}), 2^{m+1}(y - 1 + 2^{-m}))$$

for $(x, y) \in I_m \times I_m$ and $T(x, y) = \min(x, y)$ for $(x, y) \not\in \bigcup_{m \in \mathbb{N} \cup \{0\}} I_m \times I_m$ then the family $(x_T^{(n)})_{n \in \mathbb{N}}$ is equicontinuous at the point $x = 1$, i.e., $T$ is a t-norm of $H$-type.
Proposition 3. ([4]) If a continuous t-norm \( T \) is Archimedean than it can not be a t-norm of \( H \)-type.

A method of construction a new t-norm from a system of given t-norms is given in the following theorem, see [4, 7].

Theorem 4. Let \((T_k)_{k \in K}\) be a family of t-norms and let \(((\alpha_k, \beta_k))_{k \in K}\) be a family of pairwise disjoint open subintervals of the unit interval \([0,1]\) (i.e., \( K \) is an at most countable index set). Consider the linear transformations \( \varphi_k : [\alpha_k, \beta_k] \to [0,1], k \in K \) given by

\[
\varphi_k(u) = \frac{u - \alpha_k}{\beta_k - \alpha_k}.
\]

Then the function \( T : [0,1]^2 \to [0,1] \) defined by

\[
T(x,y) = \begin{cases} 
\varphi_k^{-1}(T_k(\varphi_k(x), \varphi_k(y))) & \text{if } (x,y) \in (\alpha_k, \beta_k)^2, \\
\min(x,y) & \text{otherwise,}
\end{cases}
\]

is a triangular norm, which is called the ordinal sum of \((T_k)_{k \in K}\) and will be denoted by \( T = (< (\alpha_k, \beta_k), T_k >)_{k \in K} \).

The following proposition was proved in [12].

Proposition 5. A continuous t-norm \( T \) is of \( H \)-type if and only if \( T = (< (\alpha_k, \beta_k), T_k >)_{k \in K} \) and \( \sup \beta_k < 1 \) or \( \sup \alpha_k = 1 \).

Remark 6. If \( T = (< (\alpha_k, \beta_k), T_k >)_{k \in K} \) and \( \sup \beta_k < 1 \) or \( \sup \alpha_k = 1 \), then \( T \) is of \( H \)-type for any summands \( T_k \) (not only for continuous and Archimedean summands \( T_k, k \in K \), see [12]). Hence, if

\[
T = ( < (1 - 2^{-k}, 1 - 2^{-k-1}), \bar{T} > )_{k \in \mathbb{N} \cup \{0\}}
\]

we have \( \sup \alpha_k = \sup (1 - 2^{-k}) = 1 \) (cf. Example 2).

For an arbitrary t-norm of \( H \)-type we have by [4] the following characterization.

Theorem 7. Let \( T \) be a t-norm. Then (i) and (ii) hold, where:

(i) Suppose that there exists a strictly increasing sequence \((b_n)_{n \in \mathbb{N}}\) from the interval \([0,1)\) such that \( \lim_{n \to \infty} b_n = 1 \) and \( T(b_n, b_n) = b_n \). Then \( T \) is of \( H \)-type.

(ii) If \( T \) is continuous and of \( H \)-type, then there exists a sequence \((b_n)_{n \in \mathbb{N}}\) as in (i).

From the proof of the above theorem it follows that the condition of continuity of whole sequence \((x_T^{(n)})_{n \in \mathbb{N}}\) can be replaced by the condition that the function \( \delta_T(x) = T(x, x) (x \in [0,1]) \) is right-continuous on an interval \([b,1)\) for \( b < 1 \).
Theorem 8. Let $T$ be a t-norm such that the function $\delta_T(x) = T(x, x)$ ($x \in [0, 1]$) is right-continuous on an interval $[b, 1)$ for $b < 1$. Then $T$ is a t-norm of $H$-type if and only if there exists a sequence $(b_n)_{n\in\mathbb{N}}$ from the interval $(0, 1)$ of idempotents of $T$ such that $\lim_{n\to\infty} b_n = 1$.

In particular, for continuous t-norms the following characterization holds, [4].

Theorem 9. Let $T$ be a continuous t-norm. Then the following are equivalent:

a) $T$ is not of $H$-type.

b) There exist $a_T \in [0, 1)$ and a continuous strictly increasing and surjective mapping $\varphi_{a_T} : [a_T, 1] \to [0, 1]$ such that

$$T(x, y) = \varphi_{a_T}^{-1}(\varphi_{a_T}(x) \ast \varphi_{a_T}(y)),$$

for every $x, y \geq a_T$,

where the operation $\ast$ is either $T_P$ or $T_L$, where $T_P(x, y) = xy$ and $T_L(x, y) = \max(x + y - 1, 0)$.

3. COUNTABLE EXTENSION OF t-NORMS

An arbitrary t-norm $T$ can be extended (by associativity) in a unique way to an $n$-ary operation taking for $(x_1, \ldots, x_n) \in [0, 1]^n, n \in \mathbb{N}$, the values $T(x_1, \ldots, x_n)$ which is defined by

$$\bigotimes_{i=1}^0 x_i = 1, \quad \bigotimes_{i=1}^n x_i = T \left( \bigotimes_{i=1}^{n-1} x_i, x_n \right) = T(x_1, \ldots, x_n).$$

Specially, we have $T_L(x_1, \ldots, x_n) = \max \left( \sum_{i=1}^n x_i - (n - 1), 0 \right)$ and $T_M(x_1, \ldots, x_n) = \min(x_1, \ldots, x_n)$.

We can extend $T$ to a countable infinitary operation taking for any sequence $(x_n)_{n\in\mathbb{N}}$ from $[0, 1]$ the values

$$\bigotimes_{i=1}^\infty x_i = \lim_{n\to\infty} \bigotimes_{i=1}^n x_i. \quad (1)$$

The limit on the right side of (1) exists since the sequence $(\bigotimes_{i=1}^n x_i)_{n\in\mathbb{N}}$ is non-increasing and bounded from below.

