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RISK OBJECTIVES IN TWO–STAGE STOCHASTIC PROGRAMMING MODELS

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In applications of stochastic programming, optimization of the expected outcome need not be an acceptable goal. This has been the reason for recent proposals aiming at construction and optimization of more complicated nonlinear risk objectives.

We will survey various approaches to risk quantification and optimization mainly in the framework of static and two-stage stochastic programs and comment on their properties. It turns out that polyhedral risk functionals introduced in Eichorn and Römisch [17] have many convenient features. We shall complement the existing results by an application of contamination technique to stress testing or robustness analysis of stochastic programs with polyhedral risk objectives with respect to the underlying probability distribution. The ideas will be illuminated by numerical results for a bond portfolio management problem.

Keywords: two-stage stochastic programs, polyhedral risk objectives, robustness, contamination, bond portfolio management problem

AMS Subject Classification: 90C15, 91B28

1. INTRODUCTION

55 years ago, stochastic programming was introduced to deal with uncertain values of coefficients which were observed in applications of mathematical programming. These uncertainties were modeled as random and the assumption of complete knowledge of the probability distribution of random parameters became a standard.

The classical stochastic programming (SP) models aim at hedging against consequences of possible *realizations* of random parameters – *scenarios* – so that the final *expected* outcome or position is the best possible.

The common SP model

$$\min_{x \in \mathcal{X}} \mathrm{E}_P f(x, \omega) \tag{1}$$

is identified by the probability distribution P of random parameter ω with sample space Ω, by the set $\mathcal{X}$ whose elements x are interpreted as decisions and by a random objective $f = f(x, \omega)$, the loss or cost caused by decision x when scenario ω occurs. E_P is the expectation operator under P.

In most cases, it is sufficient to consider $\mathcal{X}$ and Ω as closed subsets of Euclidean spaces. On the other hand, the structure of f may be quite complicated, e. g.

for multistage problems. For convex $\mathcal{X}$, a frequent assumption is that f is lower semicontinuous and convex in x, i. e. f is a *convex normal integrand.*

Evidently, there are many choices of the "input" $[P, \mathcal{X}, f]$ and stability, robustness or worst-case analysis of the obtained optimal decision x^* is needed. We will allow only P to be subject to some variation, whereas $\mathcal{X}$ and f will be kept fixed; this covers also the frequent case of incomplete knowledge of probability distribution. We shall confine ourselves to the case when varying P does not depend on x, that is, when the accepted decision does not influence the probability distribution P. (The opposite case, when P in (1) may depend on decision x is also of importance and is known as stochastic program with *endogenous uncertainty*; also the case when $\mathcal{X}$ is not fixed but may depend on P occurs in practice, e. g. in problems with probability constraints.)

In the present paper we shall discuss stochastic programs with expectation criterion replaced by a much general concept. This is connected with the current tendency – to spell out explicitly the concern for *hedging against risks* connected with the chosen (not necessarily optimal) decision x. However, the concept of risk is hard to define. In practice, risk is frequently linked with the fact that the (random) outcome of a decision is not known precisely, that it may deviate from the expectation, etc. Here, the "outcome" may be the ex post observed value of the random objective $f(x^*, \omega)$ in (1) for a chosen decision $x^* \in \mathcal{X}$. Another way of looking at risk is related with an assessment of potential losses.

Similarly as the expected outcome criterion, also risk objectives/functionals will be allowed to depend on the probability distribution P, but not on individual scenarios.

There are various types of risk. Their definitions depend on the context, on decision maker's attitude and risk may posses many different attributes. To reflect risk in stochastic decision models, it is necessary to quantify it. Explicit quantification of risk appears in finance since 1952 in works of Markowitz, Roy and others. Another classical possibility is to apply a suitable (dis)utility function to express the risk attitude of the decision maker.

