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APPROMATIONS FOR THE MAXIMUM OF STOCHASTIC PROCESSES WITH DRIFT$^1$

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If a stochastic process can be approximated with a Wiener process with positive drift, then its maximum also can be approximated with a Wiener process with positive drift.

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1. INTRODUCTION AND RESULTS

Let $X_1, X_2, \ldots$ be a sequence of independent, identically distributed random variables with

$$EX_i = \mu > 0 \text{ and } 0 < \text{var}X_1 = \sigma^2 < \infty.$$  \hspace{1cm} (1.1)

The motivation of our note is the following central limit theorem due to Teicher [6]. Let

$$S(j) = \sum_{1 \leq i \leq j} X_i$$

and

$$0 \leq \alpha < 1.$$  \hspace{1cm} (1.2)

**Theorem 1.1.** If (1.1) and (1.2) hold, then

$$\frac{1}{\sigma n^{1/2-\alpha}} \left\{ \max_{1 \leq j \leq n} \frac{S(j)}{j^\alpha} - \mu n^{1-\alpha} \right\} \xrightarrow{D} N(0,1),$$

where $N(0,1)$ denotes a standard normal random variable.

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Since
\[
\frac{1}{\sigma n^{1/2-\alpha}} \left\{ \frac{S(n)}{n^\alpha} - \mu n^{1-\alpha} \right\} \overset{D}{\to} N(0,1),
\]
Theorem 1.1 strongly suggests that
\[
\max_{1 \leq j \leq n} \frac{S(j)}{j^\alpha} - \frac{S(n)}{n^\alpha} = o_P(n^{1/2-\alpha}),
\]
i.e. $S(j)/j^\alpha$ reaches its largest value on $[1,n]$ nearly at $j = n$. Indeed, Chow and Hsiung \cite{1} proved the following result:

**Theorem 1.2.** If (1.1) and (1.2) hold, then
\[
\max_{1 \leq j \leq n} \frac{S(j)}{j^\alpha} - \frac{S(n)}{n^\alpha} = o(n^{1/2-\alpha}) \quad \text{a.s.} \quad (1.3)
\]

For generalizations of (1.3) we refer to Chow, Hsiung and Yu \cite{2}.

We show that (1.3) holds not only for partial sums of independent identically distributed random variables, but for any process if they can be approximated with a Wiener process with drift. Let $\Gamma(t)$ be a stochastic process on $\mathbb{D}[1,\infty)$.

**Theorem 1.3.** We assume that there exist a Wiener process $\{W(t), 1 \leq t < \infty\}$ and constants $\tau > 0$, $\gamma > 0$ such that
\[
\Gamma(t) - (\tau W(t) + \gamma t) = o(t^{1/\nu}) \quad \text{a.s.} \quad (t \to \infty) \quad (1.4)
\]
with some $\nu > 2$. If (1.2) holds, then
\[
\sup_{1 \leq t \leq T} \frac{\Gamma(t)}{t^\alpha} - \frac{\Gamma(T)}{T^\alpha} = o(T^{1/\nu-\alpha}) \quad \text{a.s.} \quad (T \to \infty) \quad (1.5)
\]
and
\[
\sup_{1 \leq t \leq T} \frac{\Gamma(t)}{t^\alpha} - \frac{\tau W(T) + \gamma T}{T^\alpha} = o(T^{1/\nu-\alpha}) \quad \text{a.s.} \quad (T \to \infty). \quad (1.6)
\]

Theorem 1.3 implies immediately an improvement of the rate in (1.3) under stronger moment conditions on $X_1$.

**Theorem 1.4.** If (1.1), (1.2) hold and
\[
E|X_1|^\nu < \infty \quad \text{with some } \nu > 2, \quad (1.7)
\]
then
\[
\max_{1 \leq j \leq n} \frac{S(j)}{j^\alpha} - \frac{S(n)}{n^\alpha} = o(n^{1/\nu-\alpha}) \quad \text{a.s.} \quad (n \to \infty). \quad (1.8)
\]
Theorems 1.3 and 1.4 will be proven in the next section. The following two corollaries are immediate consequences of (1.6) and the properties of the Wiener process. Let \([\cdot]\) denote the integer part function.

