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S-IMPLICATIONS AND R-IMPLICATIONS ON A FINITE CHAIN

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This paper is devoted to the study of two kinds of implications on a finite chain L: S-implications and R-implications. A characterization of each kind of these operators is given and a lot of different implications on L are obtained, not only from smooth t-norms but also from non smooth ones. Some additional properties on these implications are studied specially in the smooth case. Finally, a class of non smooth t-norms including the nilpotent minimum is characterized. Any t-norm in this class satisfies that both, its S-implication and its R-implication, agree.

Keywords: t-norm, t-conorm, finite chain, smoothness, implication operator
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1. INTRODUCTION

In fuzzy logic the most usual connectives to model conjunctions, disjunctions and negations are t-norms (T), t-conorms (S) and strong negations (N), respectively. Following this structure, the implication is performed by the so called *implication operators* or simply *implicators.* These operators are generally defined, from the basic ones T, S and N, through several ways obtaining different kinds of implication operators. The two most commonly used being,

- S-implications based on classical logic:

$$I_1(x,y) = S(N(x),y) \qquad \text{for all} \quad x,y \in [0,1]. \tag{1}$$

- R-implications based on the idea of residuation:

$$I_2(x,y) = \sup\{z \in [0,1] \mid T(x,z) \leq y\} \qquad \text{for all} \quad x,y \in [0,1]. \tag{2}$$

Many authors have studied these kinds of connectives from several points of view (see [1, 4, 6, 11, 12, 23, 24]). Recently, even some implications defined from uninorms, operators that are a generalization of t-norms and t-conorms, have been studied (see [2] and [3]).

On the other hand, the study of operators defined on a finite chain L is an area of special interest (see [5, 13, 14, 18, 19, 22]), mainly because the expert's reasonings are usually made through a set of linguistic terms or labels which usually is a finite totally ordered set L. This approach is important because numerical interpretations of these labels can be avoided. Frequently, most of the authors which work in this line try to translate well known operators on [0,1] (like t-norms and t-conorms) to the case of a finite chain L. Following this idea, a lot of different classes of operators on L are appearing. In particular, smooth t-norms and t-conorms are classified in [22], t-operators and uninorms on L with a smooth condition are characterized in [18] and non-commutative versions can be found in [13] and [19].

However, a similar study for implicators on L has not been made and only some initial ideas were introduced by the same authors in [20] and [21]. The main goal of this paper is to study two kinds of implications on L following the mentioned ideas, namely those defined from t-norms and t-conorms on L through expressions (1) and (2). From this study, both kinds of implications are characterized, several additional properties are considered in both cases and a lot of implications on L are obtained and their expressions are pointed out. It is proved that both kinds of implications agree for exactly one smooth t-norm: the Archimedean one. The last section is devoted to the case of non smooth t-norms. In this section we characterize a special kind of non smooth t-norms that includes the nilpotent minimum. Moreover, any t-norm in this class satisfies that both, its R-implication and its S-implication, agree.

2. PRELIMINARIES

We recall here the smooth t-norms and t-conorms on L, and their characterization, that will be used along the paper. From now on, consider the finite chain

$$L = \{0 = x_0 < x_1 < \ldots < x_n < x_{n+1} = 1\}$$

where $n \geq 1$. Such an L can be understood as a set of linguistic terms or "labels".

Let us also denote by $[x_i, x_j]$ the finite chain given by the subinterval of all $x_k \in L$ such that $i \leq k \leq j$.

The following two definitions are adapted from [14].

Definition 1. A function $f : L \to L$ is said to be *smooth* if it satisfies the following condition for all $i \geq 1$:

$f(x_i) = x_j$ implies that $f(x_{i-1}) = x_k$ where k is such that $j - 1 \leq k \leq j + 1$.

Definition 2. A binary operator F on L is said to be *smooth* if it is smooth in each place.

Although t-norms, t-conorms and strong negations are usually operators on [0,1], they can be defined as in [1] or [5] on any partially ordered set and, in particular, on L. Thus, we maintain the names of t-norm, t-conorm and strong negation for operators on L with the same corresponding properties. In this way, we have the following results:

Proposition 1. There is only one strong negation on L and it is given by

$$N(x_i) = x_{n+1-i} \qquad \text{for all} \quad x_i \in L \tag{3}$$

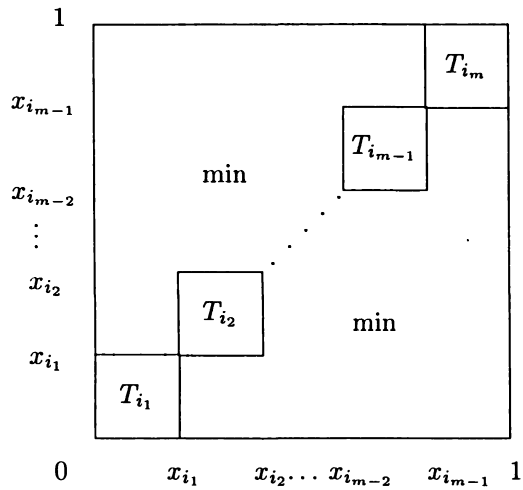

Fig. 1. Structure of smooth t-norms, where
$T_{i_{k+1}}(x_i, x_j) = x_{\max\{i_k, i+j-i_{k+1}\}}$ for $k = 0, ..., m-1$.

