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A NEW FAMILY
OF TRIVARIATE PROPER QUASI–COPULAS

MANUEL ÚBEDA-FLORES

In this paper, we provide a new family of trivariate proper quasi-copulas. As an application, we show that $W^3$ – the best-possible lower bound for the set of trivariate quasi-copulas (and copulas) – is the limit member of this family, showing how the mass of $W^3$ is distributed on the plane $x + y + z = 2$ of $[0, 1]^3$ in an easy manner, and providing the generalization of this result to $n$ dimensions.

Keywords: copula, mass distribution, quasi-copula

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1. INTRODUCTION

Let $n$ be a natural number such that $n \geq 2$. An $n$-dimensional copula (briefly, $n$-copula) is the restriction to $[0, 1]^n$ of a continuous $n$-variate distribution function whose univariate margins are uniform on $[0, 1]$. Equivalently, an $n$-copula is a function $C: [0, 1]^n \to [0, 1]$ which satisfies the following conditions:

(C1) boundary conditions: for any $(u_1, u_2, \ldots, u_n)$ in $[0, 1]^n$ it holds that $C(u_1, \ldots, u_{i-1}, 0, u_{i+1}, \ldots, u_n) = 0$ and $C(1, \ldots, 1, u_i, 1, \ldots, 1) = u_i$ for all $i \in \{1, 2, \ldots, n\}$;

(C2) the $n$-increasing property: for every $(a_1, a_2, \ldots, a_n), (b_1, b_2, \ldots, b_n) \in [0, 1]^n$, and each $n$-box $B$ in $[0, 1]^n$, i.e., $B = [a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_n, b_n]$, we have that $V_C(B) = \sum \text{sgn}(c_1, c_2, \ldots, c_n) \cdot C(c_1, c_2, \ldots, c_n) \geq 0$ – $V_C(B)$ is defined as the $C$-volume of $B$ –, where the sum is taken over all the vertices $(c_1, c_2, \ldots, c_n)$ of $B$ (i.e., each $c_k$ is equal to either $a_k$ or $b_k$) and $\text{sgn}(c_1, c_2, \ldots, c_n)$ is 1 if $c_k = a_k$ for an even number of $k$’s, and $-1$ if $c_k = a_k$ for an odd number of $k$’s.

The importance of copulas as a tool for statistical analysis and modeling stems largely from the observation that the joint distribution $H$ of a random vector $(X_1, X_2, \ldots, X_n)$ with respective one-dimensional margins $F_1, F_2, \ldots, F_n$ can be expressed by

$$H(x_1, x_2, \ldots, x_n) = C(F_1(x_1), F_2(x_2), \ldots, F_n(x_n)), \quad (x_1, x_2, \ldots, x_n) \in [-\infty, \infty]^n,$$

where $C$ is an $n$-copula that is uniquely determined on $\text{Range} F_1 \times \text{Range} F_2 \times \cdots \times \text{Range} F_n$. For a complete survey about copulas, see [14, 23, 24].
Alsina et al. [1] introduced the notion of quasi-copula in order to show that a certain class of operations on univariate distribution functions can, or cannot, be derived from corresponding operations on random variables defined on the same probability space (see also [19]). Cuculescu and Theodorescu [4] have given the characterization of an $n$-dimensional quasi-copula (or $n$-quasi-copula) as a function $Q: [0,1]^n \to [0,1]$ which satisfies condition (C1) of $n$-copulas, but instead of condition (C2), the weaker conditions:

(Q1) monotonicity: $Q$ is nondecreasing in each variable;

(Q2) Lipschitz condition: for any $(u_1,u_2,\ldots,u_n)$ and $(v_1,v_2,\ldots,v_n)$ in $[0,1]^n$, it holds that $|Q(u_1,u_2,\ldots,u_n) - Q(v_1,v_2,\ldots,v_n)| \leq \sum_{i=1}^{n} |u_i - v_i|$.

