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A CONVERGENCE OF FUZZY RANDOM VARIABLES

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In this paper, a general convergence theorem of fuzzy random variables is considered. Using this result, we can easily prove the recent result of Joo et al, which gives generalization of a strong law of large numbers for sums of stationary and ergodic processes to the case of fuzzy random variables. We also generalize the recent result of Kim, which is a strong law of large numbers for sums of levelwise independent and levelwise identically distributed fuzzy random variables.

Keywords: fuzzy number, fuzzy random variable, strong law of large numbers
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1. INTRODUCTION

In recent years, strong laws of large numbers for sums of fuzzy random variables have received much attention by several people. A SLLN for sums of independent and identically distributed fuzzy random variables was obtained by Kruse [10], and a SLLN for sums of independent fuzzy random variables was obtained by Miyakoshi and Shimbo [11], Klement, Puri and Ralescu [15]. Also, Inoue [5] obtained a SLLN for sums of independent tight fuzzy random sets, and Hong and Kim [4] proved Marcinkiewicz-type law of large numbers. Many other papers [1, 3, 7, 12, 13, 14, 15, 16, 17, 18] are related to this topic. Recently, Joo, Lee and Yoo [6] generalized a strong law of large numbers for sums of stationary and ergodic processes to the case of fuzzy random variables and Kim [8] obtained a strong law of large numbers for sums of levelwise independent and levelwise identically distributed fuzzy random variables.

In this paper, we consider a general convergence theorem of fuzzy random variables. Using this result, we can easily prove the result of Joo et al [6] and generalize the result of Kim[8]. Section 2 is devoted to describe some basic concepts of fuzzy random variables. Main results are given in Section 3.

2. PRELIMINARIES

Let $R$ denote the real line. A fuzzy number is a fuzzy set $\tilde{u} : R \rightarrow [0, 1]$ with the following properties;
(1) $\tilde{u}$ is normal, i.e., there exists $x \in R$ such that $\tilde{u}(x) = 1$.

(2) $\tilde{u}$ is upper semicontinuous.

(3) $\text{supp } \tilde{u} = \text{cl}\{x \in R|\tilde{u}(x) > 0\}$ is compact.

(4) $\tilde{u}$ is a convex fuzzy set, i.e., $\tilde{u}(\lambda x + (1 - \lambda)y) \geq \min(\tilde{u}(x), \tilde{u}(y))$ for $x, y \in R$ and $\lambda \in [0, 1]$.

Let $F(R)$ be the family of all fuzzy numbers. For a fuzzy set $\tilde{u}$, if we define

$$L_{\alpha}\tilde{u} = \begin{cases} \{x|\tilde{u}(x) \geq \alpha\}, & 0 < \alpha \leq 1, \\ \text{supp } \tilde{u}, & \alpha = 0, \end{cases}$$

then, $\tilde{u}$ is a fuzzy number if and only if $L_1 \tilde{u} \neq \phi$ and $L_0 \tilde{u}$ is a closed bounded interval for each $\alpha \in [0, 1]$. If we use this characteristic of fuzzy number, a fuzzy number $\tilde{u}$ is completely determined by the endpoints of the intervals $L_{\alpha}\tilde{u} = [u_{\alpha}^1, u_{\alpha}^2]$.

The following theorem (see Goetschel and Voxman [2]) implies that we can identify a fuzzy number $\tilde{u}$ with the parameterized representation

$$\{(u_{\alpha}^1, u_{\alpha}^2)|0 \leq \alpha \leq 1\}.$$ 

**Theorem 2.1.** For $\tilde{u} \in F(R)$, denote $u^1(\alpha) = u_{\alpha}^1$ and $u^2(\alpha) = u_{\alpha}^2$ as functions of $\alpha \in [0, 1]$. Then

(1) $u^1$ is a bounded increasing function on [0,1].

(2) $u^2$ is a bounded decreasing function on [0,1].

(3) $u^1(1) \leq u^2(1)$.

(4) $u^1$ and $u^2$ are left continuous on [0,1] and right continuous at 0.