Proposition 2. (See [22].) There is one and only one Archimedean smooth t-norm on L given by

$$T(x_i, x_j) = x_{\max\{0, i+j-(n+1)\}}. \tag{4}$$

Moreover, given any subset J of L containing $0, 1$, there is one and only one smooth t-norm on L that has J as the set of idempotent elements. In fact, if J is the set

$$J = \{0 = x_{i_0} < x_{i_1} < \ldots < x_{i_{m-1}} < x_{i_m} = 1\}$$

such a t-norm is given by

$$T(x_i, x_j) = \begin{cases} x_{\max\{i_k, i+j-i_{k+1}\}} & \text{if there is an idempotent } x_{i_k} \in J \\ & \text{such that } x_{i_k} \leq x_i, x_j \leq x_{i_{k+1}} \\ \min\{x_i, x_j\} & \text{otherwise.} \end{cases} \tag{5}$$

Although we do not deal specifically with BL-algebras, let us note that in this context, a generalization of the previous classification theorem has been proved for BL-chains in [16] and [8]. The general structure of smooth t-norms stated in the previous proposition can be viewed in Figure 1.

Smooth t-conorms have a classification theorem like the above one for t-norms which can be easily deduced by N-duality where N is the only strong negation on L given by (3). The following result follows immediately from the proposition above

Proposition 3. (See [22].) There are exactly 2^n different smooth t-norms on L.

Definition 3. A binary operator $I : L \times L \to L$ is said to be an implication operator, or an implication, if it satisfies:

- I is nonincreasing in the first place and nondecreasing in the second one. That is, if $x_i \leq x_j$ then

$$I(x_i, x_k) \geq I(x_j, x_k) \quad \text{for all } x_k \in L$$

and

$$I(x_k, x_i) \leq I(x_k, x_j) \quad \text{for all } x_k \in L$$

- $I(0,0) = I(1,1) = 1 \quad \text{and} \quad I(1,0) = 0.$

From the definition it follows that $I(x_i, 1) = 1$ and $I(0, x_i) = 1$ for all $x_i \in L$ and so the restriction of I to $\{0,1\}^2$ agrees with the classical implication. On the contrary, the symmetrical values $I(1, x_i)$ are not determined in general.

Definition 4. An implication $I : L \times L \to L$ is called a border implication if it satisfies $I(1, x_i) = x_i$ for all $x_i \in L$.

3. IMPLICATION FUNCTIONS

Since we will work with a finite chain L it is clear that expressions (1) and (2) can be rewritten in our case as follows:

$$I_{1T}(x_i, x_j) = N(T(x_i, N(x_j))) \qquad \text{for all} \quad x_i, x_j \in L \tag{6}$$

and

$$I_{2T}(x_i, x_j) = \max\{x_k \in L \mid T(x_i, x_k) \leq x_j\} \qquad \text{for all} \quad x_i, x_j \in L. \tag{7}$$

Thus, from any given t-norm T on L we can define the operators I_{1T} and I_{2T} that turn out to be border implications as the following proposition shows.

Proposition 4. Given any t-norm T, I_{1T} and I_{2T} are border implications.

Proof. The corresponding proof given in [1] applies here for the case of I_{1T}. With respect to the case of I_{2T}, all conditions follow trivially from the definition and some well known properties of t-norms. □

There are many other properties that are required on implication functions depending on the context, the most usual ones being:

P1) *Exchange principle,*

$$I(a, I(b,c)) = I(b, I(a,c)) \qquad \text{for all} \quad a,b,c \quad \text{in the domain.}$$

P2) *Contrapositive symmetry* with respect to a strong negation N,

$$I(a,b) = I(N(b), N(a)) \qquad \text{for all} \quad a,b \quad \text{in the domain.}$$

P3) $I(a, a) = 1$ for all a in the domain.

P4) $I(a, b) = 1$ if and only if $a \leq b$.

P5) $I(a, 0) = N(a)$ to be a strong negation.

P6) $I(a, b) \geq b$ for all a, b in the domain.

P7) *Generalized modus ponens*, with respect to a t-norm T:

$$T(a, I(a, b)) \leq b \qquad \text{for all} \quad a, b.$$

P8) $I(a, N(a)) = N(a)$ for all a in the domain.

All the properties above will be studied for both kinds of implications (6) and (7) derived from smooth t-norms. Also, some ones of these properties will allow us to characterize both kinds of implications in a similar way as it is done in the case of [0,1].

