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A COLLECTOR FOR INFORMATION WITHOUT PROBABILITY IN A FUZZY SETTING

Doretta Vivona and Maria Divari

In the fuzzy setting, we define a collector of fuzzy information without probability, which allows us to consider the reliability of the observers. This problem is transformed in a system of functional equations. We give the general solution of that system for collectors which are compatible with composition law of the kind “inf”.

Keywords: information measure, system of functional equations

AMS Subject Classification: 93E12, 62A10, 62F15

1. INTRODUCTION

In the subjective theory of information without probability [9, 10, 11, 12, 15] and in the crisp setting, B. Forte and others [3, 7, 8] have supposed that each group of observers $E$ provides an amount of information $J(A, E)$ from the same event $A$. Moreover they supposed that, for each $E$, the information is compositive (in the sense of [13] with the same law with an additive reliability coefficient $\lambda(E)$.

B. Forte has defined a collector as a function $\Phi$:

$$J(A, E_1 \cup E_2) = \Phi\left(\lambda(E_1), \lambda(E_2), J(A, E_1), J(A, E_2)\right)$$

for every event $A$ and disjoint groups $E_1, E_2$.

Putting $x = \lambda(E_1), y = \lambda(E_2), u = J(A, E_1), v = J(A, E_2)$, Aczél, Forte and Ng in [1, 2] gave the solution in the Shannon case:

$$\Phi(x, y, u, v) = -c \log \left(\frac{x e^{-u/c} + y e^{-v/c}}{x + y}\right),$$

where $c$ is the constant related to the Shannon information; when the information $J$ is of the kind $\land$, Benvenuti, Divari and Pandolfi obtained a more general class of solutions (see [4]).

In a previous paper [16] we have defined collectors of $\land$-composite information without probability for fuzzy sets of events, crisp sets of observers with a reliability coefficient defined in a probabilistic space.
In this paper we shall introduce fuzzy collectors for crisp groups of observers with a fuzzy V-additive measure of reliability.

Evidently, if we restrict our considerations to crisp sets, the collectors studied in [4] are recovered. One of the main aims of this paper is also to enlight interesting ideas from [4] which are not so known in the wider community.

2. PRELIMINARIES

In the setting of fuzzy sets [17], we consider the following model:

1) \( \Omega \) is an abstract space, \( \mathcal{F} \) is an algebra of fuzzy sets such that \( (\Omega, \mathcal{F}) \) is a fuzzy measurable space, the elements of \( \mathcal{F} \) are the observable events. Recall that for \( A \) and \( B \in \mathcal{F} \), whose membership functions are \( f_A \) and \( f_B \), respectively, it holds:
   \[ f_{A \cup B} = f_A \lor f_B, f_{A \cap B} = f_A \land f_B, f_A^c = 1 - f_A; \]

2) \( \mathcal{O} \) is another abstract space (space of observers), \( \mathcal{E} \) is a \( \sigma \)-algebra contained in \( \mathcal{P}(\mathcal{O}) \), whose elements are groups of observers;

3) a fuzzy V-additive measure \( \mu \) is defined on the measurable space \( (\mathcal{O}, \mathcal{E}) \): \( \mu(\emptyset) = 0, \mu(\mathcal{O}) = \bar{\mu} \in [0, +\infty] \), \( \mu \) is non-decreasing with respect to the inclusion of the elements of \( \mathcal{E} \) and \( \mu(E_1 \cup E_2) = \mu(E_1) \lor \mu(E_2) \land E_1, E_2 \in \mathcal{E} \); if \( E \in \mathcal{E} \), \( \mu(E) \) is called fuzzy reliability coefficient;

4) an information measure \( J \), called fuzzy information (see [5, 6]), linked to the group of observers, is a map \( J : \mathcal{F} \times \mathcal{E} \rightarrow \mathbb{R}^+ \) such that, fixed \( E \in \mathcal{E}, E \neq \emptyset, \neq \mathcal{O} \) for all \( A, B \in \mathcal{F} \)
   \[ 4i) \quad A \subset B \Rightarrow J(A, E) \geq J(B, E), \]
   \[ 4ii) \quad J(\emptyset, E) = +\infty, J(\Omega, E) = 0; \]

5) every information measure \( J(\cdot, E) \) is \( F_E \)-compositive i.e. for every \( E \in \mathcal{E}, E \neq \emptyset \) there exists a map \( F_E : \Gamma_E \rightarrow \mathbb{R}^+ \), where \( \Gamma_E = \{(x, y) / \exists A, B \in \mathcal{F} \text{ with } x = J(A, E), y = J(B, E), f_A \land f_B = 0 \} \) such that

\[
J(A \cup B; E) = F_E \left( J(A, E), J(B, E) \right). \tag{1}
\]

Evidently \( F_E \) is commutative, associative and \( F_E(x, +\infty) = x \), for all \( x \in \text{Ran} J(\cdot, E) \).