During the last decade, various functionals that describe risk, briefly *risk, deviation or acceptability functionals*, were introduced and their properties studied; see e. g. [20, 25, 26, 27, 31]. Their incorporating into the SP model makes the model much harder to solve. The passage to risk objectives asks also for designing suitable stress tests, for comparisons of alternative choices of risk functionals and of probability distributions and for further development of the related stability and robustness analysis methods, e. g. [19, 28]. Applicability of various output analysis techniques depends on the structure of the model, on the assumptions concerning the probability distribution, on the available data and on hardware and software facilities.

In applications, reasonable properties of risk functionals are requested. Monotonicity with respect to the pointwise partial ordering of losses and subadditivity of the risk functionals are evident requirements. Convexity allows to keep a relatively friendly structure of the problem both for computational and theoretical purposes. The polyhedral property, cf. CVaR introduced in [26], allows to rely on linear programming techniques for scenario-based stochastic linear programs with recourse and with polyhedral risk functionals, see e. g. [17, 23] and Section 2.1.

For static formulations of stochastic programs such as (1), the outcome is a decision dependent *one-dimensional* random variable. This is, in general, no more true in case of multistage stochastic programs. Modeling suitable multidimensional risk functionals and analyzing their properties has become an active area of research, cf. [25, 31].

We shall discuss various approaches to risk quantification in the framework of static or two-stage stochastic programs. A *two-stage stochastic linear program, SLP* is

$$\min_{x_1 \in \mathcal{X}_1} \left[c_1^\top x_1 + \mathrm{E}_P \varphi_1(x_1, \omega)\right] \tag{2}$$

where for a given $x_1 \in \mathcal{X}_1$ and scenario $\omega \in \Omega$, $\varphi_1(x_1, \omega)$ is the optimal value of the second-stage program

$$\text{minimize}_{x_2}\, c_2(\omega)^\top x_2 \text{ subject to } W(\omega)x_2 = h(\omega) - T(\omega)x_1,\, x_2 \geq 0. \tag{3}$$

Nonnegativity of the second-stage variables x_2 can be replaced by the requirement that x_2 belong to a convex polyhedral cone $\mathcal{X}_2$. The common assumption is that $\mathcal{X}_1$ is nonempty, convex polyhedral and c_1 is a deterministic vector.

For simplicity we will assume that *all infima are attained* and that *all expectations exist.* Consult recent books [21] or [30] for more general cases.

Alternatively (see [30]), the two-stage stochastic linear program (2)–(3) can be rewritten as

$$\min_{x_1, x_2(\cdot)} \mathrm{E}_P[c_1^\top x_1 + c_2(\omega)^\top x_2(\omega)] \tag{4}$$

subject to $x_1 \in \mathcal{X}_1$ and

$$W(\omega)x_2(\omega) = h(\omega) - T(\omega)x_1,\, x_2(\omega) \geq 0 \text{ a.s.} \tag{5}$$

We will briefly introduce two-stage stochastic linear programs with polyhedral risk functionals, cf. [17], at the place of the expectation in the objective function (4). The results of [10, 12, 16] will be then extended to robustness analysis and stress testing of polyhedral risk objectives and of their optimal values with respect to changes of probability distribution P or, for scenario-based problems, to stress testing with respect to out-of-sample scenarios. A bond portfolio management problem will serve to illustrate some of ideas.

2. RISK FUNCTIONALS

Consider again the basic stochastic decision problem (1) in which the random outcomes $f(x, \omega)$ are defined on $\mathcal{X} \times \Omega$, with $\mathcal{X} \subset \mathbb{R}^n$, $\Omega \subset \mathbb{R}^s$ nonempty, closed sets. For each $x \in \mathcal{X}$ they are measurable on $(\Omega, \mathcal{B})$. The too large linear space of all measurable functions is mostly replaced by some smaller linear subspace $\mathcal{Z}$ (e. g. by $L_p(\Omega, \mathcal{B}, P)$, $p \in [1, \infty]$).

Risk functional or risk measure is introduced as a proper function R which maps the space $\mathcal{Z}$ of allowable outcomes to the extended reals. It means that R assigns a value $R(z)$ to an uncertain outcome $z = f(x, \bullet)$ of decision x. The goal is to select decision x which minimizes the risk R.