3.1. S-implications

Given any t-norm T on L, it is obvious from expression (6) that the corresponding implication I_{1T} always satisfies properties P5) and P6). With respect to properties P1) and P2) we have the following characterization which holds in the more general framework of partially ordered sets:

Theorem 1. (See [1].) Let $I : L \times L \to L$ be a function. Then I is a border implication satisfying P1) and P2) if and only if there is a t-norm T on L such that $I = I_{1T}$.

The following example is specially interesting because of their properties, that we will see in next results.

Example 1. Let T be the only Archimedean smooth t-norm on L given by (4). Then I_{1T} is given by

$$I_{1T}(x_i, x_j) = x_{\min\{n+1, n+1+j-i\}}, \tag{8}$$

expression that we will call the Łukasiewicz implication since it reminds this implication on $[0, 1]$.

Proposition 2 allows us to obtain 2^n different implications on L from the corresponding smooth t-norms through expression (6), but many others can be derived also from non smooth t-norms as we will see in the next section. The expression of the implications I_{1T} derived from smooth t-norms is given in the next proposition.

Proposition 5. Let $T : L \times L \to L$ be a smooth t-norm with the following set of idempotent elements

$$J = \{0 = x_{i_0} < x_{i_1} < \ldots < x_{i_{m-1}} < x_{i_m} = 1\}.$$

Then the implication I_{1T} is given by

$$I_{1T}(x_i, x_j) = \begin{cases} x_{\min\{n+1-i_k, i_{k+1}+j-i\}} & \text{if there is } x_{i_k} \in J \text{ such that} \\ & x_{i_k} \leq x_i, x_{n+1-j} \leq x_{i_{k+1}} \\ \max\{x_{n+1-i}, x_j\} & \text{otherwise.} \end{cases}$$

Proof. Let us suppose first that there is $x_{i_k} \in J$ such that $x_{i_k} \leq x_i, x_{n+1-j} \leq x_{i_{k+1}}$. Then,

$$I_{1T}(x_i, x_j) = N(x_{\max\{i_k, i+n+1-j-i_{k+1}\}}) = x_{\min\{n+1-i_k, i_{k+1}+j-i\}}.$$

Otherwise, we have

$$I_{1T}(x_i, x_j) = N(\min\{x_i, x_{n+1-j}\}) = \max\{x_{n+1-i}, x_j\}. \qquad \square$$

The structure of the S-implications can be viewed in Figure 2.

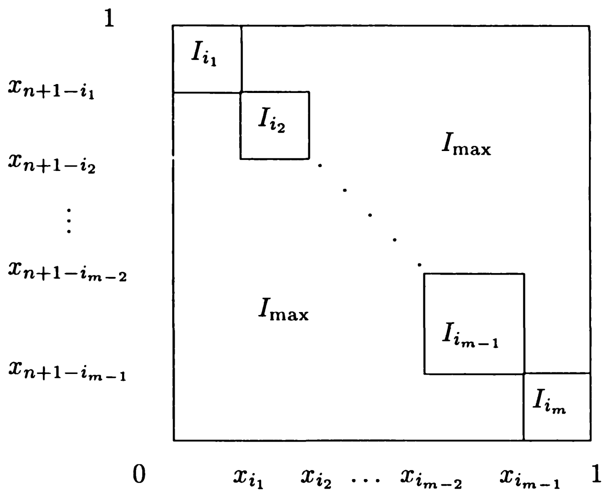

Fig. 2. Structure of S-implications, where $I_{\max}(x_i, x_j) = \max\{x_{n+1-i}, x_j\}$ and $I_{i_{k+1}}(x_i, x_j) = x_{\min\{n+1-i_k, i_{k+1}+j-i\}}$ for $k = 0, ..., m-1$.

In order to see which properties satisfy these implications let us begin with the following lemma.

Lemma 1. Let T be a smooth t-norm on L. The following statements are equivalent:

i) T is the Archimedean t-norm given by (4).

ii) $T(x_i, N(x_i)) = 0$ for all $x_i \in L$.

iii) There exists $0 < i < n+1$ such that $T(x_i, N(x_i)) = 0$.

Proof. i) $\Longrightarrow$ ii) and ii) $\Longrightarrow$ iii) are clear.

iii) $\Longrightarrow$ i) Suppose on the contrary that T is not Archimedean and let us take x_j the least idempotent element of T different from 0, 1. Then we necessarily have $x_j > \min\{x_i, N(x_i)\}$ and,

- If $x_j < \max\{x_i, N(x_i)\}$, we have from (5),

$$T(x_i, N(x_i)) = \min\{x_i, N(x_i)\} \neq 0$$

obtaining a contradiction.

- If $\max\{x_i, N(x_i)\} \leq x_j$, since x_j is the least idempotent different from 0, we have again from (5),

$$T(x_i, N(x_i)) = T(x_i, x_{n+1-i}) = x_{\max\{0, i+n+1-i-j\}} = x_{n+1-j} \neq 0$$

obtaining also a contradiction.