**Corollary 1.1.** We assume that the conditions of Theorem 1.3 are satisfied.

(i) If \(0 < \alpha < 1/2\), then
\[
\sup_{1 \leq t \leq \lceil nu \rceil + 1} \frac{\Gamma(t)}{t^\alpha} - \gamma([nu] + 1)^{1-\alpha} \xrightarrow{\mathcal{P}[0,1]} \frac{\tau W(u)}{u^\alpha}.
\]

(ii) If \(1/2 < \alpha < 1\), then
\[
\sup_{1 \leq t \leq \lceil nu \rceil + 1} \frac{\Gamma(t)}{t^\alpha} - \gamma([nu] + 1)^{1-\alpha} \xrightarrow{\mathcal{P}[1,\infty]} \frac{\tau W(u)}{u^\alpha}.
\]

(iii) For any \(0 < c_1 < c_2 < \infty\)
\[
\sup_{1 \leq t \leq \lceil nu \rceil + 1} \frac{\Gamma(t)}{t^\alpha} - \gamma([nu] + 1)^{1-\alpha} \xrightarrow{\mathcal{P}[c_1,c_2]} \frac{\tau W(u)}{u^\alpha}.
\]

**Corollary 1.2.** If the conditions of Theorem 1.3 are satisfied, then
\[
\limsup_{T \to \infty} \frac{T^\alpha}{(2T \log \log T)^{1/2}} \sup_{1 \leq t \leq T} \frac{\Gamma(t)}{t^\alpha} - \gamma T^{1-\alpha} = \tau \quad \text{a.s.}
\]

2. PROOFS

The first two lemmas show that \(\Gamma(t)/t^\alpha\) and \((\tau W(t) + \gamma t)/t^\alpha\) will reach their largest value on \([1,T]\) on the second half of this interval.

**Lemma 2.1.** If (1.2) holds and \(\gamma > 0\), then there is a random variable \(T_1\) such that
\[
\sup_{1 \leq t \leq T} \frac{\tau W(t) + \gamma t}{t^\alpha} = \sup_{T/2 \leq t \leq T} \frac{\tau W(t) + \gamma t}{t^\alpha}, \text{ if } T \geq T_1.
\]

**Proof.** By the law of iterated logarithm for \(W\) we have
\[
\frac{1}{T^{1-\alpha}} \sup_{1 \leq t \leq T} \frac{\tau W(t) + \gamma t}{t^\alpha} \to \gamma \quad \text{a.s. } (T \to \infty)
\]
and
\[
\frac{1}{T^{1-\alpha}} \sup_{1 \leq t \leq \frac{T}{2}} \frac{\tau W(t) + \gamma t}{t^\alpha} \to \left(\frac{1}{2}\right)^{1-\alpha} \gamma \quad \text{a.s. } (T \to \infty),
\]

implying the statement of Lemma 2.1. \(\square\)
Lemma 2.2. If the conditions of Theorem 1.3 are satisfied, then there is a random variable $T_2$ such that

$$
\sup_{1 \leq t \leq T} \frac{\Gamma(t)}{t^\alpha} = \sup_{T/2 \leq t \leq T} \frac{\Gamma(t)}{t^\alpha}, \text{ if } t \geq T_2.
$$

Proof. The approximation in (1.4) implies that

$$
\sup_{1 \leq t \leq T} \frac{|\Gamma(t) - (\tau W(t) + \gamma t)|}{t^\alpha} = O(\max(1, T^{1/\nu - \alpha})) \quad \text{a.s.}
$$

and therefore (2.2) and (2.3) yield

$$
\frac{1}{T^{1-\alpha}} \sup_{1 \leq t \leq T} \frac{\Gamma(t)}{t^\alpha} \to \gamma \quad \text{a.s. } (t \to \infty)
$$

and

$$
\frac{1}{T^{1-\alpha}} \sup_{1 \leq t \leq T/2} \frac{\Gamma(t)}{t^\alpha} \to \left(\frac{1}{2}\right)^{1-\alpha} \gamma \quad \text{a.s. } (T \to \infty).
$$