We will refer to $V_Q(B)$ – the $Q$-volume of $B$ – as the mass accumulated by $Q$ on $B$. Every $n$-quasi-copula $Q$ satisfies the inequalities

$$W^n(u_1,u_2,\ldots,u_n) = \max\left(0, \sum_{i=1}^{n} u_i - n + 1\right) \leq Q(u_1,u_2,\ldots,u_n) \leq \min(u_1,u_2,\ldots,u_n) = M^n(u_1,u_2,\ldots,u_n)$$

for every $(u_1,u_2,\ldots,u_n)$ in $[0,1]^n$. While every $n$-copula is an $n$-quasi-copula, there exist proper $n$-quasi-copulas, i.e., $n$-quasi-copulas which are not $n$-copulas. For any $n \geq 2$, $M^n$ is an $n$-copula; but $W^n$ is an $n$-copula if and only $n = 2$, and a proper $n$-quasi-copula for $n \geq 3$.

One of the most important applications of quasi-copulas in statistics is the following result ([15, 17, 21]): Every pointwise ordered set of copulas has a least upper bound and greatest lower bound in the set of quasi-copulas. Of interest are sets of copulas of random variables with a specific statistical property (see [10, 11, 17, 18]). Furthermore, since quasi-copulas are a special type of binary aggregation operators satisfying the Lipschitz condition (Q2) (see [3]), these functions are becoming popular in fuzzy set theory (for instance, see [2, 8, 9, 12]).

In the literature, we cannot find many families of proper $n$-quasi-copulas when $n \geq 3$ – for some examples (different from $W^n$), see [7, 16, 22]. Recently, the mass distribution associated with a 3-quasi-copula and the differences with respect to the bivariate case – we recall that the (positive) mass of $W^2$ is distributed uniformly in $[0,1]^2$ on the segment which joins the points $(0,1)$ to $(1,0)$, and the (infinite positive and infinite negative) mass of $W^3$ is distributed on the plane $x+y+z = 2$ of $[0,1]^3$ – have been studied in [7, 13]. Our purpose is to provide a new family of proper 3-quasi-copulas whose bivariate margins are 2-copulas – moreover, we construct the least upper bound and the greatest lower bound in the set of quasi-copulas with those margins. As an application, we prove that $W^3$ is the limit member of this new family, showing how the mass of $W^3$ is distributed on the plane $x+y+z = 2$ of $[0,1]^3$ in an easy manner. In the last section, we provide the generalization of this problem to $n$ dimensions.
2. A NEW FAMILY OF PROPER 3-QUASI-COPULAS

Let \( m \) be a natural number such that \( m \geq 2 \). We divide \([0, 1]^3\) into \( m^3 \) 3-boxes (or cubes, in this case), namely:

\[
B_{i_1i_2i_3} = \left[ \frac{i_1 - 1}{m}, \frac{i_1}{m} \right] \times \left[ \frac{i_2 - 1}{m}, \frac{i_2}{m} \right] \times \left[ \frac{i_3 - 1}{m}, \frac{i_3}{m} \right],
\]

for all \( i_1, i_2, i_3 = 1, 2, \ldots, m \). Now, we distribute \( 1/m \) of (positive) mass uniformly on each cube \( B_{i_1i_2i_3} \) such that \( i_1 + i_2 + i_3 = 2m + 1 \); \(-1/m \) of (negative) mass uniformly on each cube \( B_{i_1i_2i_3} \) such that \( i_1 + i_2 + i_3 = 2m + 2 \); and 0 on the remaining cubes. It can be easily computed that there are \( m(m + 1)/2 \) cubes with positive mass, and \( m(m - 1)/2 \) cubes with negative mass; and the sum of positive mass is \((m + 1)/2\), and the sum of negative mass is \(-(m - 1)/2\). Therefore, we have the amount of 1 of positive mass on \([0, 1]^3\) (see Figure 1 for this construction in the case \( m = 4 \)).

Note that if we project this construction on the planes \( x = 1 \), \( y = 1 \) and \( z = 1 \), we obtain a construction (similar on the three planes) with \( 1/m \) of (positive) mass distributed uniformly on each square of the form \( R_{i_1(m-i_1+1)} \), for \( i_1 = 1, 2, \ldots, m \), where

\[
R_{i_1i_2} = \left[ \frac{i_1 - 1}{m}, \frac{i_1}{m} \right] \times \left[ \frac{i_2 - 1}{m}, \frac{i_2}{m} \right],
\]

for all \( i_1, i_2 = 1, 2, \ldots, m \); and 0 on \([0, 1]^2 \setminus R_{i_1(m-i_1+1)}\).