Remark 10. An alternative approach to the infinitary extension of t-norms can be found in [10].

In the fixed point theory it is of interest to investigate the classes of t-norms $T$ and sequences $(x_n)_{n\in\mathbb{N}}$ from the interval $[0, 1]$ such that $\lim_{n\to\infty} x_n = 1$, and

$$\lim_{n\to\infty} \bigotimes_{i=n}^\infty x_i = \lim_{n\to\infty} \bigotimes_{i=1}^\infty x_{n+i} = 1. \quad (2)$$
In the classical case $T = T_P$ we have $(T_P)^n_{i=1} = \prod_{i=1}^{n} x_i$ and for every sequence $(x_n)_{n \in \mathbb{N}}$ from the interval $[0,1]$ with $\sum_{i=1}^{\infty} (1 - x_n) < \infty$ it follows that

$$\lim_{n \to \infty} (T_P)^\infty_{i=n} = \lim_{n \to \infty} \prod_{i=n}^{\infty} x_i = 1.$$ 

Namely, it is well known that

$$\prod_{i=1}^{\infty} x_i > 0 \Leftrightarrow \lim_{n \to \infty} \prod_{i=n}^{\infty} x_i = 1 \Leftrightarrow \sum_{i=1}^{\infty} (1 - x_i) < \infty.$$ 

The equivalence

$$\sum_{i=1}^{\infty} (1 - x_i) < \infty \Leftrightarrow \lim_{n \to \infty} \prod_{i=n}^{\infty} x_i = 1$$

holds also for $T \geq T_L$. Indeed

$$(T_L)^n_{i=1} x_i = \max \left( \sum_{i=1}^{n} x_i - (n - 1), 0 \right) = \max \left( \sum_{i=1}^{n} (x_i - 1) + 1, 0 \right),$$

and therefore $\sum_{i=1}^{\infty} (1 - x_i) < \infty$ holds if and only if

$$\lim_{n \to \infty} (T_L)^\infty_{i=n} x_i = \max \left( \lim_{n \to \infty} \sum_{i=n}^{\infty} (x_i - 1) + 1, 0 \right) = 1.$$ 

For $T \geq T_L$ we have $\prod_{i=1}^{n} x_i \geq (T_L)^n_{i=1} x_i$ and therefore for such a t-norm $T$ the implication

$$\sum_{i=1}^{\infty} (1 - x_i) < \infty \Rightarrow \lim_{n \to \infty} \prod_{i=n}^{\infty} x_i = 1$$

holds.

We shall need some families of t-norms given in the following example.

**Example 11.** (i) The Dombi family of t-norms $(T^D_\lambda)_{\lambda \in [0,\infty]}$ is defined by

$$T^D_\lambda(x,y) = \begin{cases} 
T_D(x,y) & \text{if } \lambda = 0, \\
T_M(x,y) & \text{if } \lambda = \infty, \\
\left(1 + \left(\left(\frac{1-x}{x}\right)^\lambda + \left(\frac{1-y}{y}\right)^\lambda\right)^{1/\lambda} - 1\right) & \text{if } \lambda \in (0,\infty). 
\end{cases}$$
(ii) The Schweizer–Sklar family of t-norms \( T_{\lambda}^{SS} \) is defined by
\[
T_{\lambda}^{SS}(x,y) = \begin{cases} 
T_M(x,y) & \text{if } \lambda = -\infty, \\
(x^\lambda + y^\lambda - 1)^{1/\lambda} & \text{if } \lambda \in (-\infty,0), \\
T_P(x,y) & \text{if } \lambda = 0, \\
\max(x^\lambda + y^\lambda - 1,0))^{1/\lambda} & \text{if } \lambda \in (0,\infty), \\
T_D(x,y) & \text{if } \lambda = \infty.
\end{cases}
\]

(iii) The Aczel–Alsina family of t-norms \( T_{\lambda}^{AA} \) is defined by
\[
T_{\lambda}^{AA}(x,y) = \begin{cases} 
T_D(x,y) & \text{if } \lambda = 0, \\
T_M(x,y) & \text{if } \lambda = \infty, \\
e^{-\left(|\log x^\lambda| + |\log y^\lambda|\right)^{1/\lambda}} & \text{if } \lambda \in (0,\infty).
\end{cases}
\]

(iv) The family \( T_{\lambda}^{SW} \) of Sugeno–Weber t-norms is given by
\[
T_{\lambda}^{SW}(x,y) = \begin{cases} 
T_D(x,y) & \text{if } \lambda = -1, \\
T_P(x,y) & \text{if } \lambda = \infty, \\
\max \left(0, \frac{x + y - 1 + \lambda xy}{1 + \lambda}\right) & \text{otherwise.}
\end{cases}
\]

The condition \( T \geq T_L \) is fulfilled by the families: 1. \( T_{\lambda}^{SS} \) for \( \lambda \in [-\infty,1] \); 2. \( T_{\lambda}^{SW} \) for \( \lambda \in [0,\infty] \).

On the other side there exists a member of the family \( T_{\lambda}^{P} \) which is incomparable with \( T_L \), and there exists a member of the family \( T_{\lambda}^{AA} \) which is incomparable with \( T_L \).

We shall give some sufficient conditions for (2).

**Proposition 12.** Let \( (x_n)_{n \in \mathbb{N}} \) be a sequence of numbers from \([0,1]\) such that \( \lim_{n \to \infty} x_n = 1 \) and t-norm \( T \) is of H-type. Then (2) holds.