Lemma 2.2 follows from (2.4) and (2.5). □

Let $F_0(t)$ be the uniform distribution function on $[0,1]$. For any $0 < \alpha < 1$, $F_\alpha(t)$ denotes the uniform distribution function on $[1,1/\alpha]$.

Lemma 2.3. Let $0 \leq \alpha < 1$ and $Y_1, Y_2, \ldots$ be independent, identically distributed random variables with distribution function $F_\alpha(t)$. Then

$$
\max_{1 \leq j \leq n} \frac{1}{j^\alpha} \sum_{1 \leq i \leq j} Y_i = \frac{1}{n^\alpha} \sum_{1 \leq i \leq n} Y_i.
$$

Proof. It is enough to show that

$$
\left(1 + \frac{1}{j}\right)^\alpha \sum_{1 \leq i \leq j} Y_i \leq \sum_{1 \leq i \leq j+1} Y_i \quad \text{for all } 1 \leq j < \infty.
$$

Since $Y_i \geq 0$, (2.6) holds if $\alpha = 0$. If $0 < \alpha < 1$, we observe that $1 \leq Y_i \leq 1/\alpha$ and

$$
\left(1 + \frac{1}{j}\right)^\alpha - 1 \leq \frac{\alpha}{j}.
$$

Hence

$$
\left\{ \left(1 + \frac{1}{j}\right)^\alpha - 1 \right\} \sum_{1 \leq i \leq j} Y_i \leq \frac{\alpha}{j} \sum_{1 \leq i \leq j} Y_i \leq 1 \leq Y_{j+1},
$$

completing the proof of (2.6). □
Lemma 2.4. If (1.2) holds and $\tau > 0$, $\gamma > 0$, then

$$\sup_{1 \leq t \leq T} \frac{\tau W(t) + \gamma t}{t^\alpha} - \frac{\tau W(T) + \gamma T}{T^\alpha} = O\left(\frac{\log T}{T^\alpha}\right) \quad \text{a.s.}$$

Proof. Let $\mu_* = \mu_*(\alpha)$ and $\sigma_* = \sigma_*(\alpha)$ be the mean and standard deviation of a random variable with distribution function $F_\alpha(t)$. Next we define

$$c = \left(\frac{\mu_* \tau}{\gamma \sigma_*}\right)^2. \quad (2.7)$$

Obviously,

$$\sup_{1 \leq t \leq T} \frac{\tau W(t) + \gamma t}{t^\alpha} = \tau \sup_{1/c \leq s \leq T/c} \frac{W(cs) + \gamma cs}{(cs)^\alpha} \quad (2.8)$$

$$= \tau c^{1/2-\alpha} \sup_{1/c \leq s \leq T/c} \frac{W_1(s) + \frac{\gamma}{\tau} c^{1/2}s}{s^\alpha},$$

where

$$W_1(s) = c^{-1/2}W(cs), \quad 0 \leq s < \infty \quad (2.9)$$

is a Wiener process. By (2.7) and (2.8) we have

$$\sup_{1 \leq t \leq T} \frac{\tau W(t) + \gamma t}{t^\alpha} = \tau \sup_{1/c \leq t \leq T/c} \frac{\sigma_* W_1(t) + \mu_* t}{t^\alpha}. \quad (2.10)$$

Using the K–M–T approximation (cf. Komlós, Major and Tusnády [3, 4] and Major [5]) we can define $Y_1^*, Y_2^*, \ldots$, a sequence of independent, identically distributed random variables with distribution function $F_\alpha(t)$ such that

$$\sum_{1 \leq i \leq t} Y_i^* - (\sigma_* W_1(t) + \mu_* t) = O(\log t) \quad \text{a.s.} \quad (t \to \infty). \quad (2.11)$$