Thus T must be Archimedean and consequently it is given by (4). □

Proposition 6. Let T be a smooth t-norm on L. The following statements are equivalent:

i) T is the Archimedean t-norm given by (4).

ii) I_{1T} satisfies P4).

iii) I_{1T} satisfies P3).

Proof. Again i) $\Longrightarrow$ ii) and ii) $\Longrightarrow$ iii) are trivial. With respect to iii) $\Rightarrow$ i), note that $I_{1T}(x_i, x_i) = 1$ for all $x_i \in L$ if and only if $T(x_i, N(x_i)) = 0$ for all $x_i \in L$ and then Lemma 1 ends the proof. □

Another interesting property is P8), extensively studied on [0,1] in [6]. In our case we have:

Proposition 7. Let T be any t-norm on L. Then I_{1T} satisfies P8) if and only if $T = \min$. That is, when I_{1T} is the so called Kleene–Dienes implication

$$I_{1T}(x_i, x_j) = \max\{x_{n+1-i}, x_j\}.$$

Proof. $I_{1T}(x_i, N(x_i)) = N(x_i) \Longleftrightarrow N(T(x_i, x_i)) = N(x_i) \Longleftrightarrow T(x_i, x_i) = x_i$, for all $x_i \in L$, and this happens if and only if $T = \min$. □

With respect to the generalized modus ponens we have:

Proposition 8. Let T be a smooth t-norm on L. Then I_{1T} satisfies P7) if and only if T is the Archimedean t-norm given by (4).

Proof. It is clear tl.at I_{1T} satisfies P7) when T is given by (4). Conversely, just take $b = 0$ and $a = x_i$ in property P7) to obtain $T(x_i, N(x_i)) = 0$ for all $x_i \in L$ and then apply Lemma 1. □

Finally, with respect to the smoothness condition we have:

Proposition 9. Let T be any t-norm on L. Then the implication I_{1T} is smooth if and only if so is T.

Proof. Note that, for any t-norm T on L, we have

$$I_{1T}(x_i, x_j) = x_k \Longleftrightarrow N(T(x_i, x_{n+1-j})) = x_k \Longleftrightarrow T(x_i, x_{n+1-j}) = x_{n+1-k}$$

and from this equivalence the proposition follows trivially. □

3.2. R-implications

It is obvious from the definition that all implications obtained by residuation from expression (7) satisfy property P6) as well as property P4) and consequently, also P3). Since they satisfy P4) they can never satisfy P8) (the same proof given in [6] for [0,1] works here). Moreover, from expression (7) it is obvious that they also satisfy the generalized modus ponens. On the other hand, they also satisfy P1), in fact we have the following characterization of these implications:

Theorem 2. Let $I : L \times L \to L$ be a function. Then I is a border implication satisfying P1) and P4) if and only if there is a t-norm T on L such that $I = I_{2T}$.

Proof. If there is a t-norm T on L such that $I = I_{2T}$, we already know that I is a border implication and clearly satisfies P4). With respect to the exchange principle, let us prove first that

$$I_{2T}(x_i, I_{2T}(x_j, x_k)) = I_{2T}(T(x_i, x_j), x_k). \tag{9}$$

To do this, it suffices to prove that the sets A and B given by

$$A = \{x_l \in L \mid T(x_i, x_l) \leq I_{2T}(x_j, x_k)\}$$

and

$$B = \{x_l \in L \mid T(T(x_i, x_j), x_l) \leq x_k\}$$

agree. However, from the definition of I_{2T} it is obvious that an element $x_l \in L$ satisfies

$$T(x_i, x_l) \leq I_{2T}(x_j, x_k)$$

if and only if it satisfies

$$T(x_j, T(x_i, x_l)) \leq x_k$$

and consequently we have $A = B$. Now, the exchange principle follows from equation (9) and the commutativity of T.

Conversely, suppose that I is a border implication satisfying P1) and P4) and let us define $T : L \times L \to L$ as follows:

$$T(x_i, x_j) = \min\{x_k \in L \mid I(x_i, x_k) \geq x_j\}$$

It is easy to see that such T is nondecreasing in each place and has $x_{n+1} = 1$ as neutral element. To prove that T is a t-norm it remains only commutativity and associativity:

- To see commutativity we only need to prove the following equality:

$$\{x_k \in L \mid I(x_i, x_k) \geq x_j\} = \{x_k \in L \mid I(x_j, x_k) \geq x_i\}.$$

Note however that

$$x_j \leq I(x_i, x_k) \Longleftrightarrow I(x_j, I(x_i, x_k)) = 1 \Longleftrightarrow I(x_i, I(x_j, x_k)) = 1$$

by property P1). Finally, we have

$$I(x_i, I(x_j, x_k)) = 1 \Longleftrightarrow x_i \leq I(x_j, x_k)$$

and thus the two considered sets agree.