**Proof.** Since t-norm \( T \) is of H-type for every \( \lambda \in (0,1) \) there exists \( \delta(\lambda) \in (0,1) \) such that
\[
x \geq \delta(\lambda) \Rightarrow \prod_{i=1}^{p} x > 1 - \lambda
\]
for every \( p \in \mathbb{N} \). Since \( \lim_{n \to \infty} x_n = 1 \) there exists \( n_0(\lambda) \in \mathbb{N} \) such that \( x_n \geq \delta(\lambda) \) for every \( n \geq n_0(\lambda) \). Hence
\[
\prod_{i=1}^{p} x_{n+i} \geq \prod_{i=1}^{p} \delta(\lambda) > 1 - \lambda,
\]
for every $n \geq n_0(\lambda)$ and every $p \in \mathbb{N}$. This means that (2) holds. 

**Remark 13.** If $T$ is a t-norm such that there exists a sequence $(x_n)_{n \in \mathbb{N}}$ from the interval $(0,1)$ such that $\lim_{n \to \infty} x_n = 1$ and $\lim_{n \to \infty} \prod_{i=n}^{\infty} x_i = 1$, then $T$ is continuous at the point $(1,1)$. Indeed, let $\lambda \in (0,1)$ be given. Then there exists $n_0(\lambda) \in \mathbb{N}$ such that

$$\prod_{i=n_0(\lambda)}^{\infty} x_i > 1 - \lambda.$$  

Since $T(x_{n_0(\lambda)}, x_{n_0(\lambda)+1}) \geq \prod_{i=n_0(\lambda)}^{\infty} x_i > 1 - \lambda$ we obtain that $x, y \geq \max(x_{n_0(\lambda)}, x_{n_0(\lambda)+1})$ implies $T(x, y) > 1 - \lambda$.

For some families of t-norms we shall characterize the sequences $(x_n)_{n \in \mathbb{N}}$ from $(0,1]$, which tend to 1 and for which (2) holds.

**Lemma 14.** Let $T$ be a strict t-norm with an additive generator $t$, and the corresponding multiplicative generator $\theta$. Then we have

$$\prod_{i=1}^{\infty} x_i = t^{-1} \left( \sum_{i=1}^{\infty} t(x_i) \right)$$

or

$$\prod_{i=1}^{\infty} x_i = \theta^{-1} \left( \prod_{i=1}^{\infty} \theta(x_i) \right).$$

The preceding lemma and the continuity of the generators of strict t-norms imply the following proposition.

**Proposition 15.** Let $T$ be a strict t-norm with an additive generator $t$, and the corresponding multiplicative generator $\theta$. For a sequence $(x_n)_{n \in \mathbb{N}}$ from the interval $(0,1)$ such that $\lim_{n \to \infty} x_n = 1$ the condition

$$\lim_{n \to \infty} \sum_{i=n}^{\infty} t(x_i) = 0,$$

or the condition

$$\lim_{n \to \infty} \prod_{i=n}^{\infty} \theta(x_i) = 1,$$

holds if and only if (2) is satisfied.
Example 16. Let \((T^D_\lambda)^{\infty}_{x \in (0, \infty)}\) be the Dombi family of t-norms and \((x_n)_{n \in \mathbb{N}}\) be a sequence of elements from \((0, 1]\) such that \(\lim_{n \to \infty} x_n = 1\). Then we have the following equivalence:

\[
\sum_{i=1}^{\infty} \left( \frac{1 - x_i}{x_i} \right)^{\lambda} < \infty \iff \lim_{n \to \infty} (T^D_\lambda)^{\infty}_{i=n} x_i = 1.
\]

For a t-norm \(T^D_\lambda, \lambda \in (0, \infty)\), the multiplicative generator \(\theta^D_\lambda\) is given by

\[
\theta^D_\lambda(x) = e^{-\left(\frac{1-x}{x}\right)^{\lambda}}
\]

and therefore with the property \(\theta^D_\lambda(1) = 1\). Hence

\[
\prod_{i=n}^{\infty} \theta^D_\lambda(x_i) = \prod_{i=n}^{\infty} e^{-\left(\frac{1-x_i}{x_i}\right)^{\lambda}}
= e^{-\sum_{i=n}^{\infty} \left(\frac{1-x_i}{x_i}\right)^{\lambda}},
\]

and therefore the above equivalence follows by Proposition 15. Since \(\lim_{n \to \infty} x_n = 1\), we have that

\[
\left(\frac{1-x_n}{x_n}\right)^{\lambda} \sim (1-x_n)^{\lambda} \text{ as } n \to \infty.
\]

Hence

\[
\sum_{n=1}^{\infty} (1-x_n)^{\lambda} < \infty \iff \sum_{n=1}^{\infty} \left(\frac{1-x_n}{x_n}\right)^{\lambda} < \infty,
\]

which implies the equivalence

\[
\sum_{n=1}^{\infty} (1-x_n)^{\lambda} < \infty \iff \lim_{n \to \infty} (T^D_\lambda)^{\infty}_{i=n} x_i = 1.
\]

Example 17. Let \((T^{AA}_\lambda)^{\infty}_{\lambda \in (0, \infty)}\) be the Aczél–Alsina family of t-norms given by

\[
T^{AA}_\lambda(x, y) = e^{-\left(|\log x|^{\lambda} + |\log y|^{\lambda}\right)^{1/\lambda}}
\]

and \((x_n)_{n \in \mathbb{N}}\) be a sequence of elements from \((0, 1]\) such that \(\lim_{n \to \infty} x_n = 1\). Then we have the following equivalence

\[
\sum_{i=1}^{\infty} (1-x_i)^{\lambda} < \infty \iff \lim_{n \to \infty} (T^{AA}_\lambda)^{\infty}_{i=n} x_i = 1.
\]

For a t-norm \(T^{AA}_\lambda, \lambda \in (0, \infty)\), the multiplicative generator \(\theta^{AA}_\lambda\) is given by

\[
\theta^{AA}_\lambda(x) = e^{-\left(-\log x\right)^{\lambda}}
\]
and therefore with the property $\theta^{AA}_{\lambda}(1) = 1$. Hence

$$\prod_{i=n}^{\infty} \theta^{AA}_{\lambda}(x_i) = \prod_{i=n}^{\infty} e^{-(\log x_i)^\lambda} = e^{-\sum_{i=n}^{\infty} (\log x_i)^\lambda}.$$ 

Since $\lim_{i \to \infty} x_i = 1$ and $\log x_i \sim x_i - 1$ as $i \to \infty$ by Proposition 15, the above equivalence follows.