(5) If $v^1$ and $v^2$ satisfy above (1) – (4), then there exists a unique $\tilde{v} \in F(R)$ such that $v_{\alpha}^1 = u^1(\alpha), v_{\alpha}^2 = u^2(\alpha)$.

The addition and scalar multiplication on $F(R)$ are defined as usual;

$$(\tilde{u} + \tilde{v})(z) = \sup \min(\tilde{u}(x), \tilde{v}(y)),$$

$$(\lambda \tilde{u})(z) = \begin{cases} \tilde{u}(z/\lambda), & \lambda \neq 0, \\ \tilde{0}, & \lambda = 0, \end{cases}$$

for $\tilde{u}, \tilde{v} \in F(R)$ and $\lambda \in R$, where $\tilde{0} = I_{\{0\}}$ is the characteristic function of $\{0\}$. It follows that if $\tilde{u} = \{(u_{\alpha}^1, u_{\alpha}^2)|0 \leq \alpha \leq 1\}$ and $\tilde{v} = \{(v_{\alpha}^1, v_{\alpha}^2)|0 \leq \alpha \leq 1\}$, then

$$\tilde{u} + \tilde{v} = \{(u_{\alpha}^1 + v_{\alpha}^1, u_{\alpha}^2 + v_{\alpha}^2)|0 \leq \alpha \leq 1\}$$

$$\lambda \tilde{u} = \{(\lambda u_{\alpha}^1, \lambda u_{\alpha}^2)|0 \leq \alpha \leq 1\} \text{ for } \lambda \geq 0.$$
Now, we define the metric $d_\infty$ on $F(R)$ by
\[ d_\infty(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} h(L_\alpha \tilde{u}, L_\alpha \tilde{v}), \]
where $h$ is Hausdorff metric defined as
\[ h(L_\alpha \tilde{u}, L_\alpha \tilde{v}) = \max(|u_1^1 - v_1^1|, |u_2^2 - v_2^2|). \]
The norm of $\tilde{u} \in F(R)$ is defined by
\[ \|\tilde{u}\| = d_\infty(\tilde{u}, \tilde{0}) = \max(|u_0^1|, |u_0^2|). \]
Then it is well-known that $F(R)$ is complete but nonseparable with respect to the metric $d_\infty$. Joo and Kim [7] introduced a metric $d_s$ in $F(R)$ which makes it a separable metric space as follows.

**Definition 2.1.** Let $T$ denote the class of strictly increasing, continuous mappings of $[0,1]$ onto itself. For $u, v \in F(R)$, we define
\[ d_s(\tilde{u}, \tilde{v}) = \inf \{ \varepsilon : \text{there exists a } t \in T \text{ such that} \]
\[ \sup_{0 \leq \alpha \leq 1} |t(\alpha) - \alpha| \leq \varepsilon \text{ and } d_\infty(\tilde{u}, t \circ \tilde{v}) \leq \varepsilon \}, \]
where $t \circ \tilde{v}$ denotes the composition of $\tilde{v}$ and $t$.

### 3. MAIN RESULTS

Throughout this section, we assume that the space $F(R)$ is considered as the metric space endowed with the metric $d_s$, unless otherwise stated. Also, we denote by $B_s$ the Borel $\sigma$-field of $F(R)$ generated by the metric $d_s$.