- To see associativity, using the equality

$$T(T(x_i, x_j), x_k) = T(x_k, T(x_i, x_j)),$$

it suffices to show that sets A and B given by

$$A = \{x_l \in L \mid I(x_k, x_l) \geq T(x_i, x_j)\}$$

and

$$B = \{x_l \in L \mid I(x_i, x_l) \geq T(x_j, x_k)\}$$

agree. Note that from the definition of T we can deduce that

$$I(x_i, T(x_i, x_j)) \geq x_j \tag{10}$$

and

$$T(x_i, I(x_i, x_j)) \leq x_j. \tag{11}$$

Thus, when $x_l \in A$ we have $I(x_k, x_l) \geq T(x_i, x_j)$ and consequently

$$I(x_i, I(x_k, x_l)) \geq I(x_i, T(x_i, x_j)).$$

Now, by the exchange principle and inequality (10), $I(x_k, I(x_i, x_l)) \geq x_j$ and then

$$T(x_j, x_k) \leq T(I(x_k, I(x_i, x_l)), x_k) = T(x_k, I(x_k, I(x_i, x_l)) \leq I(x_i, x_l)$$

where the last inequality is due to (11). These reasonings prove the inclusion $A \subseteq B$, and the other one follows similarly.

We have proved that the defined T is a t-norm and from its definition it follows trivially that $I = I_{2T}$. □

Remark 1. For this kind of implications we have, like in the case of [0,1], that

$$T(x_i, x_j) \leq x_k \Longleftrightarrow I_{2T}(x_i, x_k) \geq x_j.$$

Note that, since R-implications satisfy property P4) we obtain

$$\max\{I_{2T}(x_i, x_j), I_{2T}(x_j, x_i)\} = 1 \quad \text{for all} \quad x_i, x_j \in L.$$

This fact, jointly with the previous remark, ensures that for any t-norm T on L, $(L, \min, \max, T, I_{2T}, 0, 1)$ is an MTL-algebra (see [10]). Moreover, when we deal with smooth t-norms the divisibility condition ($x \leq y$ implies that there is $z \in L$ such that $T(y, z) = x$) holds (see [22] or [13]), and consequently we actually have a BL-algebra (see [15] for a basic reference on BL-algebras).

A similar result of the above one but in [0,1] can be found in [4] where an additional hypothesis on continuity is needed. However, for this kind of implications, contrapositive symmetry fails in general. In this way we have the following result.

Proposition 10. Let T be a smooth t-norm on L. The following statements are equivalent:

i) T is the Archimedean t-norm given by (4).

ii) The implication functions I_{1T} and I_{2T} agree.

iii) I_{2T} satisfies contrapositive symmetry with respect to N.

Proof. i) $\Longrightarrow$ ii). If T is given by (4), a straightforward computation shows that I_{2T} is given by expression (8) and consequently agrees with I_{1T}.

ii) $\Longrightarrow$ iii). If $I_{2T} = I_{1T}$ then clearly I_{2T} satisfies contrapositive symmetry by Theorem 1.

iii) $\Longrightarrow$ i). If I_{2T} satisfies contrapositive symmetry, let us prove that

$$I_{2T}(x_i, x_j) = N(T(x_i, N(x_j))) \quad \text{for all } x_i, x_j \in L. \tag{12}$$

Suppose that $I_{2T}(x_i, x_j) = x_k$, then from Remark 1 above we have $T(x_i, x_k) = T(x_k, x_i) \leq x_j$ and consequently $I_{2T}(x_k, x_j) \geq x_i$. Now, by contrapositive symmetry

$$I_{2T}(N(x_j), N(x_k)) \geq x_i,$$

and then $T(N(x_j), x_i) \leq N(x_k)$ or equivalently

$$x_k = I_{2T}(x_i, x_j) \leq N(T(x_i, N(x_j))).$$

This proves one inequality of (12) and the other follows similarly. Finally, this equation shows that $I_{2T} = I_{1T}$ but then I_{1T} satisfies P4) and Proposition 6 proves that T must be given by (4). □

As for the remaining properties we have:

Proposition 11. Let T be a smooth t-norm, then I_{2T} satisfies P5) if and only if T is given by (4).

Proof. Just note that I_{2T} satisfies P5) if and only if $T(x_i, N(x_i)) = 0$ and then apply Lemma 1. □

In the context of BL-algebras, property P5) is widely studied. In fact, BL-algebras satisfying that the negation induced by their residual implication is involutive, that is, a strong negation, are usually called MV-algebras (see [9]). Thus, given any smooth t-norm T, the BL-algebra $(L, \min, \max, T, I_{2T}, 0, 1)$ becomes an MV-algebra if and only if T is the t-norm given by (4).

For R-implications, the smoothness condition is not satisfied in general as it is proved in the following proposition.