Throughout this paper we deal with universal composition rule \( F = \land, \)

\[
J(A \cup B, E) = F[J(A, E), J(B, E)] = J(A, E) \land J(B, E). \tag{2}
\]

Note that due the idempotency of the operator \( \land \) we need not to require the disjointness \( f_A \land f_B = 0 \) in the above equality (2).

We call \( \land \)-compositive fuzzy information a fuzzy information \( J \) which satisfies (2) for every \( E \in \mathcal{E} \).
3. COLLECTOR OF $\wedge$-COMPOSITE FUZZY INFORMATION

In the previous paper [16] we have defined a collector for crisp sets.

Here, in the fuzzy setting, we give the definition of collector which we shall call *fuzzy collector*.

**Definition 3.1.** A fuzzy collector for a given reliability measure $\mu$ is a continuous function $\Psi$

$$\Psi: \Sigma \rightarrow \mathbb{R}^+$$

where $\Sigma \subset \left(0, \overline{\mu}\right] \times \mathbb{R}_+^+$, $\overline{\mu} = \mu(\emptyset)$, such that for every pair of two disjoint groups $E_1$ and $E_2$ of observers with reliability coefficients $\mu(E_1)$ and $\mu(E_2)$ it holds

$$J(A, E_1 \cup E_2) = \Psi \left( \mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2) \right). \quad (3)$$

4. PROPERTIES OF A FUZZY COLLECTOR $\Psi$

In this section we present the properties if a fuzzy collector is expressed by $\Psi$. They are:

(i) (commutativity):

$$\Psi \left( \mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2) \right) = \Psi \left( \mu(E_2), J(A, E_2), \mu(E_1), J(A, E_1) \right),$$

\forall A \in \mathcal{F}, E_1, E_2 \in \mathcal{E}, \text{ as } J(A, E_1 \cup E_2) = J(A, E_2 \cup E_1);

(ii) (associativity):

$$\Psi \left( \mu(E_1) \lor \mu(E_2), \Psi \left( \mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2) \right), \mu(E_3), J(A, E_3) \right)$$

$$= \Psi \left( \mu(E_1), J(A, E_1), \mu(E_2) \lor \mu(E_3), \Psi \left( \mu(E_2), J(A, E_2), \mu(E_3), J(A, E_3) \right) \right),$$

\forall A \in \mathcal{F}, E_1, E_2, E_3 \in \mathcal{E}, \text{ as } J(A, (E_1 \cup E_2) \cup E_3) = J(A, E_1 \cup (E_2 \cup E_3));

(iii) (universal value $J(\emptyset, E) = +\infty$):

$$\Psi \left( \mu(E_1), +\infty, \mu(E_2), +\infty \right) = +\infty,$$

as $J(\emptyset, E_1 \cup E_2) = +\infty$;

(iv) (universal value $J(\Omega, E) = 0$):

$$\Psi \left( \mu(E_1), 0, \mu(E_2), 0 \right) = 0,$$

as $J(\Omega, E_1 \cup E_2) = 0$.

If the information of the group of observers is $\wedge$-compositive in the sense of (2) we can add another property:
(v) (compatibility condition between the \(\land\)-compositivity of \(J\) and the collector \(\Psi\):)
\[
\Psi \left( \mu(E_1), \left[ J(A, E_1) \land J(B, E_1) \right], \mu(E_2), \left[ J(A, E_2) \land J(B, E_2) \right] \right) \\
= \Psi \left( \mu(E_1), J(A, E_1), \mu(E_2), J(A, E_2) \right) \land \Psi \left( \mu(E_1), J(B, E_1), J(B, E_2), \mu(E_2), \right) \\
\forall A, B \in \mathcal{F}, E_1, E_2 \in \mathcal{E}.
\]

In fact, from (2) it is \(J(A \cup B, E_1 \cup E_2) = J(A, E_1 \cup E_2) \land J(B, E_1 \cup E_2)\), and, on the other hand, from (3), we get \(J(A \cup B, E_1 \cup E_2) = \Psi \left( \mu(E_1), J(A \cup B, E_1), J(A \cup B, E_2) \right) \land \Psi \left( \mu(E_1), J(A, E_1) \land J(B, E_1), \mu(E_2), \right) \).