Let $(\Omega, A, P)$ be a probability space. A fuzzy number valued function $\tilde{X} : \Omega \to F(R)$ is called a fuzzy random variable if it is measurable, i.e.,
\[ \tilde{X}^{-1}(B) = \{ \omega : \tilde{X}(\omega) \in B \} \in A \text{ for every } B \in B_s. \]
If we denote $\tilde{X}(\omega) = \{(X_1^1(\omega), X_2^2(\omega))|0 \leq \alpha \leq 1\}$, then it is known that $\tilde{X}$ is a fuzzy random variable if and only if for each $\alpha \in [0,1]$, $X_1^1$ and $X_2^2$ are random variables in the usual sense. A fuzzy random variable $\tilde{X} = \{(X_1^1, X_2^2)|0 \leq \alpha \leq 1\}$ is called integrable if for each $\alpha \in [0,1]$, $X_1^1$ and $X_2^2$ are integrable, equivalently, $\int \|\tilde{X}\| \, dP < \infty$. In this case, the expectation of $\tilde{X}$ is the fuzzy number $E\tilde{X}$ defined by
\[ E\tilde{X} = \{ (EX_1^1, EX_2^2) | 0 \leq \alpha \leq 1 \} \]
Theorem 3.1. Let \( \{X_n\} = \{(X_{n\alpha}, X_{n\alpha}) \mid 0 \leq \alpha \leq 1\} \) be a sequence of fuzzy random variables and \( \tilde{u} = \{(u_{\alpha}^1, u_{\alpha}^2) \mid 0 \leq \alpha \leq 1\} \) be a fuzzy number with \( \|\tilde{u}\| < \infty \). Suppose that

1. \( X_{n\alpha}^1 \to u_{\alpha}^1 \) a.s. and \( X_{n\alpha}^2 \to u_{\alpha}^2 \) a.s. for any \( \alpha \in [0, 1] \)
2. \( X_{n\alpha}^1+ \to u_{\alpha}^1+ \) a.s. and \( X_{n\alpha}^2- \to u_{\alpha}^2- \) a.s. for every discontinuity point of \( u_{\alpha}^1 \) and \( u_{\alpha}^2 \), respectively.

Then we have

\[
\lim_{n \to \infty} d_{\infty}(X_n, \tilde{u}) = 0 \ a.s.
\]

We need the following lemma given in [6].

Lemma 3.1. Let \( u = \{(u_{\alpha}^1, u_{\alpha}^2) \mid 0 \leq \alpha \leq 1\} \) with \( \|u\| < \infty \) and \( \varepsilon > 0 \) be given.

1. Then there exists a partition \( 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1 \) of \([0, 1]\) such that \( u_{\alpha_i}^1 - u_{\alpha_{i-1}}^1 \leq \varepsilon \) for all \( i = 1, 2, \ldots, r \).
2. Similar statements hold for \( u_{\alpha}^2 \).

Proof of Theorem 3.1. Let \( \varepsilon > 0 \) be arbitrary fixed. By Lemma 3.1, there exists a partition \( 0 = \alpha_0 < \alpha_1 < \ldots < \alpha_r = 1 \) of \([0, 1]\) such that \( u_{\alpha_i}^1 - u_{\alpha_{i-1}}^1 \leq \varepsilon \) for all \( i = 1, 2, \ldots, r \). Let \( A_k = \{X_{n\alpha_k}^1 \to u_{\alpha_k}^1 \) and \( X_{n\alpha_k}^1+ \to u_{\alpha_k}^1+ \) for all discontinuity points of \( u_{\alpha}^1 \} \) and \( A_{\varepsilon} = \cap_{k=1}^r A_k \), then by the assumption \( P(A_k) = 1 \), \( k = 1, 2, \ldots, r \), and hence \( P(A_{\varepsilon}) = 1 \). Then for any given \( w \in A_{\varepsilon} \), there exists \( N(w) \) such that for \( n \geq N(w) \)

\[
\sup_{k=1,2,\ldots,r} \{|X_{n\alpha_k}^1(w) - u_{\alpha_k}^1|, |X_{n\alpha_k}^1+(w) - u_{\alpha_k}^1+|\} \leq \varepsilon.
\]

Now, let \( \alpha \in (\alpha_{k-1}, \alpha_k] \), then for \( n \geq N(w) \),

\[
X_{n\alpha}^1(w) - u_{\alpha}^1 \leq X_{n\alpha_k}^1(w) - u_{\alpha_k}^1 \leq u_{\alpha_k}^1+ - u_{\alpha_k-1}^1 \leq 2\varepsilon
\]

and

\[
u_{\alpha}^1 - X_{n\alpha}^1(w) \leq u_{\alpha_k}^1 - X_{n\alpha_k}^1+(w) \leq u_{\alpha_k}^1 - (u_{\alpha_k-1}^1 - \varepsilon) \leq 2\varepsilon.
\]