If \((u_1, u_2, u_3)\) is a point in \([0, 1]^3\), and \( Q_m(u_1, u_2, u_3) \) is the mass spread on \([0, u_1] \times [0, u_2] \times [0, u_3]\), then \( Q_m \) is a proper 3-quasi-copula – whose three bivariate margins are 2-copulas –, as the following result shows.
**Theorem 2.1.** For each natural number $m \geq 2$, let $Q_m: [0,1]^3 \to [0,1]$ be the function defined by

$$Q_m(u_1,u_2,u_3) = \begin{cases} 
0, & (u_1,u_2,u_3) \in B_1, \\
m^2 \prod_{j=1}^{3} \left( u_j - \frac{i_j - 1}{m} \right), & (u_1,u_2,u_3) \in B_2, \\
m^3 \sum_{k=1}^{3} \prod_{j=1}^{3} \left( u_j - \frac{i_j - 1}{m} \right) - m^2 \prod_{j=1}^{3} \left( u_j - \frac{i_j - 1}{m} \right), & (u_1,u_2,u_3) \in B_3, \\
u_1 + u_2 + u_3 - 2, & \text{otherwise},
\end{cases}$$

where $B_1 = \{B_{i_1,i_2,i_3} : i_1 + i_2 + i_3 \leq 2m\}$, $B_2 = \{B_{i_1,i_2,i_3} : i_1 + i_2 + i_3 = 2m+1\}$, and $B_3 = \{B_{i_1,i_2,i_3} : i_1 + i_2 + i_3 = 2m+2\}$. Then, $Q_m$ is a proper 3-quasi-copula for every $m \geq 2$ whose three bivariate margins (which are the same) are the 2-copula $C_m^{(2)}$ given by

$$C_m^{(2)}(v_1,v_2) = \begin{cases} 
0, & (v_1,v_2) \in R_1, \\
m \prod_{j=1}^{2} \left( v_j - \frac{i_j - 1}{m} \right), & (v_1,v_2) \in R_2, \\
v_1 + v_2 - 1, & \text{otherwise},
\end{cases}$$

where $R_1 = \{R_{i_1,i_2} : i_1 + i_2 \leq m\}$ and $R_2 = \{R_{i_1,i_2} : i_1 + i_2 = m+1\}$.

**Proof.** Suppose $m$ is a fixed natural number such that $m \geq 2$, and let $(u_1,u_2,u_3)$ be a point in $[0,1]^3$. First, we show that $Q_m$ is well-defined. Let $B_{i_1,i_2,i_3} \in B_2$ and $B_{j_1,j_2,j_3} \in B_3$ be two cubes in $[0,1]^3$ such that $i_2 = j_2$ and $i_3 = j_3$ (all the other cases can be proved in a similar manner). Then we have that $j_1 = 1 + i_1$. Since

$$Q_m(u_1,u_2,u_3) = m^2 \left( u_1 - \frac{i_1 - 1}{m} \right) \prod_{k=2}^{3} \left( u_k - \frac{j_k - 1}{m} \right),$$

in particular, we obtain that

$$Q_m\left(\frac{i_1}{m},u_2,u_3\right) = m^2 \left( \frac{i_1}{m} - \frac{i_1 - 1}{m} \right) \prod_{k=2}^{3} \left( u_k - \frac{j_k - 1}{m} \right) = m \prod_{k=2}^{3} \left( u_k - \frac{j_k - 1}{m} \right);$$
and since

\[
Q_m(u_1, u_2, u_3) = m \sum_{k=1}^{3} \prod_{j=1}^{3} \left( u_j - \frac{i_j - 1}{m} \right) - m^2 \prod_{j=1}^{2} \left( u_j - \frac{i_j - 1}{m} \right),
\]

\[
(u_1, u_2, u_3) \in \prod_{k=1}^{3} \left[ \frac{j_k - 1}{m}, \frac{j_k}{m} \right],
\]

in particular, we obtain that

\[
Q_m \left( \frac{i_1}{m}, u_2, u_3 \right) = Q_m \left( \frac{j_1 - 1}{m}, u_2, u_3 \right) = m \prod_{k=2}^{3} \left( u_k - \frac{j_k - 1}{m} \right).
\]