For t-norms $T^{SW}_{\lambda}$, $\lambda \in (-1, \infty]$ we have the following proposition.

**Proposition 18.** Let $(x_n)_{n \in \mathbb{N}}$ be a sequence from $(0, 1)$ such that the series $\sum_{n=1}^{\infty} (1 - x_n)$ is convergent. Then for every $\lambda \in (-1, \infty]$

$$\lim_{n \to \infty} (T^{SW}_{\lambda})_{i=n}^{\infty} x_i = 1.$$

**Proof.** An additive generator of $T^{SW}_{\lambda}$ for $\lambda \in (-1, 0)$ is given by

$$t^{SW}_{\lambda}(x) = -\log \left( \frac{1 + \lambda x}{1 + \lambda} \right) \cdot \frac{1}{\log(1 + \lambda)}.$$

We shall prove that for some $n_1 \in \mathbb{N}$ and every $p \in \mathbb{N}$

$$\prod_{i=1}^{p} \theta^{SW}_{\lambda}(x_{n+i-1}) = \exp \left( \sum_{i=1}^{p} \log \left( \frac{1 + \lambda x_{n+i-1}}{1 + \lambda} \right) \cdot \frac{1}{\log(1 + \lambda)} \right) > e^{-1} \quad (4)$$

for every $n \geq n_1$ since in this case

$$(T^{SW}_{\lambda})_{i=1}^{p} x_{n+i-1} = (\theta^{SW}_{\lambda})^{-1} \left( \prod_{i=1}^{p} \theta^{SW}_{\lambda}(x_{n+i-1}) \right). \quad (5)$$

We have to prove that for some $n_1 \in \mathbb{N}$ and every $p \in \mathbb{N}$

$$-\frac{1}{\log(1 + \lambda)} \sum_{i=0}^{p} \log \left( \frac{1 + \lambda x_{n+i-1}}{1 + \lambda} \right) < 1 \text{ for every } n > n_1, \quad (6)$$

since (6) implies (4). From $\lim_{n \to \infty} (1 - x_n) = 0$ it follows that

$$\log \left( 1 + \frac{\lambda}{1 + \lambda} (x_n - 1) \right) \sim \frac{\lambda}{1 + \lambda} (x_n - 1)$$

and therefore the series

$$-\frac{1}{\log(1 + \lambda)} \sum_{n=1}^{\infty} \log \left( 1 + \frac{\lambda}{1 + \lambda} (x_n - 1) \right)$$
is convergent. Hence it follows that there exists $n_1 \in \mathbb{N}$ such that (4) holds for every $n \geq n_1$ and every $p \in \mathbb{N}$, and this implies (5).

The above proposition holds also for $\lambda \geq 0$ since in this case $T^{SW}_\lambda \geq T_L$. □

It is of special interest for the fixed point theory in probabilistic metric spaces to investigate condition (2) for a special sequence $(1 - q^n)_{n \in \mathbb{N}}$ for $q \in (0, 1)$.

**Proposition 19.** If for a t-norm $T$ there exists $q_0 \in (0, 1)$ such that

$$\lim_{n \to \infty} T\left(\prod_{i=n}^{\infty} (1 - q_0^i)\right) = 1,$$

then

$$\lim_{n \to \infty} T\left(\prod_{i=n}^{\infty} (1 - q^i)\right) = 1,$$

for every $q \in (0, 1)$.

**Proof.** If $q < q_0$ then $1 - q^n > 1 - q_0^n$ for every $n \in \mathbb{N}$ and therefore (7) implies

$$\lim_{n \to \infty} T\left(\prod_{i=n}^{\infty} (1 - q^i)\right) \geq \lim_{n \to \infty} T\left(\prod_{i=n}^{\infty} (1 - q_0^i)\right) = 1.$$

Now suppose that $q > q_0$. First, we consider the special case when $q^2 = q_0$, i.e., $\sqrt{q_0} = q > q_0$. Then

$$\prod_{i=2m}^{\infty} (1 - q^i) \geq T\left(\prod_{i=m}^{\infty} (1 - q^{2i}), \prod_{i=m}^{\infty} (1 - q^{2i+1})\right) \geq T\left(\prod_{i=m}^{\infty} (1 - q_0^i), \prod_{i=m}^{\infty} (1 - q_0^i)\right)$$

and since $T$ by Remark 13 is continuous at $(1, 1)$ it follows that

$$\lim_{m \to \infty} \prod_{i=2m}^{\infty} (1 - q^i) \geq T(1, 1) = 1.$$

Therefore

$$\lim_{m \to \infty} \prod_{i=2m+1}^{\infty} (1 - q^i) \geq \lim_{m \to \infty} \prod_{i=2m}^{\infty} (1 - q^i) = 1.$$

Now we consider an arbitrary $q > q_0$ from the interval $(0, 1)$. Since for $q > q_0$ there exists $m \in \mathbb{N}$ such that $q_0^{2^m} > q$ we reduce this situation on the case of the $m$-iterations of the preceding procedure. □
Definition 20. We say that a t-norm $T$ is geometrically convergent (briefly $g$-convergent, in [4] called $q$-convergent for some $q \in (0,1)$) if
\[
\lim_{n \to \infty} \prod_{i=n}^{\infty} (1 - q^i) = 1.
\]
for every $q \in (0,1)$.