Risk functionals should posses some natural properties, cf. [20]. Let us recall some of them.

Definition 1. For $z_1, z_2 \in \mathcal{Z}$ the pointwise partial ordering $z_1 \preceq z_2$ is defined by $z_2(\omega) \geq z_1(\omega) \, \forall \, \omega \in \Omega$.

Definition 2. A mapping $R : \mathcal{Z} \to \mathbb{R}$ is called risk functional if it fulfills the following conditions for all $z, z_1, z_2 \in \mathcal{Z}$:

If $z_1 \preceq z_2$ then $R(z_1) \leq R(z_2)$; *monotonicity wrt. pointwise ordering.*

If $m \in \mathbb{R}$, then $R(z + m) = R(z) + m$; *translation equivariance.*

Definition 3. Risk functional R is called *convex* if it satisfies for arbitrary $z_1, z_2 \in \mathcal{Z}$

$R(\lambda z_1 + (1 - \lambda) z_2) \leq \lambda R(z_1) + (1 - \lambda) R(z_2)$ for all $0 \leq \lambda \leq 1$

and is called *homogeneous* if R is a positively homogeneous function, i. e. $R(\lambda z) = \lambda R(z)$ for all $z \in \mathcal{Z}$ and $\lambda \geq 0$.

When risk functional R is positively homogeneous, convexity is equivalent to

subadditivity: $R(z_1 + z_2) \leq R(z_1) + R(z_2)$ for arbitrary $z_1, z_2 \in \mathcal{Z}$,

and in terminology of [3], R is called *coherent risk functional* or measure.

The concept of risk functional can be introduced without any reference to a probability measure on $(\Omega, \mathcal{B})$ and for *convex lower semicontinuous risk functionals, a dual representation* is possible and useful. With reference to [20, 25, 31] we will not elaborate here on the general case of infinite dimensional probability spaces $\mathcal{Z} = L_p(\Omega, \mathcal{B}, P)$, $p \in [1, +\infty)$ and will focus mainly on the scenario-based problems:

For *finite sample spaces*, say $\Omega = \{\omega^1, \dots, \omega^S\}$, the space $\mathcal{Z}$ is finite dimensional, $\mathcal{Z} = \mathbb{R}^S$. The probability distribution P is fully defined by probabilities p_s of scenarios $\omega^s \, \forall \, s$.

The above definitions imply that the classes of coherent and convex risk functionals are closed with respect to convex linear combinations.

Proposition 1. Let R_k, $k = 1, \dots, K$, be coherent (convex) risk functionals. Then $R = \sum_k \alpha_k R_k$ with $\alpha_k \geq 0 \, \forall \, k$, $\sum_k \alpha_k = 1$ is also a coherent (convex) functional.

This simple observation is behind the *spectral risk measures* introduced in [1].

Comment. In parallel, *deviation (risk) functionals*, cf. [27], were designed for applications to problems involving risk of uncertainty in the random position in the sense of its nonconstancy. Particular examples are variance and standard deviation, mean-absolute deviation and their one-side versions. The idea is to design a general class of functionals which (except for symmetry) obey axioms taken from properties of standard deviation, and relate them to risk functionals [27] under suitable additional conditions. Risk and deviation functionals can be then studied simultaneously within the framework of acceptability functionals as done e. g. in [25].

2.1. Polyhedral risk functionals

Contrary to expectation-type objectives, risk functionals need not be linear in the probability distribution P and their introduction means that complexity of the resulting risk minimizing problems increases essentially over complexity of the initial expectation-type stochastic program (1). Consequently, many known results will no more be valid. That's why [17] introduce a class of *convex polyhedral risk functionals*. Briefly, a polyhedral risk functional is defined as the optimal value of a certain stochastic linear program with fixed recourse; the only random coefficients appear on the right-hand sides.