To prove the boundary conditions, suppose \( u_2 = u_3 = 1 \) (the cases \( u_1 = u_2 = 1 \) and \( u_1 = u_3 = 1 \) use similar arguments) in a cube \( B_{i_1i_2i_3} \in B_3 \) (all the remaining cases can be proved in a similar manner). Thus \( i_2 = i_3 = m \), and hence \( i_1 = 2 \). Then, we obtain that

\[
Q_m(u_1, 1, 1) = m \left[ 2 \left( u_1 - \frac{1}{m} \right) \left( 1 - \frac{m-1}{m} \right) + \left( 1 - \frac{m-1}{m} \right)^2 \right] - m^2 \left( u_1 - \frac{1}{m} \right) \left( 1 - \frac{m-1}{m} \right)^2 = u_1.
\]

In what follows, let \((u'_1, u_2, u_3)\) and \((u_1, u_2, u_3)\) be two points in a cube \( B_{i_1i_2i_3} \) such that \( u'_1 > u_1 \) (the case \( u'_1 = u_1 \) is trivial in the following). We now check that \( Q_m \) is nondecreasing in the first variable and satisfies the Lipschitz condition (Q2) in the same variable (the cases for the other two variables can be proved in a similar manner) and in each cube \( B_{i_1i_2i_3} \). We consider two cases (the remaining cases are trivial).

(i) Suppose \( B_{i_1i_2i_3} \in B_2 \). Then we have

\[
Q_m(u'_1, u_2, u_3) - Q_m(u_1, u_2, u_3) = m^2(u'_1 - u_1) \prod_{j=2}^{3} \left( u_j - \frac{i_j - 1}{m} \right).
\]

It is trivial that \( Q_m(u'_1, u_2, u_3) - Q_m(u_1, u_2, u_3) \geq 0 \). On the other hand, we have that \( Q_m(u'_1, u_2, u_3) - Q_m(u_1, u_2, u_3) \leq u'_1 - u_1 \) if, and only if, \( m^2 \cdot \prod_{j=2}^{3}(u_j - (i_j - 1)/m) \leq 1 \). Since \( 0 \leq u_j - (i_j - 1)/m \leq 1/m \), for \( j = 2, 3 \), the result follows.

(ii) Suppose now \( B_{i_1i_2i_3} \in B_3 \). Then, we have that

\[
Q_m(u'_1, u_2, u_3) - Q_m(u_1, u_2, u_3) = m(u'_1 - u_1) \left[ \sum_{j=2}^{3} \left( u_j - \frac{i_j - 1}{m} \right) - m \prod_{j=2}^{3} \left( u_j - \frac{i_j - 1}{m} \right) \right].
\]

Thus, \( Q_m(u'_1, u_2, u_3) - Q_m(u_1, u_2, u_3) \geq 0 \) if, and only if,
\[ m \prod_{j=2}^{3} \left( u_j - \frac{i_j - 1}{m} \right) \leq \sum_{j=2}^{3} \left( u_j - \frac{i_j - 1}{m} \right). \quad (3) \]

Suppose \( u_2 - (i_2 - 1)/m > 0 \) and \( u_3 - (i_3 - 1)/m > 0 \) (the cases with the equality are trivial), then inequality (3) is equivalent to \( m \leq \sum_{j=2}^{3} (u_j - (i_j - 1)/m)^{-1}. \) Since \( u_2 \in ((i_2 - 1)/m, i_2/m] \), we have that \( u_2 \leq i_2/m = (i_2 - 1)/m + 1/m \), thus \( u_2 - (i_2 - 1)/m \leq 1/m \) (and similarly for \( u_3 \)); whence the result follows.

On the other hand, we have that \( Q_m(u_1', u_2, u_3) - Q_m(u_1, u_2, u_3) \leq u_1' - u_1 \) holds if, and only if, \( m \prod_{j=2}^{3} (u_j - (i_j - 1)/m) \geq \sum_{j=2}^{3} (u_j - (i_j - 1)/m) + 1/m. \) Since \( \prod_{j=2}^{3} (u_j - (i_j - 1)/m) \geq 0 \), i.e., \( u_2u_3 - u_2i_3/m - u_3i_2/m + i_2i_3/m^2 \geq 0 \), we have that \( u_2u_3 - u_2(i_3 - 1)/m - u_3(i_2 - 1)/m + (i_2 - 1)(i_3 - 1)/m^2 \geq u_2/m + u_3/m - i_2/m^2 - i_3/m^2 + 1/m^2 \); whence the result follows.