Since $\lim_{n \to \infty} (1 - q^n) = 1$ and $\sum_{n=1}^{\infty} (1 - (1 - q^n))^s < \infty$ for every $s > 0$ it follows that all t-norms from the family
\[
\bigcup_{\lambda \in (0,\infty)} \{T^D_\lambda\} \bigcup \bigcup_{\lambda \in (0,\infty)} \{T^A_\lambda\} \bigcup \bigcup_{\lambda \in (-1,\infty]} \{T^{SW}_\lambda\}
\]
are $g$-convergent, where $T^H$ is the class of all t-norms of $H$-type.

The following example shows that not every strict t-norm is $g$-convergent.

Example 21. Let $T$ be the strict t-norm with an additive generator $t(x) = -\frac{1}{\log(1-x)}$. In this case the series $\sum_{i=1}^{\infty} t(1 - q^i)$ for any $q \in (0,1)$ is not convergent since
\[
\sum_{i=1}^{\infty} t(1 - q^i) = -\sum_{i=1}^{\infty} \frac{1}{\log(q^i)} = -\sum_{i=1}^{\infty} \frac{1}{i \log q}.
\]

In the following two propositions we shall give sufficient conditions for a t-norm $T$ to be $g$-convergent.

Proposition 22. Let $T$ and $T_1$ be strict t-norms and $t$ and $t_1$ their additive generators, respectively, and there exists $b \in (0,1)$ such that $t(x) \leq t_1(x)$ for every $x \in (b,1]$. If $T_1$ is $g$-convergent, then $T$ is $g$-convergent.

Proof. Since $T_1$ is $g$-convergent we have $\lim_{n \to \infty} (T_1)_{i=n}^{\infty} (1 - q^i) = 1$. Therefore
\[
\lim_{n \to \infty} \sum_{i=n}^{\infty} t_1(1 - q^i) = 0. \quad (8)
\]
Since there exists $n_0 \in N$ such that $1 - q^{n_0} \in (b,1]$ we have by the condition of the proposition that
\[
t(1 - q^n) \leq t_1(1 - q^n) \text{ for every } n \geq n_0.
\]
Therefore, by (8) $\lim_{n \to \infty} \sum_{i=n}^{\infty} t(1 - q^i) = 0$, i.e., $T$ is $g$-convergent. \qed
Proposition 23. Let $T$ be a strict t-norm with a generator $t$ which has a bounded derivative on an interval $(b, 1)$ for some $b \in (0, 1)$. Then $T$ is $g$-convergent.

Proof. By the Lagrange mean value theorem we have for every $x \in (b, 1)$ that

$$t(x) - t(1) = t'(\xi)(x - 1)$$

for some $\xi \in (x, 1)$, and therefore

$$\sum_{i=0}^{\infty} t(1 - q^i) \leq M \sum_{i=0}^{\infty} q^i,$$

where $M = \sup_{x \in (b, 1)} |t'(x)|$, and $1 - q^i \in (b, 1)$. □

Proposition 24. Let $T$ be a t-norm and $\psi : (0, 1] \to [0, \infty)$. If for some $\delta \in (0, 1)$ and every $x \in [0, 1]$, $y \in [1 - \delta, 1]$

$$|T(x, y) - T(x, 1)| \leq \psi(y)$$

then for every sequence $(x_n)_{n \in \mathbb{N}}$ from the interval $[0, 1]$ such that $\lim_{n \to \infty} x_n = 1$ and

$$\sum_{n=1}^{\infty} \psi(x_n) < \infty,$$

relation (2) holds.

For the proof see [4].

Corollary 25. Let $T$ and $\psi$ be as in Proposition 25. If for some $q \in (0, 1)$,

$$\sum_{n=1}^{\infty} \psi(1 - q^n) < \infty$$

then $T$ is $g$-convergent.

Proof. Since $\lim_{n \to \infty} (1 - q^n) = 1$ by Proposition 25 we obtain that

$$\lim_{n \to \infty} \prod_{i=n}^{\infty} (1 - q^n) = 1.$$

Example 26. Let $\alpha > 0$, $p > 1$ and $z_{\alpha, p} : (0, 1] \times [0, 1] \to [0, \infty)$ be defined in the following way:

$$z_{\alpha, p}(x, y) = \begin{cases} y - \frac{\alpha}{|\ln(1 - x)|^p} & \text{if } (x, y) \in (0, 1) \times [0, 1], \\ y & \text{if } (x, y) \in \{1\} \times [0, 1]. \end{cases}$$

In this case the function $z_{\alpha, p}$ is equal to zero on the curve which connects the points $(1, 0)$ and $(1 - e^{-\alpha^{1/p}}, 1)$, where $1 - e^{-\alpha^{1/p}} < 1$. 
Let $T$ be a t-norm such that $T(x, y) \geq z_{\alpha, p}(x, y)$ for every $(x, y) \in [1 - \delta, 1] \times [0, 1]$. Then for every $(x, y) \in [0, 1] \times [1 - \delta, 1)$

$$|T(x, y) - T(x, 1)| = |T(y, x) - T(1, x)| \leq |z_{\alpha, p}(y, x) - z_{\alpha, p}(1, x)| \leq \frac{\alpha}{|\ln(1 - y)|^p},$$

i.e., (9) holds for

$$\psi(y) = \begin{cases} \frac{\alpha}{|\ln(1 - y)|^p} & \text{if } y \in [1 - \delta, 1), \\ 0 & \text{if } y = 1. \end{cases}$$

Since

$$\sum_{n=1}^{\infty} \psi(1 - q^n) = \sum_{n=1}^{\infty} \frac{\alpha}{|\ln(q^n)|^p} = \sum_{n=1}^{\infty} \frac{\alpha}{n^p|\ln(q)|^p} < \infty,$$

$T$ is $g$-convergent.