Definition 4. (Eichorn and Römisch [17]) A risk functional R on $L_p(\Omega, \mathcal{B}, P)$, $p \in [1, +\infty)$, is called polyhedral if there exist $k_1, k_2 \in \mathbb{N}, d_1, w_1 \in \mathbb{R}^{k_1}, d_2, w_2 \in \mathbb{R}^{k_2}$, a nonempty polyhedral set $\mathcal{Y}_1 \subseteq \mathbb{R}^{k_1}$, and a polyhedral cone $\mathcal{Y}_2 \subseteq \mathbb{R}^{k_2}$ such that

$$R(z, P) = \inf\{d_1^\top y_1 + \mathrm{E}_P d_2^\top y_2(\omega)\} \tag{6}$$

subject to

$$y_1 \in \mathcal{Y}_1, y_2 \in L_p(\Omega, \mathcal{B}, P), y_2(\omega) \in \mathcal{Y}_2 \text{ a.s.} \tag{7}$$

$$w_1^\top y_1 + w_2^\top y_2(\omega) = z(\omega) \text{ a.s.} \tag{8}$$

for every $z \in L_p(\Omega, \mathcal{B}, P)$.

It is easy to show that the above definition includes as a special case the Conditional Value at Risk (cf. [26])

$$\mathrm{CVaR}_\alpha(z, P) = \min_v \left\{ v + \frac{1}{1-\alpha} \mathrm{E}_P[z(\omega) - v]^+ \right\}; \tag{9}$$

set in (6) – (8) $\mathcal{Y}_1 = \mathbb{R}$, $\mathcal{Y}_2 = \mathbb{R}_+^2$, $d_1 = 1$, $d_2^\top = (\frac{1}{1-\alpha}, 0)$, $w_1 = 1$, $w_2^\top = (1, -1)$.

Various important properties can be proved for convex polyhedral risk functionals at the place of the objective function in (1), cf. [17, 19], namely:

- Optimization of scenario-based two-stage stochastic programs with convex polyhedral risk objectives reduces to solution of linear programs.
- Generalization to multidimensional / multistage polyhedral risk functionals is possible.
- There are promising applications, e. g. [18] or Chapter 6 of [25].
- One may exploit various qualitative and quantitative sensitivity results with respect to changes of the probability distribution P that were derived for two-stage stochastic programs of the form (6) – (8).

As we shall demonstrate in the next Section, for polyhedral risk functionals and for their minimal values, also stress tests with respect to changes of the probability distribution P, e. g. with respect to out-of-sample scenarios, can be developed. To do so we shall firstly summarize the results valid for the one-period case and deal with two-stage stochastic programs with polyhedral risk objective. Recall that the random objectives, e. g. $f(x,\omega)$ in (1), are interpreted as *losses* connected with the first-stage decision if realization ω occurs.

According to Definition 4, one-period, static polyhedral risk functional $R(z,P)$ is the optimal value of a two-stage expectation-type stochastic program which in turn can be rewritten with a fixed set $\mathcal{Y}_1$ of the first-stage decision variables y_1. The objective function (6) is convex in z and linear in the probability distribution P.

The random objective $c_1^\top x_1 + c_2(\omega)^\top x_2(\omega)$ of the two-stage program, see (4), enters the constraint (8) in Definition 4 of the polyhedral risk functional $R(z,P)$ at the place of argument $z(\omega)$. We minimize then $R(c_1^\top x_1 + c_2(\omega)^\top x_2(\omega), P)$ with respect to $x_1 \in \mathcal{X}_1$ and the second-stage constraints (5) instead of minimizing the expectation $\mathrm{E}_P(c_1^\top x_1 + c_2(\omega)^\top x_2(\omega))$.