By Lemmas 2.1, 2.2 and (2.10) there is random variable $T_0$ such that

$$\sup_{1/c \leq t \leq T/c} \frac{\sigma_* W_1(t) + \mu_* t}{t^\alpha} = \sup_{T/(2c) \leq t \leq T/c} \frac{\sigma_* W_1(t) + \mu_* t}{t^\alpha} \quad (2.10)$$

and

$$\sup_{1/c \leq t \leq T/c} \frac{1}{t^\alpha} \sum_{1 \leq i \leq t} Y_i^* = \sup_{T/(2c) \leq t \leq T/c} \frac{1}{t^\alpha} \sum_{1 \leq i \leq t} Y_i^*,$$

if $T \geq T_0$. Hence (2.11) yields, as $T \to \infty$,

$$\sup_{1/c \leq t \leq T/c} \frac{\sigma_* W_1(t) + \mu_* t}{t^\alpha} - \sup_{1/c \leq t \leq T/c} \frac{1}{t^\alpha} \sum_{1 \leq i \leq t} Y_i^* = O(T^{-\alpha} \log T) \quad \text{a.s.} \quad (2.12)$$
Putting together Lemma 2.3 and (2.11) we conclude

$$\sup_{1/\varepsilon \leq t \leq T/c} \frac{1}{t^\alpha} \sum_{1 \leq i \leq t} Y_i^* = \left( \frac{T}{c} \right)^{-\alpha} \sum_{1 \leq i \leq T/c} Y_i^* = \left( \frac{T}{c} \right)^{-\alpha} \left\{ \sigma W_1 \left( \frac{T}{c} \right) + \mu \frac{T}{c} \right\} + O(T^{-\alpha} \log T) \text{ a.s.}$$

(2.13)

\((T \to \infty)\). Next we use (2.7), (2.9) and (2.10) to obtain

$$\left( \frac{T}{c} \right)^{-\alpha} \left\{ \sigma W_1 \left( \frac{T}{c} \right) + \mu \frac{T}{c} \right\}$$

$$= \left( \frac{T}{c} \right)^{-\alpha} \left\{ \sigma c^{-1/2} W(T) + \mu \frac{T}{c} \right\}$$

$$= \frac{1}{T^\alpha} c^{\alpha - 1/2} \frac{\sigma}{\sigma_*} \left\{ W(T) + \frac{\mu_*}{\sigma_*} c^{-1/2} T \right\}$$

$$= \frac{1}{T^\alpha} c^{\alpha - 1/2} \frac{\sigma}{\tau} \left\{ \tau W(T) + \gamma T \right\}.$$

(2.14)

Lemma 2.4 now follows from (2.8) and (2.12) – (2.14).

Proof of Theorem 1.3. Using (1.4) and Lemmas 2.1 and 2.2 we get that

$$\sup_{0 \leq t \leq T} \frac{\Gamma(t)}{t^\alpha} - \sup_{1 \leq t \leq T} \frac{\tau W(t) + \gamma t}{t^\alpha} = o(T^{1/\nu - \alpha}) \text{ a.s.}$$

Hence Theorem 1.3 follows from Lemma 2.4.

Proof of Theorem 1.4. By the K–M–T approximation there is a Wiener process \(\{W(t), 0 \leq t < \infty\}\) such that

$$S(t) - (\sigma W(t) + \mu t) = o(t^{1/\nu}) \text{ a.s. (} t \to \infty).$$

Hence (1.4) holds and the result follows from Theorem 1.3.