4. FIXED POINT THEORY IN PROBABILISTIC METRIC SPACES

Let $\Delta^+$ be the set of all distribution functions $F$ such that $F(0) = 0$ ($F$ is a nondecreasing, left continuous mapping from $\mathbb{R}$ into $[0, 1]$ such that $\sup_{x \in \mathbb{R}} F(x) = 1$).

The ordered pair $(S, \mathcal{F})$ is said to be a probabilistic metric space if $S$ is a nonempty set and $\mathcal{F} : S \times S \to \Delta^+$ ($\mathcal{F}(p, q)$ is written by $F_{p,q}$ for every $(p, q) \in S \times S$) satisfies the following conditions:

1. $F_{u,v}(x) = 1$ for every $x > 0 \Rightarrow u = v$ ($u, v \in S$).
2. $F_{u,v} = F_{v,u}$ for every $u, v \in S$.
3. $F_{u,v}(x) = 1$ and $F_{v,w}(y) = 1 \Rightarrow F_{u,w}(x + y) = 1$ for $u, v, w \in S$ and $x, y \in \mathbb{R}_+ = [0, \infty)$.

A Menger space is a triple $(S, \mathcal{F}, T)$, where $(S, \mathcal{F})$ is a probabilistic metric space, $T$ is a t-norm and the following inequality holds

$$F_{u,v}(x + y) \geq T(F_{u,w}(x), F_{w,v}(y))$$

for every $u, v, w \in S$ and every $x > 0, y > 0$.

The $(\varepsilon, \lambda)$-topology in $S$ is introduced by the family of neighbourhoods

$$\mathcal{U} = \{U_v(\varepsilon, \lambda) : (v, \varepsilon, \lambda) \in S \times \mathbb{R}_+ \times (0, 1)\},$$

where

$$U_v(\varepsilon, \lambda) = \{u : u \in S, F_{u,v}(\varepsilon) > 1 - \lambda\}.$$
4.1. Probabilistic q-contraction and g-convergent t-norms

Definition 27. ([15]) Let \((S, \mathcal{F})\) be a probabilistic metric space. A mapping \(f : S \to S\) is a probabilistic \(q\)-contraction \((q \in (0, 1))\) if

\[
F_{f_{p_1}, f_{p_2}}(x) \geq F_{p_1, p_2} \left( \frac{x}{q} \right)
\]

(10)

for every \(p_1, p_2 \in S\) and every \(x \in \mathbb{R}\).

By Remark 13 each \(g\)-convergent t-norm \(T\) satisfies the condition \(\sup_{x < 1} T(x, x) = 1\), which ensures the metrizability of the \((\varepsilon, \lambda)\)-topology.

Theorem 28. Let \((S, \mathcal{F}, T)\) be a complete Menger space and \(f : S \to S\) a probabilistic \(q\)-contraction such that for some \(p \in S\) and \(k > 0\)

\[
\sup_{x > 0} x^k (1 - F_{p, f_p}(x)) < \infty.
\]

(11)

If t-norm \(T\) is \(g\)-convergent, then there exists a unique fixed point \(z\) of the mapping \(f\) and \(z = \lim_{n \to \infty} f^n p\).

Proof. Let \(\mu \in (q, 1)\) and \(\delta = q/\mu < 1\). We shall prove that \((f^n p)_{n \in \mathbb{N}}\) is a Cauchy sequence. Choose \(\varepsilon > 0\) and \(\lambda \in (0, 1)\) and prove that there exists \(n_0(\varepsilon, \lambda) \in \mathbb{N}\) such that

\[
F_{f^n p, f^{n+m} p}(\varepsilon) > 1 - \lambda \text{ for every } n \geq n_0(\varepsilon, \lambda) \text{ and every } m \in \mathbb{N}.
\]

Since the series \(\sum_{i=1}^{\infty} \delta^i\) is convergent, there exists \(n_1 = n_1(\varepsilon) \in \mathbb{N}\) such that \(\sum_{i=n_1}^{\infty} \delta^i \leq \varepsilon\). Let \(n > n_1\). Then we have

\[
F_{f^n p, f^{n+m} p}(\varepsilon) \geq F_{f^n p, f^{n+m} p} \left( \sum_{i=n}^{\infty} \delta^i \right)
\]

\[
\geq F_{f^n p, f^{n+m} p} \left( \sum_{i=n}^{n+m-1} \delta^i \right)
\]

\[
\geq T \left( T \left( \cdots T \left( F_{f^n p, f^{n+1} p}(\delta^n), F_{f^{n+1} p, f^{n+2} p}(\delta^{n+1}) \right), \cdots, F_{f^{n+m-1} p, f^{n+m} p}(\delta^{n+m-1}) \right) \right),
\]

\( (m-1)\)-times

\[
\geq T \left( T \left( \cdots T \left( F_{p, f_p}(\frac{1}{\mu^n}), F_{p, f_p}(\frac{1}{\mu^{n+1}}), \cdots, F_{p, f_p}(\frac{1}{\mu^{n+m-1}}) \right) \right), \cdots, F_{p, f_p}(\frac{1}{\mu^{n+m-1}}) \right).\]

\( (m-1)\)-times

Let \(M > 0\) be such that

\[
x^k (1 - F_{p, f_p}(x)) \leq M \text{ for every } x > 0.
\]

(12)
Suppose that $n_2$ is such that
\[ 1 - M(\mu^k)^n \in [0, 1) \text{ for every } n \geq n_2. \] (13)