Thus, we have proved that \( Q_m \) is a 3-quasi-copula. Now, since
\[
V_{Q_m} \left( \left[ \frac{1}{m}, \frac{2}{m} \right] \times \left[ \frac{m-1}{m}, 1 \right] \times \left[ \frac{m-1}{m}, 1 \right] \right) = Q_m \left( \frac{2}{m}, 1, 1 \right) - Q_m \left( \frac{1}{m}, 1, 1 \right) - Q_m \left( \frac{2}{m}, 1, \frac{m-1}{m} \right) - Q_m \left( \frac{2}{m}, \frac{m-1}{m}, \frac{m-1}{m} \right) + Q_m \left( \frac{1}{m}, \frac{m-1}{m}, 1 \right) + Q_m \left( \frac{1}{m}, \frac{m-1}{m}, \frac{m-1}{m} \right) - Q_m \left( \frac{1}{m}, \frac{m-1}{m}, \frac{m-1}{m} \right) = \frac{2}{m} - \frac{3}{m} = -\frac{1}{m},
\]
we conclude that \( Q_m \) is a proper 3-quasi-copula.

Finally, since (as it is easy to check) the bivariate margins – or of higher dimension – of any \( n \)-quasi-copula are quasi-copulas, the three bivariate margins of \( Q_m \) – i.e., \( Q_m(u_1, u_2, 1), Q_m(u_1, 1, u_3) \) and \( Q_m(1, u_2, u_3) \) – given by (2) are 2-copulas since the mass (only positive) of \( C_m^{(2)} \) is distributed uniformly on \([0, 1]^2\), which completes the proof. \( \square \)

From Theorem 2.1, we first note that the 2-copulas given by (2) are a special type of orthogonal grid constructions of copulas studied in [6] with \( W^2 \) as background copula, and \( \Pi^2 \) – the copula of independent random variables, i.e., \( \Pi^2(u, v) = uv \) for all \((u, v)\) in \([0, 1]^2\) – as foreground copula.

We also observe that there does not exist a 3-copula whose three bivariate margins are \( C_m^{(2)}(u_1, u_2), C_m^{(2)}(u_1, u_3) \) and \( C_m^{(2)}(u_2, u_3) \) – this is related to the problem of the compatibility of three 2-copulas (for more details, see [5, 20]). The following result shows this fact.

**Proposition 2.1.** For any natural number \( m \geq 2 \), there does not exist a 3-copula whose three bivariate margins are the 2-copula \( C_m^{(2)} \) given by (2).
Proof. Suppose $C$ is a 3-copula whose three bivariate margins are $C^{(2)}_m$. Let $B = [1/2, 1]^3$. Then we have that

$$V_C(B) = C(1, 1, 1) - C\left(\frac{1}{2}, 1, 1\right) - C\left(1, \frac{1}{2}, 1\right) - C\left(1, 1, \frac{1}{2}\right) + C\left(\frac{1}{2}, \frac{1}{2}, 1\right) + C\left(\frac{1}{2}, 1, \frac{1}{2}\right) + C\left(1, \frac{1}{2}, \frac{1}{2}\right) - C\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$$

$$= 1 - \frac{3}{2} + 3 \cdot C^{(2)}_m\left(\frac{1}{2}, \frac{1}{2}\right) - C\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right).$$

If $m$ is even, it is easy to check that $V_C(B) = -1/2$ for every $m \geq 2$; and if $m$ is odd, we have that $V_C(B) = (1 - m)/(2m) < 0$ for every $m \geq 3$. In both cases we obtain a contradiction; therefore, $C$ is not a 3-copula, which completes the proof. □

We also note that $Q_m$ is not the unique proper 3-quasi-copula whose three bivariate margins are $C^{(2)}_m$ (for methods of constructing Lipschitz aggregation operators, see [2]). In fact, for any natural number $m \geq 2$, and given $C^{(2)}_m(u_1, u_2)$, $C^{(2)}_m(u_1, u_3)$ and $C^{(2)}_m(u_2, u_3)$, $(u_1, u_2, u_3) \in [0, 1]^3$, we can construct an infinite number of proper 3-quasi-copulas whose three bivariate margins are $C^{(2)}_m$, as the following example shows.