5. SYSTEM OF FUNCTIONAL EQUATIONS

Put \(\mu(E_1) = x, \mu(E_2) = y, \mu(E_3) = z\), with \(x, y, z \in [0, 1]\). The function \(\Psi\) given in (3) is defined in the domain \(\Sigma^2 = ([0, \mu] \times \mathbb{R}^+)^2\). Moreover we set \(J(A, E_1) = u, J(A, E_2) = v, J(B, E_1) = u', J(B, E_2) = v', J(A, E_3) = w\).

Now we rewrite the conditions \([(i)-(v)]\) in order to obtain a system of functional equations. The equations are:

\[
\begin{align*}
(i') & \quad \Psi \left( x, u, y, v \right) = \Psi \left( y, v, x, u \right) \\
(ii') & \quad \Psi \left( x, u, y \lor z, \Psi(y, v, z, w) \right) = \Psi \left( x \lor y, \Psi(x, y, u, v), z, w \right) \\
(iii') & \quad \Psi \left( x, +\infty, y, +\infty \right) = +\infty \\
(iv') & \quad \Psi \left( x, 0, y, 0 \right) = 0 \\
(v') & \quad \Psi \left( x, u \land u', y, v \land v' \right) = \Psi \left( x, u, y, v \right) \land \Psi \left( x, u', y, v' \right).
\end{align*}
\]

In the setting of crisp sets, an analogous system was studied and solved by Benvenuti–Divari–Pandolfi in [4]. We study the system \([(i')-(v')]\) and we give the general solution step by step.

**Theorem 5.1. Main Theorem.** The function \(\Psi \left( x, u, y, v \right)\) is solution of the system \([(i')-(v')]\) if and only if
\[
\Psi \left( x, u, y, v \right) = g(x, y, u) \land g(y, x, v) \tag{4}
\]
where the function \(g : [0, \mu]^2 \times \mathbb{R} \to \mathbb{R}\) fulfills the following properties:

\(\alpha\) \(g\) is non decreasing with respect to \(u\) and continuous,

\(\beta\) \(g(x, y, +\infty) = +\infty\),

\(\gamma\) \(g(x, y, 0) \land g(y, x, 0) = 0\),

\(\delta\) \(g[x \lor z, y, g(x, z, u)] = g(x, y \lor z, u)\).
Proof. Putting $g(x, y, u) = \Psi(x, u, y, +\infty)$, from $(v')$ for $u' = v$, we have the 
(4). It is easy to verify that every function $\Psi$ with the form (4) and the properties 
[(\alpha) - (\delta)] is a solution of the system $[(v') - (v')]$. □

For every function $g(x, y, u)$ which satisfies the properties $[(\alpha) - (\delta)]$, we can prove 
the following Lemmas.

Lemma 5.2. For every function $g(x, y, u)$ which satisfies the properties $[(\alpha) - (\delta)]$, we have $g(x, y, 0) = 0$.