Proof of Corollary 1.1. Assume that \(0 \leq \alpha < 1/2\). By Theorem 1.3 there is a Wiener process \(\{W(t), 0 \leq t < \infty\}\) such that

$$n^{\alpha - 1/2} \sup_{0 \leq u \leq 1} \left| \frac{\Gamma(t)}{t^\alpha} - \frac{\tau W([nu] + 1) + \gamma([nu] + 1)}{([nu] + 1)^\alpha} \right| = o(n^{1/\nu - 1/2}) \text{ a.s.}$$

Hence (1.9) is proven if

$$n^{\alpha - 1/2} \frac{W([nu] + 1)}{([nu] + 1)^\alpha} \overset{D[0,1]}{\to} \frac{W(u)}{u^\alpha}.$$

(2.15)
Obviously,

\[
\sup_{0 \leq u \leq \epsilon \leq u \leq [n\epsilon] + 1} \frac{|W([nu] + 1)|}{([nu] + 1)^\alpha} \leq \sup_{0 \leq u \leq [n\epsilon] + 1} \frac{|W(u)|}{u^\alpha}
\]

and by the scale transformation of \( W \) we have

\[
n^{\alpha-1/2} \sup_{0 \leq u \leq [n\epsilon] + 1} \frac{|W(u)|}{u^\alpha} \overset{\mathcal{D}}{=} \sup_{0 \leq u \leq [n\epsilon] + 1} \frac{|W(u/n)|}{(u/n)^\alpha} = \sup_{0 \leq u \leq ((n\epsilon) + 1)/n} \frac{|W(u)|}{u^\alpha}.
\]

By the law of the iterated logarithm for \( W \) at 0 we have

\[
\lim_{\epsilon \to 0} \lim_{n \to \infty} \sup \left\{ \sup_{0 \leq u \leq [n\epsilon] + 1} \frac{|W(u)|}{u^\alpha} > \delta \right\} = 0 \text{ for all } \delta > 0.
\] (2.16)

The scale transformation of \( W \) and the almost sure continuity of \( W(u)/u^\alpha \) on \([c_1, c_2], 0 < c_1 \leq c_2 \) yield

\[
n^{\alpha-1/2} \frac{|W([nu] + 1)|}{([nu] + 1)^\alpha} \overset{\mathcal{D}[c_1, c_2]}{=} \frac{W(u)}{u^\alpha}.
\] (2.17)

Clearly, (2.15) follows from (2.16) and (2.17).

Assume that \( 1/2 < \alpha < 1 \). Using again Theorem 1.3 there is a Wiener process \( \{W(t), 0 < t < \infty\} \) such that

\[
n^{\alpha-1/2} \sup_{1 \leq u < \infty} \left| \sup_{1 \leq t \leq [nu] + 1} \frac{\Gamma(t)}{t^\alpha} - \frac{\tau W([nu] + 1) + \gamma([nu] + 1)}{([nu] + 1)^\alpha} \right| = o(1) \text{ a.s.}
\]

Hence (1.10) is proven if we show that

\[
n^{\alpha-1/2} \frac{|W([nu] + 1)|}{([nu] + 1)^\alpha} \overset{\mathcal{D}[1, \infty]}{=} \frac{W(u)}{u^\alpha}.
\] (2.18)

For any \( T > 0 \) we have that

\[
\sup_{T \leq u < \infty} \frac{|W([nu] + 1)|}{([nu] + 1)^\alpha} \leq \sup_{[nT] \leq u < \infty} \frac{|W(u)|}{u^\alpha}
\]

and by the scale transformation of \( W \) we have

\[
n^{\alpha-1/2} \sup_{[nT] \leq u < \infty} \frac{|W(u)|}{u^\alpha} \overset{\mathcal{D}}{=} \sup_{[nT]/n \leq u < \infty} \frac{|W(u)|}{u^\alpha}.
\]

The law of the iterated logarithm for \( W \) at \( \infty \) yields that

\[
\sup_{T \leq u < \infty} \frac{|W(u)|}{u^\alpha} \to 0 \text{ a.s. } (T \to \infty).
\] (2.19)

Now (2.18) follows from (2.17) and (2.19).

Theorem 1.3 and (2.17) imply immediately (1.11). \( \square \)

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REFERENCES


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