Example 2.1. For every $(u_1, u_2, u_3)$ in $[0, 1]^3$, consider the function $Q$ given by

$$Q(u_1, u_2, u_3) = \lambda \cdot Q_U(u_1, u_2, u_3) + (1 - \lambda) \cdot Q_L(u_1, u_2, u_3),$$

where

$$Q_U(u_1, u_2, u_3) = \min(C^{(2)}_m(u_1, u_2), C^{(2)}_m(u_1, u_3), C^{(2)}_m(u_2, u_3))$$

and

$$Q_L(u_1, u_2, u_3) = \max(0, C^{(2)}_m(u_1, u_2) + u_3 - 1, C^{(2)}_m(u_1, u_3) + u_2 - 1, C^{(2)}_m(u_2, u_3) + u_1 - 1),$$

with $\lambda \in [0, 1]$. $Q_L$ and $Q_U$ are two proper 3-quasi-copulas – whose three bivariate margins are $C^{(2)}_m$ – which satisfy the inequalities $Q_L(u_1, u_2, u_3) \leq Q_m(u_1, u_2, u_3) \leq Q_U(u_1, u_2, u_3)$ for every $(u_1, u_2, u_3)$ in $[0, 1]^3$ (see [22]). Observe that $Q_L(u_1, u_2, u_3) \neq Q_m(u_1, u_2, u_3) \neq Q_U(u_1, u_2, u_3)$ for some $(u_1, u_2, u_3)$ in $[0, 1]^3$ and for every $m \geq 2$. For instance, if $i$ is a real number such that $3i = 2m + 1$, after some elementary algebra we have that

$$Q_m\left(\frac{i}{m}, \frac{i}{m}, \frac{i}{m}\right) = \frac{1}{m} < \frac{m + 2}{3m} = Q_U\left(\frac{i}{m}, \frac{i}{m}, \frac{i}{m}\right)$$

for any $m \geq 2$. Moreover, if we suppose that $i_1 = 1$ and $i_2 = i_3 = m$, then we have that

$$Q_L\left(\frac{1}{2m}, 1 - \frac{1}{2m}, 1 - \frac{1}{2m}\right) = 0 < \frac{1}{8m} = Q_m\left(\frac{1}{2m}, 1 - \frac{1}{2m}, 1 - \frac{1}{2m}\right)$$

for every $m \geq 2$. 


3. APPROXIMATION OF $W^3$

In this section we show that $W^3$ is the limit member of the family of the proper 3-quasi-copulas defined by (1).

**Theorem 3.1.** Let $\varepsilon > 0$. For $m$ sufficiently large, there exists a proper 3-quasi-copula $Q_m$ given by (1) such that $|Q_m(u_1, u_2, u_3) - W^3(u_1, u_2, u_3)| < \varepsilon$ for all $(u_1, u_2, u_3)$ in $[0, 1]^3$.

**Proof.** Let $m$ be a natural number such that $m \geq 6/\varepsilon$. We first prove that

$$Q_m\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = W^3\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = \max\left(0, \frac{i_1 + i_2 + i_3}{m} - 2\right),$$

for every $i_1, i_2, i_3 = 1, 2, \ldots, m$. For that, we consider the following four cases:

(i) If $i_1 + i_2 + i_3 < 2m$, then we have that

$$Q_m\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = 0 \quad \text{and} \quad W^3\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = \max\left(0, \frac{i_1 + i_2 + i_3}{m} - 2\right) = 0.$$

(ii) If $i_1 + i_2 + i_3 = 2m + 1$, then we have that

$$Q_m\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = m^2 \prod_{j=1}^{3} \left(\frac{i_j}{m} - \frac{i_j - 1}{m}\right) = \frac{1}{m}$$

and

$$W^3\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = \max\left(0, \frac{2m + 1}{m} - 2\right) = \frac{1}{m}.$$