From (12) it follows that
\[ F_{p, f_p} \left( \frac{1}{\mu^n} \right) > 1 - M(\mu^k)^n \text{ for every } n \in \mathbb{N} \]
and by (13) for $n \geq \max(n_1, n_2)$
\[ F_{f^n, f_{n+m}}(\varepsilon) \geq T \left( T \left( \cdots \left( T \left( 1 - M(\mu^k)^n, 1 - M(\mu^k)^{n+1} \right), \ldots, 1 - M(\mu^k)^{n+m-1} \right) \right) \right). \]

Let $s_0$ be such that $M(\mu^k)^{s_0} < \mu^k$. Then for every $n \in \mathbb{N}$
\[ 1 - M(\mu^k)^{n+s_0} \geq 1 - (\mu^k)^{n+1} \]
and therefore for $n \geq \max(n_1, n_2)$ and $m \in \mathbb{N}$
\[ F_{f^{n+s_0}, f^{n+s_0+m}}(\varepsilon) \geq T \left( T \left( \cdots \left( T \left( 1 - M(\mu^k)^{n+s_0}, 1 - M(\mu^k)^{n+s_0+1} \right), \ldots, 1 - M(\mu^k)^{n+s_0+m-1} \right) \right) \right) \geq \prod_{i=n+1}^{\infty} (1 - (\mu^k)^i). \]

Since $T$ is $g$-convergent we conclude that $(f^n p)_{n \in \mathbb{N}}$ is a Cauchy sequence. Let $z = \lim_{n \to \infty} f^n p$. By the continuity of the mapping $f$ it follows that $fz = z$. \qed

**Corollary 29.** Let $(S, \mathcal{F}, T)$ be a complete Menger space such that $T$ is a strict $t$-norm with a multiplicative generator $\theta$, and $f : S \to S$ a probabilistic $g$-contraction such that for some $k > 0$ and $p \in S$ (11) holds. If there exists $\mu \in (0, 1)$ such that
\[ \lim_{n \to \infty} \prod_{i=n}^{\infty} \theta(1 - \mu^i) = 1, \]
then there exists a unique fixed point $x$ of the mapping $f$ and $x = \lim_{n \to \infty} f^n p$.

Let
\[ \mathcal{T} = \bigcup_{\lambda \in (0, \infty)} \{ T_\lambda^p \} \cup \bigcup_{\lambda \in (0, \infty)} \{ T_\lambda \mathcal{A} \}. \]
Corollary 30. Let \((S, T, T)\) be a complete Menger space such that \(T \geq T_1\) for some \(T_1 \in T\) and \(f : S \to S\) a probabilistic \(q\)-contraction such that for some \(k > 0\) and \(p \in S\) (11) holds. Then there exists a unique fixed point \(x\) of the mapping \(f\) and \(x = \lim_{n \to \infty} f^n p\).

From the proof of Theorem 28 it follows that \(f : S \to S\) has a unique fixed point if (11) and the condition that \(T\) is \(g\)-convergent is replaced by the condition

\[
\lim_{n \to \infty} \prod_{i=n}^{\infty} F_{p,fp} \left( \frac{1}{\mu^i} \right) = 1 \quad (\mu \in (0, 1)).
\]  

(14)

Using Examples 16 and 17 and Proposition 18 we obtain a fixed point theorem, where the condition (11) is replaced by the condition

\[
\sup_{x > 1} \ln^k x (1 - F_{p,fp}(x)) < \infty,
\]  

(15)

for some \(k > 0\), which under some additional conditions implies (14).

Theorem 31. Let \((S, T, T)\) be a complete Menger space and \(f : S \to S\) a probabilistic \(q\)-contraction. Suppose that one of the following two conditions is satisfied:

(i) \(T \in \{ T^D_\lambda, T^{AA}_\lambda \}\) for some \(\lambda > 0\) and there exists \(p \in S\) such that (15) holds, where \(k \lambda > 1\).

(ii) \(T = T^S_\lambda\) for some \(\lambda \in (-1, \infty]\) and there exists \(p \in S\) such that (15) holds, where \(k > 1\).

Then there exists a unique fixed point \(z\) of the mapping \(f\) and \(z = \lim_{n \to \infty} f^n p\).

Proof. (i) Suppose that \(\sup_{x > 1} \ln^k x (1 - F_{p,fp}(x)) < \infty\), i.e., that there exists \(M > 0\) such that

\[
\ln^k x (1 - F_{p,fp}(x)) < M \quad \text{for every } x > 1.
\]  

(16)

Relation (16) implies that

\[
F_{p,fp} \left( \frac{1}{\mu^n} \right) \geq 1 - \frac{M}{\ln^k \left( \frac{1}{\mu^n} \right)} = 1 - \frac{M}{n^k |\ln \mu|^k} \quad (\mu \in (0, 1)).
\]

Suppose that \(1 - \frac{M}{n^k |\ln \mu|^k} > 0\) for every \(n \geq n_0\). Then

\[
\prod_{i=n}^{\infty} F_{p,fp} \left( \frac{1}{\mu^i} \right) \geq \prod_{i=n}^{\infty} \left( 1 - \frac{M}{n^k |\ln \mu|^k} \right) \quad \text{for every } n \geq n_0.
\]

By Examples 16 and 17

\[
\lim_{n \to \infty} \prod_{i=n}^{\infty} \left( 1 - \frac{M}{n^k |\ln \mu|^k} \right) = 1
\]
since for $k\lambda > 1$
\[ \sum_{i=1}^{\infty} \frac{M^\lambda}{i^{k\lambda} \ln|\mu|^{\frac{k\lambda}{i}}} < \infty. \]

Hence (14) holds.

(ii) If $T = T^{SW}_\lambda$ for some $\lambda \in (-1, \infty)$ and (16) holds for some $k > 1$ then (14) holds, since by Proposition 18, $\sum_{i=1}^{\infty} \frac{M^\lambda}{i^{k\lambda} \ln|\mu|^{\frac{k\lambda}{i}}} < \infty$ implies (14).