It turns out that the optimal value of the resulting two-stage stochastic program with a polyhedral risk objective is equal to the optimal value of the following *extended two-stage stochastic linear program*

$$\text{minimize } \{d_1^\top y_1 + \mathrm{E}_P d_2^\top y_2(\omega)\} \tag{10}$$

subject to

$$y_1 \in \mathcal{Y}_1, y_2 \in L_p(\Omega, \mathcal{B}, P), y_2(\omega) \in \mathcal{Y}_2 \text{ a.s.} \tag{11}$$

$$x_1 \in \mathcal{X}_1, x_2 \in L_p(\Omega, \mathcal{B}, P),\ W(\omega)x_2(\omega) = h(\omega) - T(\omega)x_1,\ x_2(\omega) \geq 0 \text{ a.s.} \tag{12}$$

$$w_1^\top y_1 + w_2^\top y_2(\omega) = c_1^\top x_1 + c_2(\omega)^\top x_2(\omega) \text{ a.s.} \tag{13}$$

see [17] – a result comparable with the "optimization short-cut" for minimization of CVaR in [26]. The resulting stochastic program is of the expectation type with random entries $c_2(\omega)$ in the extended recourse matrix.

Hence, two-stage stochastic linear programs with polyhedral risk objectives can be transformed into expectation-based stochastic linear programs and they can be rewritten in a form similar to (2) – (3) with *fixed* sets of decisions $\mathcal{X}_1, \mathcal{Y}_1$, independent of P.

If the recourse matrix $W \equiv W(\omega)$ in (12) is non-random, the special form of the extended recourse matrix in (12) – (13) allows to apply the stability results of [29] under standard assumptions of relatively complete recourse, dual feasibility and compactness of the first-stage solution set of (6) – (8), see Proposition 4.2 of [17]. For stochastic programs of this type, robustness analysis or stress testing of results with respect to perturbations of probability distribution P has been developed, cf. [7, 8, 10, 11] and will be briefly surveyed and applied in the next Section.

To complete this part, consider now *scenario-based problems.*

Let $\Omega = \{\omega^1, \dots, \omega^S\}$ and scenario probabilities are $p_1, \dots, p_S$. Then (6) – (8) reads

$$R(z,P) = \inf \left\{ d_1^\top y_1 + \sum_{s=1}^{S} p_s d_2^\top y_2^s \right\} \tag{14}$$

subject to

$$y_1 \in \mathcal{Y}_1, y_2^s \in \mathcal{Y}_2,\ w_1^\top y_1 + w_2^\top y_2^s = z^s\ \forall\, s. \tag{15}$$

Substituting for $z^s = c_1^\top x_1 + c_2^{s\top} x_2^s$ – the random objective in the scenario-based version of the two-stage stochastic program (3)–(4), we get the *minimum risk solution* $x_R(P)$ by solving the *extended* two-stage stochastic linear program

$$\varphi_R(P) := \inf_{x_1, y_1} \left\{ d_1^\top y_1 + \sum_{s=1}^{S} p_s d_2^\top y_2^s \right\} \tag{16}$$

subject to $x_1 \in \mathcal{X}_1,\ y_1 \in \mathcal{Y}_1,\ y_2^s \in \mathcal{Y}_2$

$$W^s x_2^s = h^s - T^s x_1, x_2^s \geq 0, \qquad w_1^\top y_1 + w_2^\top y_2^s = c_1^\top x_1 + c_2^{s\top} x_2^s \,\forall\, s \tag{17}$$

or, equivalently,

$$\varphi_R(P) = \inf_{x_1, y_1} \left\{ d_1^\top y_1 + \sum_{s=1}^{S} p_s \varphi^s(x_1, y_1) \right\} \tag{18}$$

subject to $x_1 \in \mathcal{X}_1$, $y_1 \in \mathcal{Y}_1$ and with

$$\begin{aligned} \varphi^s(x_1, y_1) \ = \ \inf \Big\{ & d_2^\top y_2^s \,|\, W^s x_2^s = h^s - T^s x_1,\ x_2^s \geq 0, y_2^s \in \mathcal{Y}_2,\ w_1^\top y_1 + w_2^\top y_2^s \\ & = c_1^\top x_1 + c_2^{s\top} x_2^s \Big\}. \end{aligned} \tag{19}$$

Multiperiod/multistage polyhedral risk functionals have been defined in the same spirit – as the optimal values of certain multistage stochastic linear programs. They aim to capture the multiple dimensions of the risk which cannot be always tied only with the outcome at the horizon; see Definition 3.7 of [17] and Section 3.3.5 of [25].