(iii) If $i_1 + i_2 + i_3 = 2m + 2$, then we have that

$$Q_m\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = 3m\left(\frac{1}{m}\right)^2 - m^2 \left(\frac{1}{m}\right)^3 = \frac{2}{m}$$

and

$$W^3\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = \max\left(0, \frac{2m + 2}{m} - 2\right) = \frac{2}{m}.$$

(iv) If $i_1 + i_2 + i_3 > 2m + 2$, then we have that

$$Q_m\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = \frac{i_1 + i_2 + i_3}{m} - 2$$

and

$$W^3\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) = \max\left(0, \frac{i_1 + i_2 + i_3}{m} - 2\right) = \frac{i_1 + i_2 + i_3}{m} - 2.$$
Now, let \((u_1, u_2, u_3)\) be a point in \([0, 1]^3\). We have \(|u_1 - i_1/m| < 1/m\), \(|u_2 - i_2/m| < 1/m\), and \(|u_3 - i_3/m| < 1/m\) for some \((i_1, i_2, i_3)\). Then

\[
\begin{align*}
|Q_m(u_1, u_2, u_3) - W^3(u_1, u_2, u_3)| &\leq \left|Q_m\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) - Q_m\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right)\right| \\
&\quad + \left|Q_m\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) - W^3\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right)\right| \\
&\quad + \left|W^3\left(\frac{i_1}{m}, \frac{i_2}{m}, \frac{i_3}{m}\right) - W^3(u_1, u_2, u_3)\right| \\
&\leq 2\left|u_1 - \frac{i_1}{m}\right| + 2\left|u_2 - \frac{i_2}{m}\right| + 2\left|u_3 - \frac{i_3}{m}\right| < \frac{6}{m} \leq \varepsilon,
\end{align*}
\]

which completes the proof. \(\square\)

As a consequence of Theorem 3.1, for \(m\) sufficiently large \((m \to \infty)\), the mass of \(W^3\) is distributed on the plane \(x + y + z = 2\) of \([0, 1]^3\) with subsets with arbitrarily large \(W^3\)-volume and subsets with arbitrarily small \(W^3\)-volume (see also \([13, 14]\)).

4. CONCLUSION

In this paper, we have defined a new family of proper 3-quasi-copulas for which \(W^3\) is the limit member of that family. Although our study is restricted to the trivariate case – for the sake of simplicity –, similar results can be obtained in higher dimensions – with a tedious algebra – by defining families of proper \(n\)-quasi-copulas in a similar manner. Let \(m\) be a natural number such that \(m \geq 2\), and suppose \(n \geq 3\). We divide \([0, 1]^n\) into \(m^n\) \(n\)-boxes, namely:

\[
B_{i_1i_2\ldots i_n} = \left[\frac{i_1-1}{m}, \frac{i_1}{m}\right] \times \left[\frac{i_2-1}{m}, \frac{i_2}{m}\right] \times \cdots \times \left[\frac{i_n-1}{m}, \frac{i_n}{m}\right],
\]

for all \(i_1, i_2, \ldots, i_n = 1, 2, \ldots, m\). Now, we distribute \(1/m\) of (positive) mass uniformly on each \(n\)-box \(B_{i_1i_2\ldots i_n}\) such that \(i_1+i_2+\cdots+i_n = (n-1)m+1\); \(-1/m\) of (negative) mass uniformly on each \(n\)-box \(B_{i_1i_2\ldots i_n}\) such that \(i_1+i_2+\cdots+i_n = (n-1)m+2\); and 0 on the remaining \(n\)-boxes. For example, if \(n = 4\), the number of 4-boxes with positive mass is \(\sum_{i=2}^{m+1} i\), and the number of 4-boxes with negative mass is \(\sum_{i=2}^{m+1} i - m\); then, the amount of positive and negative mass can be easily computed. Therefore, \(W^n\) – whose (infinite positive and infinite negative) mass is distributed on the set \(\{(x_1, x_2, \ldots, x_n) \in [0, 1]^n \mid x_1 + x_2 + \cdots + x_n = n-1\}\) – is the member limit of this family of proper \(n\)-quasi-copulas.

Finally, we note that the family introduced in this paper (and its generalization to \(n\)-dimensions) could be much interesting in applications, especially in the construction of aggregation operators to fitting a data set.
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