**Remark 32.** It is obvious by Proposition 18 that in the case (ii) the condition (15) can be replaced by the Tardiff's condition (see [16])
\[ \int_{1}^{\infty} \ln u \, dF_{p,f}(u) < \infty. \]

### 4.2. An application to random operator equations

Special non-additive measures, so called decomposable measures, see [11], generate a probabilistic metric space ([4]) on which Theorem 28 implies a random fixed point theorem.

**Definition 33.** Let $S$ be a t-conorm. An $S$-decomposable measure $m$ is a set function $m : \mathcal{A} \to [0, 1]$ such that $m(\emptyset) = 0$ and
\[ m(A \cup B) = S(m(A), m(B)) \]
whenever $A, B \in \mathcal{A}$ and $A \cap B = \emptyset$.

**Example 34.** Taking $S_L$ t-conorm, $\Omega = \mathbb{N}$, $\mathcal{A} = 2^\mathbb{N}$ and $m(E) = \min(|E|/N, 1)$ for a fixed natural number $N$, where $|E|$ is the cardinal number of $E$, we obtain that $m$ is $S_L$-decomposable measure.

**Definition 35.** Let $S$ be a left-continuous t-conorm. A set function $m : \mathcal{A} \to [0, 1]$ is $\sigma$-$S$-decomposable measure if $m(\emptyset) = 0$ and
\[ m\left( \bigcup_{i=1}^{\infty} A_i \right) = \bigvee_{i=1}^{\infty} m(A_i) \]
for every sequence $(A_i)_{i \in \mathbb{N}}$ from $\mathcal{A}$ whose elements are pairwise disjoint set.

The set function considered in Example 34 is $\sigma$-$S_L$-decomposable.

An $S$-decomposable measure $m$ is monotone, which means that $A, B \in \mathcal{A}$, $A \subseteq B$ implies $m(A) \leq m(B)$. A measure $m$ is of (NSA)-type (see [17]) if and only if $s \circ m$ is a finite additive measure, where $s$ is an additive generator of the t-conorm $S$ (see [17]), which is continuous, non-strict, and Archimedean, and with respect to which $m$ is decomposable ($s(1) = 1$). If $(\Omega, \mathcal{A}, m)$ is a measure space and $(M, d)$ is a separable metric space, by $\mathcal{S}$ we shall denote the set of all the equivalence classes of measurable mappings $X : \Omega \to M$. An element from $\mathcal{S}$ will be denoted by $\tilde{X}$ if \{X(\omega)\} $\in \tilde{X}$. The following proposition is proved in [14].
Proposition 36. Let \((\Omega, \mathcal{A}, m)\) be a measure space, where \(m\) is a continuous \(S\)-decomposable measure of (NSA)-type with monotone increasing generator \(s\). Then \((\mathcal{S}, \mathcal{F}, T)\) is a Menger space, where \(\mathcal{F}\) and t-norm \(T\) are given in the following way:

\[
(F^S, F^T)(u) = m(\{\omega \mid \omega \in \Omega, d(X(\omega), Y(\omega)) < u\}) = m(\{d(X, Y) < u\})
\]

(for every \(X, Y \in S, u \in \mathbb{R}\)),

\[
T(x, y) = s^{-1}(\max(0, s(x) + s(y) - 1)), \text{ for every } x, y \in [0, 1].
\]

Let \(f: \Omega \times M \to M\) be a continuous random operator. Then for every measurable mapping \(X: \Omega \to M\), the mapping \(\omega \mapsto f(\omega, X(\omega))(\omega \in \Omega)\) is measurable. If \(X: \Omega \to M\) is a measurable mapping let \((\hat{f}X)(\omega) = f(\omega, X(\omega)), \omega \in \Omega, X \in \mathcal{X}\). Hence \(\hat{f}: S \to S\).

Corollary 37. Let \((\Omega, \mathcal{A}, m)\) be a measure space, where \(m\) is a continuous \(S\)-decomposable measure of (NSA)-type, \(s\) is a monotone increasing additive generator of \(S\), \((M, d)\) a complete separable metric space and \(f: \Omega \times M \to M\) a continuous random operator such that for some \(q \in (0, 1)\)

\[
m(\{\omega \mid \omega \in \Omega, d((\hat{f}X)(\omega), (\hat{f}Y)(\omega)) < u\}) \geq m\left(\{\omega \mid \omega \in \Omega, d(X(\omega), Y(\omega)) < \frac{u}{q}\}\right)
\]

for every measurable mappings \(X, Y: \Omega \to M\) and every \(u > 0\). If there exists a measurable mapping \(U: \Omega \to M\) such that for some \(k > 0\)

\[
\sup_{x > 0} x^k (1 - m(\{d(U, fU) < x\})) < \infty
\]

and t-norm \(T\) defined by

\[
T(x, y) = s^{-1}(\max(0, s(x) + s(y) - 1)), x, y \in [0, 1],
\]

is \(g\)-convergent, then there exists a random fixed point of the operator \(f\).

Corollary 38. Let \((\Omega, \mathcal{A}, m)\) be a measure space, where \(m\) is a continuous \(S^W\)-decomposable measure of (NSA)-type for some \(\lambda \in (-1, \infty)\), \((M, d)\) a complete separable metric space and \(f: \Omega \times M \to M\) a continuous random operator such that for some \(q \in (0, 1)\) \((17)\) holds for every measurable mappings \(X, Y: \Omega \to M\) and every \(u > 0\). If there exists a measurable mapping \(U: \Omega \to M\) such that for some \(k > 1\)

\[
\sup_{x > 1} \ln^k x (1 - m(\{d(U, fU) < x\})) < \infty,
\]

then there exists a random fixed point of the operator \(f\).
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