Hence

\[
\sup_{\alpha \in (\alpha_{k-1}, \alpha_k]} |X_{n\alpha}^1(w) - u_{\alpha}^1| \leq 2\varepsilon.
\]

Since \( k \) is arbitrary, we have

\[
\sup_{\alpha \in [0, 1]} |X_{n\alpha}^1(w) - u_{\alpha}^1| \leq 2\varepsilon.
\]

Let \( A = \cap_{n=1}^\infty A_{\frac{1}{n}} \), then \( P(A) = 1 \) and for any \( w \in A \)

\[
\lim_{n \to \infty} \sup_{0 \leq \alpha \leq 1} |X_{n\alpha}^1(w) - u_{\alpha}^1| = 0.
\]
Similarly, it can be proved that
\[
\lim_{n \to \infty} \sup_{0 \leq \alpha \leq 1} |X_{n\alpha}^2 - u^2_{\alpha}| = 0, \text{ a.s.}
\]
which completes the proof. \(\Box\)

Recently, Kim [8] proved a SLLN for sums of levelwise independent and identically distributed fuzzy random variables. But his result is a special case of Theorem 1. If \(\tilde{X}_n\) is a sequence of levelwise independent and levelwise identically distributed random variables with \(E||\tilde{X}_1|| < \infty\), then, it is easy to check that both \(\{X_{n\alpha+}^1\}\) and \(\{X_{n\alpha-}^2\}\) for \(\alpha \in [0, 1]\) are independent and identically distributed random variables, respectively, with \(E|X_{n\alpha+}^1| < \infty\) and \(E|X_{n\alpha-}^2| < \infty\). And it is also easy to check that for any \(\alpha \in [0, 1]\)

\[
\frac{1}{n} \sum_{i=1}^{n} X_{\alpha+}^1 \xrightarrow{a.s.} EX_{\alpha+}^1
\]

and

\[
\frac{1}{n} \sum_{i=1}^{n} X_{\alpha-}^2 \xrightarrow{a.s.} EX_{\alpha-}^2
\]

by Kolmogorov’s strong law of large numbers and Monotone Convergence Theorem. It is also noted that the set of discontinuity point of \(EX_{\alpha+}^1\) and \(EX_{\alpha-}^2\) is at most countable. Now, using Theorem 1 we have the following generalized result of Kim [8] as a corollary.

**Corollary 3.1.** Let \(\tilde{X}_n\) be a sequence of levelwise independent and levelwise identically distributed fuzzy random variables, with \(E||\tilde{X}_1|| < \infty\). Then we have

\[
d_{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, E\tilde{X}_1 \right) \rightarrow 0 \text{ a.s.}
\]

**Remark.** The condition that \(EX_{\alpha+}^1\) and \(EX_{\alpha-}^2\) are continuous as functions of \(\alpha\) in Kim’s result is not needed.

Recently Joo et al [6] proved a SLLN for sums of stationary and ergodic fuzzy random variables. With similar arguments as above, noting that for each \(\alpha \in [0, 1]\), \(\{X_{n\alpha}^1\}, \{X_{n\alpha+}^1\}, \{X_{n\alpha}^2\}\) and \(\{X_{n\alpha-}^2\}\) are sequences of stationary and ergodic random variables under the assumption that \(\{\tilde{X}_n\}\) is a sequence of stationary and ergodic fuzzy random variables, we also have Joo’s result as a corollary by Theorem 1.

**Corollary 3.2.** Let \(X_n\) be a sequence of stationary fuzzy random variables. If \(\{\tilde{X}_n\}\) is ergodic and \(E||\tilde{X}_1|| < \infty\), then

\[
d_{\infty} \left( \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i, E\tilde{X}_1 \right) \rightarrow 0 \text{ a.s.}
\]
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