3. ROBUSTNESS ANALYSIS OF RISK OBJECTIVES

3.1. Contamination technique

Our approach is based on the contamination technique for stochastic programs (1). It was developed in a series of papers as one of tools for analysis of robustness of the optimal value with respect to deviations from the assumed probability distribution P and/or its parameters; e. g. [7, 8, 10].

For construction of contamination bounds, let us repeat the formulation of the stochastic program.

$$\min_{x \in \mathcal{X}} F(x, P) := \mathrm{E}_P f(x, \omega) = \int_\Omega f(x, \omega) P(\mathrm{d}\omega) \tag{20}$$

with $\mathcal{X}$ independent of P, i. e. the form of (1).

Via contamination, robustness analysis with respect to changes in P gets reduced to a much simpler analysis with respect to a scalar parameter λ :

Assume that (20) was solved for a probability distribution P and denote $\varphi(P)$ the optimal value and $\mathcal{X}^*(P)$ the set of optimal solutions. Possible changes in probability distribution P are modeled using contaminated distributions P_λ,

$$P_\lambda := (1 - \lambda) P + \lambda Q,\ \lambda \in [0, 1]$$

with Q another *fixed* probability distribution. Limiting the analysis to the selected direction only, the results are directly applicable but they are less general than quantitative stability results with respect to arbitrary (but small) changes in P summarized e. g. in [28].

The objective function in (20) is linear in P, hence

$$F(x,\lambda) := \int_\Omega f(x,\omega)P_\lambda(\mathrm{d}\omega) = (1-\lambda)F(x,P) + \lambda F(x,Q)$$

is linear in λ. Suppose that the stochastic program (20) has an optimal solution for all considered distributions P_λ, $0 \le \lambda \le 1$. Then the optimal value function

$$\varphi(\lambda) := \min_{x\in\mathcal{X}} F(x,\lambda)$$

is concave on $[0,1]$ which implies its continuity and existence of directional derivatives in $(0,1)$. Continuity at the point $\lambda = 0$ is a property related with stability results for the stochastic program in question. In general, one needs a nonempty, bounded set of optimal solutions $\mathcal{X}^*(P)$ of the initial stochastic program (20). This assumption together with stationarity of derivatives $\frac{\mathrm{d}F(x,\lambda)}{\mathrm{d}\lambda} = F(x,Q) - F(x,P)$ are used to derive the form of the directional derivative

$$\varphi'(0^+) = \min_{x\in\mathcal{X}^*(P)} F(x,Q) - \varphi(0) \tag{21}$$

which enters the upper bound for the optimal value function $\varphi(\lambda)$

$$\varphi(0) + \lambda\varphi'(0^+) \ge \varphi(\lambda) \ge (1-\lambda)\varphi(0) + \lambda\varphi(1),\ \lambda \in [0,1]; \tag{22}$$

see [7, 10] and references therein.

If $x^*(P)$ is the *unique* optimal solution of (20), $\varphi'(0^+) = F(x^*(P),Q) - \varphi(0)$, i. e. the *local change of the optimal value function caused by a small change of* P *in direction* $Q - P$ *is the same as that of the objective function at* $x^*(P)$. If there are multiple optimal solutions, each of them leads to an upper bound $\varphi'(0^+) \le F(x(P),Q) - \varphi(0)$, $x(P) \in \mathcal{X}^*(P)$. Consequently, contamination bounds (22) are relaxed to

$$(1-\lambda)\varphi(P) + \lambda F(x(P),Q) \ge \varphi(P_\lambda) \ge (1-\lambda)\varphi(P) + \lambda\varphi(Q) \tag{23}$$

valid with an arbitrary $x(P) \in \mathcal{X}^*(P)$ and $\lambda \in [0,1]$.

Contamination bounds (22), (23) are global bounds which hold true for all $\lambda \in [0,1]$. They *quantify* the change in the optimal value due to the considered perturbations of (20) which is a true robustness result. The probability distribution P_λ can be also understood as a result of contamination of Q by P and alternative contamination bounds can be constructed accordingly. They differ in the upper bound, now constructed as

$$\lambda\varphi(Q) + (1-\lambda)F(x(Q),P).$$

Its joint application together with the initial upper bound (23) leads to a substantially tighter bound.