Proposition 12. Let T be a smooth t-norm. Then I_{2T} is smooth if and only if T is given by (4).

Proof. If T is given by (4), we have $I_{2T} = I_{1T}$ by Proposition 10, and then Proposition 9 proves that I_{2T} is smooth. Conversely, since I_{2T} satisfies P4) we have $I_{2T}(x_1, x_1) = 1$ and $I_{2T}(x_1, x_0) < 1$, but smoothness implies that $I_{2T}(x_1, x_0) = x_n$. Consequently, $T(x_1, x_n) = 0$ and so Lemma 1 ends the proof. □

Note that each smooth t-norm defines through expression (7) a new implication operator on L which general expression can be viewed in the following proposition:

Proposition 13. Let $T : L \times L \to L$ be a smooth t-norm with the following set of idempotent elements

$$J = \{0 = x_{i_0} < x_{i_1} < \ldots < x_{i_{m-1}} < x_{i_m} = 1\}.$$

Then the implication I_{2T} is given by

$$I_{2T}(x_i, x_j) = \begin{cases} 1 & \text{if } x_i \leq x_j \\ x_{i_{k+1}+j-i} & \text{if there is } x_{i_k} \in J \text{ such that } x_{i_k} \leq x_j < x_i \leq x_{i_{k+1}} \\ x_j & \text{otherwise.} \end{cases}$$

Proof. It is clear from property P4) that $I_{2T}(x_i, x_j) = 1$ if $x_i \leq x_j$. On the other hand, when $x_i > x_j$, let us distinguish two cases:

- If there is $x_{i_k} \in J$ such that $x_{i_k} \leq x_j < x_i \leq x_{i_{k+1}}$, then

$$T(x_i, x_{i_{k+1}+j-i}) = x_{\max\{i_k, i+i_{k+1}+j-i-i_{k+1}\}} = x_{\max\{i_k, j\}} = x_j$$

  whereas for any value $k > i_{k+1} + j - i$ we obtain similarly $T(x_i, x_k) > x_j$. Thus, $I_{2T}(x_i, x_j) = x_{i_{k+1}+j-i}$.

- In any other case we have $T(x_i, x_j) = \min\{x_i, x_j\} = x_j$ whereas $T(x_i, x_k) > x_j$ for any $k > j$ and consequently $I_{2T}(x_i, x_j) = x_j$. □

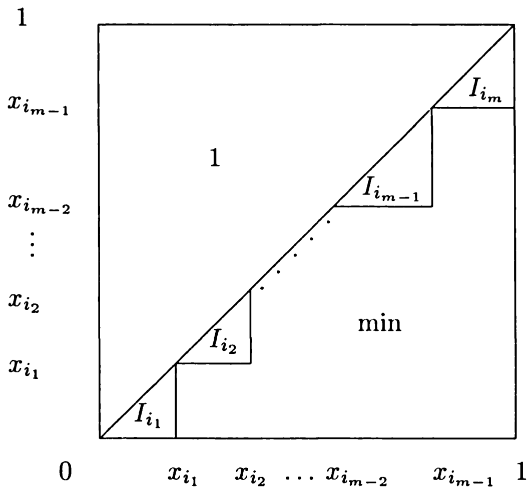

Fig. 3. Structure of R-implications, where
$I_{i_{k+1}}(x_i, x_j) = x_{i_{k+1}+j-i}$ for $k = 0, \ldots, m-1$.

The structure of the R-implications can be viewed in Figure 3.

Since all these implications are different of those given in Proposition 5 except for the case of the only Archimedean smooth t-norm, as it is proved in Proposition 10, we obtain the following result.

Proposition 14. There are exactly $2^{n+1} - 1$ different implications on L obtained through expressions (6) and (7) from smooth t-norms.

4. NON SMOOTH t-NORMS

We have seen in the section above that a lot of implications of the forms I_{1T} and I_{2T} can be derived from smooth t-norms. But, from Proposition 4, it is clear that the same can be made from non smooth ones. Let us give several examples showing that some well known implications on [0,1], translated to L, can be obtained in this way, whereas another ones can not.

Example 2. i) We have already proved that the Łukasiewicz implication can be obtained as I_{1T} as well as I_{2T} when T is the only Archimedean smooth t-norm.

ii) We know from Proposition 7 that the Kleene–Dienes implication equals $I_{1\,\min}$, but since it does not satisfy P4), there is no t-norm T on L such that I_{2T} gives this implication.

iii) On the contrary, the so called Gödel implication

$$I(x_i, x_j) = \begin{cases} 1 & \text{if } i \leq j \\ x_j & \text{otherwise} \end{cases}$$

equals $I_{2\,\min}$ whereas there is no t-norm T on L such that I_{1T} gives this implication.
iv) Finally, it is easy to see that the Gaines–Rescher implication

$$I(x_i, x_j) = \begin{cases} 1 & \text{if } i \leq j \\ x_0 & \text{otherwise} \end{cases}$$

is different from I_{1T} and from I_{2T} for all t-norms T on L.