Proof. From $(\delta)$, for $u = 0$ it is 
$$g(x \lor z, y, g(x, z, 0)) = g(x, y \lor z, 0),$$
and then, changing $x$ with $z$
$$g(z \lor x, y, g(z, x, 0)) = g(z, y \lor x, 0).$$
Because of $(\gamma)$, either $g(x, z, 0) = 0$ or $g(z, x, 0) = 0$, from $(\alpha)$, (4) and (5) we get
$$g(x \lor z, y, 0) = g(x, y \lor z, 0) \land g(z, y \lor x, 0),$$
and from (7) for $y = 0$
$$g(x \lor z, 0, 0) = g(x, z, 0) \land g(x, z, 0)$$
i.e., due to $(\gamma)$,
$$g(x \lor z, 0, 0) = 0 \forall x, z.$$ 
Finally, from (8) and (9), for $x = z$, we get
$$g(z, z, 0) = 0.$$ 
For $x \leq z$
$$g(z, x, 0) = g(x, z, 0) \land g(z, x, 0),$$
so we obtain, due to $(\gamma)$,
$$g(z, x, 0) = 0 \forall x \leq z.$$ 
Putting in (7) $y = x$ and for $x > z$
$$g(x, x, 0) = g(x, x, 0) \land g(z, x, 0),$$
$$g(x \lor z, x, 0) = g(x, x \lor z, 0) \land g(z, x \lor x, 0).$$
From $(\delta)$, for $u = 0$
$$g(x \lor z, y, g(x, z, 0)) = g(x, y \lor z, 0).$$
By contradiction we suppose $g(z, x, 0) = \lambda > 0$, i.e. $g(x, y, \lambda) = g(x, y \lor z, 0)$. For 
y $> z$, we get $g(x, y, \lambda) = g(x, y, 0)$: this is impossible as $g$ is non-decreasing with 
respect to $u$, then
$$g(z, x, 0) = 0 \forall x, z.$$ □
Lemma 5.3. For every function $g$ which enjoys $[(\alpha) - (\delta)]$, we have
\[ g(x, 0, u) = u \text{ in } [0, \bar{u}] \times \mathbb{R}^+. \tag{13} \]

Proof. As, from $(\gamma)$ and $(\delta)$, $g(x, 0, 0) = 0$ and $g(x, 0, +\infty) = +\infty$, for every $v \in \mathbb{R}^+$ there exists $u$ such that $g(x, 0, u) = v$.

From $(\gamma)$, for $y = z = 0$, we have $g(x, 0, g(x, 0, v)) = g(x, 0, u)$. \[ \square \]

Lemma 5.4. Every function $g$ which satisfies $[(\alpha) - (\delta)]$ has the following representation:
\[ g(x, y, u) = h[x \lor y, h^{-1}(x, u)] (x, y, u) \in \mathbb{R}^+ \tag{14} \]
with $h : [0, 1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$, continuous, non decreasing with respect to $u$ and $h^{-1}$ its pseudo-inverse [14], defined by $h^{-1}(x, v) = \text{Inf}\{\xi / h(x, \xi) = v\}$.

Proof. Putting $h(x, u) = g(0, x, u)$, for $(\alpha)$ and $(\beta)$ the function $h$ is continuous, monotone and $h(x, 0) = 0$, $h(x, +\infty) = +\infty$, therefore its pseudo-inverse $h^{-1}$ is defined on $[0, 1] \times \mathbb{R}^+$. From $(\delta)$, for $x = 0$ and $u = h^{-1}(z, v)$, we have $g(z, y, g(0, z, h^{-1}(z, v))) = g(0, y \land z, h^{-1}(z, v))$, i.e. $g(z, y, g(0, z, h^{-1}(z, v))) = h(y \land z, h^{-1}(z, v))$. The thesis follows from $h(z, h^{-1}(z, v)) = v$. \[ \square \]

Remark. We observe that continuity of $g$ and condition $(\beta)$ imply that $h(x, u) = g(0, x, u)$ is not (definitely) null or constant (unless $= +\infty$). Indeed, if we hold the following situation: $g(x, y, u) = \frac{x u}{x \lor y}$ (with $0 \cdot +\infty = 0$), then we couldn't find $h^{-1}$, but clearly $g(0, x, u) = 0$, contrary to $(\beta)$.

This situation corresponds to the following example:
Let $O = \{1, 2, \ldots, n\}$ be the set of observers, $\mu(E) = \max E$ and $J(A, E) = -\log \mu(A) / \mu(E)$. So, we have: $g(x, y, u) = \frac{x u}{x \lor y}$, $h(x, u) = 0$ and the collector is: $\Psi(x, u, y, v) = \frac{x u \lor y v}{x \lor y v}$.

Lemma 5.5. For every function $g$ which satisfies $[(\alpha) - (\delta)]$, the corresponding function $h$ given by (14) enjoys the following properties:
\[ h(0, v) = v \in \mathbb{R}^+ \tag{15} \]
and
\[ h(x, u) = h(x, v) \Rightarrow h(y, u) = h(y, v) \forall y > x. \tag{16} \]

Proof. The condition (15) follows from the definition of the function $h$ and from Lemma 5.4. Now, we shall prove the (16): in $(\delta)$ setting $x = 0$ it is $g(z, y, g(0, z, u)) = g(0, y \land z, u)$ and for (14) we get $h(z \lor y, h^{-1}(z, h(z, h^{-1}(0, u)))) = h(y \lor z, h^{-1}(0, u))$, i.e. $h(z \lor y, h^{-1}(z, h(z, u))) = h(y \lor z, u)$.