Concavity of the optimal value function $\varphi(\lambda)$ is important for constructing the bounds. It does not hold true, in general, when the set $\mathcal{X}$ depends on the probability distribution P. In such cases and under additional assumptions, only local stability results can be proved, cf. [8]. On the other hand the results can be generalized to objective functions $F(x,P)$ convex in x and *concave* in P, see [10, 11]. To derive these generalizations, it is again necessary to analyze persistence and stability properties of the parametrized problems $\min_{x\in\mathcal{X}} F(x,P_\lambda)$ and to derive the form of the directional derivative. Under the assumptions listed above, the optimal value function $\varphi(\lambda)$ remains concave on $[0,1]$. Additional assumptions are needed to get the existence of the derivative

$$\varphi'(0^+) = \min_{x\in\mathcal{X}^*(P)} \frac{\mathrm{d}}{\mathrm{d}\lambda} F(x,P_\lambda)|_{\lambda=0^+},$$

see e. g. [10] and references therein.

For expected (dis)utility models, stress testing and robustness analysis via contamination with respect to changes in the probability distribution P is a straightforward procedure because the objective function is linear in P. It can be extended to stress testing for convex risk or deviation functionals via contamination: When the risk or deviation functionals are *concave with respect to the probability distribution* P, they are concave with respect to the parameter λ of the contaminated probability distributions $(1-\lambda)P+\lambda Q$. Hence, contamination bounds for their value and for optimal value of the risk minimization models with respect to changes in P can be obtained, provided that the directional derivatives exist. This applies also to the mean-variance model, cf. [12].

Particularly, stress testing and robustness analysis of polyhedral risk functionals and of their minimal values follow similarly as for CVaR in [16].

3.2. Contamination for static polyhedral risk functionals

To get the form (20), we rewrite the polyhedral risk functional $R(z,P)$ defined by (6)–(8):

$$R(z,P) = \min_{y_1\in\mathcal{Y}_1} \{d_1^\top y_1 + \mathrm{E}_P \vartheta(y_1,\omega)\} \tag{24}$$

where for a given $y_1 \in \mathcal{Y}_1$ and scenario $\omega \in \Omega$, $\vartheta(y_1,\omega)$ is the optimal value of the second-stage program

$$\text{minimize } d_2^\top y_2 \text{ subject to } w_1^\top y_1 + w_2^\top y_2 = z(\omega),\ y_2 \in \mathcal{Y}_2; \tag{25}$$

compare with (2)–(3). Notice that for CVaR_α, $\vartheta(y_1,\omega) = \frac{1}{1-\alpha}[z(\omega)-y_1]^+$.

Denote

$$G(y_1,P) = d_1^\top y_1 + \mathrm{E}_P \vartheta(y_1,\omega)$$

the objective function in (24). It is linear in P, hence for the contaminated probability distribution

$$P_\lambda = (1-\lambda)P + \lambda Q,\ \lambda\in[0,1],$$

the objective function $G(y_1,\lambda) := G(y_1,P_\lambda)$ is linear in λ. Its optimal value

$$R(z,\lambda) = R(z,P_\lambda) = \min_{y_1\in\mathcal{Y}_1} G(y_1,P_\lambda)$$

is a concave function of λ. This allows to construct the contamination bounds (22)

$$R(z,P)+\lambda R'(z,0^+)\geq R(z,P_\lambda)\geq(1-\lambda)R(z,P)+\lambda R(z,Q). \tag{26}$$

The directional derivative needed in (26) follows the usual pattern (21):

$$R'(z,0^+)=\min_{y_1\in\mathcal{Y}_1^*(P)}G(y_1,Q)-R(z,P) \tag{27}$$

with the set $\mathcal{Y}_1^*(P)$ of optimal first-stage solutions $y_1(P)$ of (24)−(25) obtained for probability distribution P. To avoid minimization in (27), one may again choose an *arbitrary* optimal solution $y_1(P)\in\mathcal{Y}_1^*(P)$ and to get an upper bound

$$R'(z,0^+)\leq G(y_1(P),Q)-R(z,P)$$

for the derivative in (27). To evaluate this bound means to get in addition the expected second-stage costs $G(y_1,Q)$ for an already found optimal first-stage solution $y_1(P)$ with respect to the alternative probability distribution Q.