We have proved that among all the smooth t-norms only the Archimedean one satisfies that the corresponding implicators I_{1T} and I_{2T} agree. However, among the non smooth t-norms it is easy to find new examples satisfying this property, like the well known nilpotent minimum, given by

$$T(x_i, x_j) = \begin{cases} x_0 & \text{if } i + j \leq n + 1 \\ \min\{x_i, x_j\} & \text{otherwise.} \end{cases}$$

From this t-norm we obtain, via I_{1T} and I_{2T}, the so called R_0−implication which is extensively studied in the case of [0,1] in [23].

Proposition 15. Let T be the nilpotent minimum t-norm, then $I_{1T} = I_{2T} = R_0$, where

$$R_0(x_i, x_j) = \begin{cases} x_{n+1} & \text{if } i \leq j \\ \max\{x_{n+1-i}, x_j\} & \text{otherwise.} \end{cases}$$

Proof. It is a straightforward computation from the definitions. □

A clear generalization of the nilpotent minimum appears when one replaces the min t-norm by any smooth t-norm T as follows:

Definition 5. Given a t-norm T and the strong negation N, define the operator $T_{(N)} : L \times L \to L$ by

$$T_{(N)}(x_i, x_j) = \begin{cases} x_0 & \text{if } i + j \leq n + 1 \\ T(x_i, x_j) & \text{otherwise.} \end{cases}$$

Let T and T' be t-norms, T is said to be similar to T' with respect to N, denoted by $T \leftrightarrow_N T'$, if $T_{(N)} = T'_{(N)}$.

The operator $T_{(N)}$ on [0,1] as well as the nilpotent minimum appears for the first time in [12] and it is extensively studied in [17]. Moreover, similar operations but taking N a non-necessarily involutive negation, are studied in [7] generalizing the results in [17].

As in [0,1], given any t-norm T, the operator $T_{(N)}$ on L is clearly commutative, nondecreasing and such that $T_{(N)}(x_i, x_{n+1}) = x_i$ for all $x_i \in L$, only associativity condition may fail in order to obtain a t-norm. The following theorem characterizes the smooth t-norms T for which $T_{(N)}$ is also a t-norm.

Theorem 3. Let T be a smooth t-norm, then $T_{(N)}$ is a t-norm if and only if there is an $x_k \in L$ such that $N(x_k) \leq x_k$ and $T \leftrightarrow_N T_{J_k}$ where T_{J_k} stands for the only smooth t-norm with set of idempotents given by $J_k = [x_0, N(x_k)] \cup [x_k, x_{n+1}]$. In this case the expression for $T_{(N)}$ is given by

$$T_{(N)}(x_i, x_j) = \begin{cases} x_0 & \text{if } i+j \leq n+1 \\ x_{i+j-k} & \text{if } i+j > n+1 \text{ and } n+1-k \leq i, j \leq k \\ \min\{x_i, x_j\} & \text{otherwise.} \end{cases} \tag{13}$$

Proof. It is a straightforward computation to show that the operator $T_{(N)}$ given by expression (13) is associative and consequently a t-norm, since the other properties are obvious. Conversely, suppose that T is a smooth t-norm such that $T_{(N)}$ is a t-norm and let us prove that $T \leftrightarrow_N T_{J_k}$ and that $T_{(N)}$ is given by expression (13) in several steps:

- First we prove that if x_j is an idempotent element of T with $x_{n+1-j} \leq x_j$ then x_{j+1} also is idempotent. Suppose on the contrary that x_{j+1} is not idempotent. Then, since x_j is idempotent and $x_{n+1-j} \leq x_j < x_{j+1}$, we have

$$T_{(N)}(T_{(N)}(x_{n+1-j}, x_{j+1}), x_{j+1}) = T_{(N)}(x_{n+1-j}, x_{j+1}) = x_{n+1-j}$$

whereas, since x_{j+1} is not idempotent, by the definition of $T_{(N)}$, we have

$$T_{(N)}(x_{n+1-j}, T_{(N)}(x_{j+1}, x_{j+1})) = T_{(N)}(x_{n+1-j}, x_j) = x_0$$

contradicting the associativity of $T_{(N)}$.