If $h(z, v) = h(z, u)$ with $v < u$, from definition of $h^{-1}$, we have $h^{-1}(z, h(z, u)) = \text{Inf}\{\xi / h(z, \xi) = h(z, u)\} = v' \leq v$ and therefore $h(y \land z, v') = h(y \land z, u)$.

If $v > u$, for the monotonicity of the function $h$ and the arbitrary of $y$, we obtain the (16). \[ \square \]
Lemma 5.6. The expression (14) with the function \( h(x, u) \) satisfying the conditions of the Lemmas 5.4 and 5.5 gives the general form of the continuous solutions of the system \( (\alpha) \).  

Proof. We shall, now, verify that every function \( g(x, y, u) \) defined by (14) 
\[
g(x, y, u) = h(x \lor y, h^{-1}(x, u))
\]
with \( h(x, u) \) satisfying the conditions of the Lemmas 5.4 and 5.5 is solution of the system \( (\alpha) \). In fact, for the properties of \( h \) in Lemma 5.5, the properties \( (\alpha) \) and \( (\beta) \) are verified. The \( (\delta) \) becomes \( g(x \lor z, y, g(x, z, u) = g(x, y \lor z, u) \) and then 
\[
h \left( x \lor z \lor y, h^{-1}(x \lor z, h(x \lor z, h^{-1}(x, u))) \right) = h \left( x \lor z \lor y, h^{-1}(x, u) \right).
\]  
Putting \( h^{-1}(x, u) = v \), the (17) becomes \( h(x \lor z \lor y, h^{-1}(x \lor z, h(x \lor z, v))) = h(x \lor z \lor y, v) \). Moreover \( h^{-1}(x \lor z, h(x \lor z, v)) = \inf \{ \xi / h(x \lor z, \xi) = h(x \lor z, v) \} = v' \leq v \), with \( h(x \lor z, v') = h(x \lor z, v) \). For the (16), as \( x \lor z \lor y \geq x \lor z \) and 
\[
h(x \lor y \lor z, v') = h(x \lor y \lor z, v) \), we have the \( (\delta) \). \( \Box \)

Summarizing the previous Lemmas, we obtain the following main result:

Theorem 5.7. The general solution of the system \( (\iota') \) is the function 
\[
\Psi(x, y, u, v) = h \left( x \lor y, h^{-1}(x, u) \land h^{-1}(y, v) \right)
\]
where \( h : [0,1] \times \mathbb{R}^+ \rightarrow \mathbb{R}^+ \) satisfies the following conditions:
- \( h(x, \cdot) \) is non-decreasing, continuous, \( h(x, 0) = 0 \), \( h(x, +\infty) = +\infty \), \( \forall x \in (0,\bar{\mu}] \),
- \( h(x, u) = h(x, v) \Rightarrow h(y, u) = h(y, v) \) for every \( y > x \).

Example: Let \( h(x, u) = e^x u \), this function satisfies the hypotheses of the Theorem above; its pseudo-inverse is \( h^{-1}(x, v) = \frac{v}{e^x} \). Then the function \( g \) is 
\[
g(x, y, u) = h(x \lor y, h^{-1}(x, u)) = e^{x \lor y} h^{-1}(x, u) = e^{x \lor y} u e^{-x} = u e^{(x \lor y) - x}.
\]
Then the collector \( \Psi \) has the following expression: 
\[
\Psi \left( x, y, u, v \right) = \begin{cases} 
\begin{align*}
g(x, y, u) \land g(y, x, v) \
u e^{(x \lor y) - x} \land v e^{(y \lor x) - y} = e^{x \lor y} \left( \frac{u}{e^x} \land \frac{v}{e^y} \right).
\end{align*}
\end{cases}
\]

Let \( J \) be an information measure on crisp sets such that \( J(E) = e^{-\lambda(E)} \) with \( \lambda \) a fuzzy measure \( \lor \)-additive and \( J(A, E) \) an information depending on the set of observers.

From (3) and (18), we get 
\[
J(A, E_1 \cup E_2) = \frac{J(A, E_1) J(E_1) \land J(A, E_2) J(E_2)}{J(E_1 \cup E_2)}.
\]
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