- Now, let x_k be the least idempotent of T such that $x_{n+1-k} \leq x_k$. Then
  - If $x_{n+1-k} = x_k$ we clearly have $T \leftrightarrow_N \min$.
  - If $x_{n+1-k} < x_k$ then T must be an ordinal sum with an Archimedean term on an interval $[x_\ell, x_k]$ for some $x_\ell < x_{n+1-\ell}$ due to the minimality of x_k. Let us prove in this step that $x_{\ell-1} \leq x_{n+1-k}$. To do this, note that if $x_{\ell-1} > x_{n+1-k}$ we would have:

$$T_{(N)}(T_{(N)}(x_{n+1-\ell}, x_{\ell+1}), x_{k-1}) = T_{(N)}(x_{\max\{\ell, n-\ell+\ell+1-k\}}, x_{k-1})$$
$$= T_{(N)}(x_\ell, x_{k-1}) = x_{\max\{\ell, \ell+k-1-k\}} = x_{\max\{\ell, \ell-1\}} = x_\ell$$

whereas

$$T_{(N)}(x_{n+1-\ell}, T_{(N)}(x_{\ell+1}, x_{k-1})) = T_{(N)}(x_{n+1-\ell}, x_{\max\{\ell, \ell+1+k-1-k\}}) = x_0$$

obtaining a contradiction.

- Finally, let us prove jointly in this step that when $x_{\ell-1} \leq x_{n+1-k}$, we have $T \leftrightarrow_N T_{J_k}$ and $T_{(N)}$ is given by expression (13).
  - From the definition we have that $T_{(N)}(x_i, x_j)$ and $(T_{J_k})_{(N)}(x_i, x_j)$ vanish when $x_j \leq N(x_i) = x_{n+1-i}$.
  - Whenever $x_{n+1-k} \leq x_i, x_j \leq x_k$ and $x_j > x_{n+1-i}$, we have

    $$(T_{J_k})_{(N)}(x_i, x_j) = x_{\max\{n+1-k, i+j-k\}} = x_{i+j-k}$$

    whereas

    $$T_{(N)}(x_i, x_j) = x_{\max\{\ell, i+j-k\}} = x_{i+j-k}$$

    since $x_{i+j-k} > x_{n+1-k} \geq x_{\ell-1}$.
  - It is clear that $T_{(N)}(x_i, x_j)$ and $(T_{J_k})_{(N)}(x_i, x_j)$ agree with the minimum otherwise.

Thus, the proof is complete. □

Again, as it happened for the nilpotent minimum, the implicators $I_{1T_{(N)}}$ and $I_{2T_{(N)}}$ are the same, for any smooth t-norm T such that $T_{(N)}$ is a t-norm:

Proposition 16. Let T be any smooth t-norm such that $T_{(N)}$ is a t-norm, then $I_{1T_{(N)}} = I_{2T_{(N)}}$ and their common expression I is given by

$$I(x_i, x_j) = \begin{cases} x_{n+1} & \text{if } i \leq j \\ x_{k+j-i} & \text{if } n+1-k \leq j < i \leq k \\ \max\{x_{n+1-i}, x_j\} & \text{otherwise.} \end{cases} \tag{14}$$

P r o o f. A straightforward computation, based on similar reasonings to those used in Propositions 5 and 13, shows that $I_{1T_{(N)}}$ and $I_{2T_{(N)}}$ are given by (14). □

The structure of the t-norms $T_{(N)}$ given by expression (13) as well as their derived implications given by (14) can be viewed in Figure 4.

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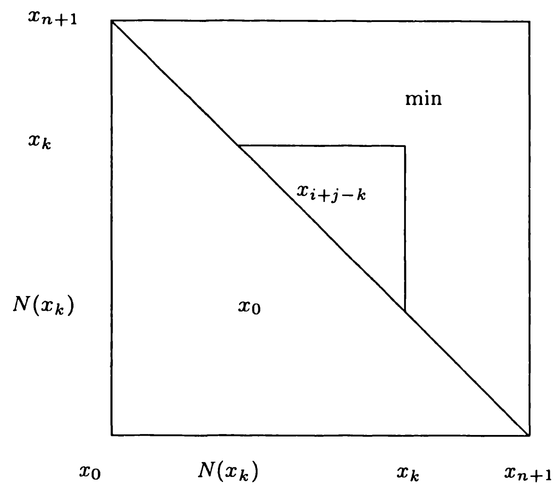

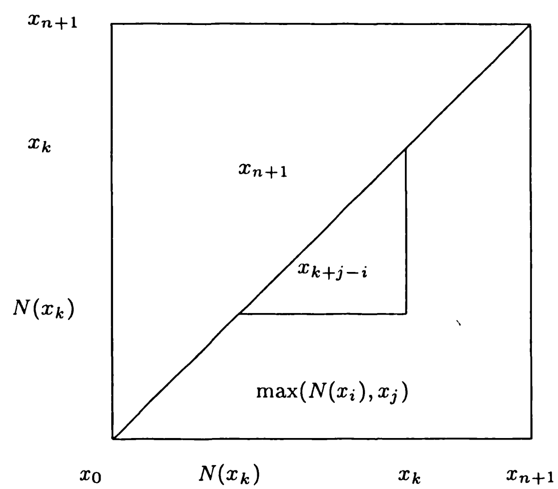

Fig. 4. A general t-norm $T_{(N)}$ (top) and its derived implication (bottom).

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