3.3. Contamination of minimal static polyhedral risk

The minimal risk follows by solving the extended two-stage SLP (10)−(13) or by minimizing with respect to (12) the polyhedral risk functional $R(z,P)$ defined by (6)−(8); just the right-hand side $z(\omega)$ is replaced by the random objective $c_1^\top x_1+c_2(\omega)^\top x_2(\omega)$ of the two-stage SLP. See [17] for details.

For both possibilities, contamination bounds for the minimal risk value can be constructed (and are equal). Indeed, the extended two-stage SLP can be rewritten as

$$\varphi_R(P):=\min_{x_1\in\mathcal{X}_1,y_1\in\mathcal{Y}_1}[d_1^\top y_1+\mathrm{E}_P\varrho(x_1,y_1,\omega)] \tag{28}$$

with

$$\varrho(x_1,y_1,\omega)=\min d_2^\top y_2$$

subject to constraints (11)−(13) to be satisfied with a fixed $x_1\in\mathcal{X}_1$, $y_1\in\mathcal{Y}_1$ and scenario ω.

Hence, to get the contamination bounds (22) for $\varphi_R(\lambda):=\varphi_R(P_\lambda)$ it is necessary to solve (28) for the contaminating probability distribution Q to get the minimal risk $\varphi_R(Q)$ and to evaluate the derivative $\varphi_R'(0^+)$ or its upper bound. The additional numerical effort can be reduced again to evaluating the objective function in (28) at an arbitrary optimal solution $x_{1R}(P)$, $y_{1R}(P)$ of (28) obtained for the initial probability distribution P but taking the expectation with respect to the contaminating probability distribution Q. The result is

$$d_1^\top y_{1R}(P)+\mathrm{E}_Q\varrho(x_{1R}(P),y_{1R}(P),\omega).$$

For scenario-based problems with Q carried by scenarios ω^σ, $\sigma=1,\dots,S'$, with probabilities q_σ one has to compute the minimal second-stage costs ϱ for all additional scenarios ω^σ

$$\varrho^\sigma(x_{1R}(P),y_{1R}(P)):=\min d_2^\top y_2^\sigma$$

subject to

$$W^\sigma x_2^\sigma = h^\sigma - T^\sigma x_{1R}(P),\ x_2^\sigma \geq 0,\ y_2^\sigma \in \mathcal{Y}_2,\ w_1^\top y_{1R}(P) + w_2^\top y_2^\sigma = c_1^\top x_{1R}(P) + c_2^{\sigma\top} x_2^\sigma$$

and their weighted average

$$\sum_{\sigma=1}^{S'} q_\sigma \varrho^\sigma(x_{1R}(P), y_{1R}(P)).$$

An extension of these approaches to robustness analysis of multiperiod polyhedral risk functionals is more involved. Some possibilities based on [9, 13] are presented in [14].

4. APPLICATION: BOND PORTFOLIO MANAGEMENT PROBLEM

The main purpose of the considered bond portfolio management problem is to preserve the value of a bond portfolio of a risk averse or risk neutral institutional investor over time. For simplicity, neither liabilities nor external cash flows are taken into account and the interest rate evolution is assumed to be the only factor which drives the prices of the considered bonds. It means that given a sequence of equilibrium future short term interest rates r_t valid for the time interval $(t, t+1], t = 0, \ldots, T-1$, the fair price of the jth bond at time t just after the coupon was paid equals the total cash flow $f_{j\tau}, \tau = t+1, \ldots, T$, generated by this bond in subsequent time instances discounted to t:

$$P_{jt}(r) = \sum_{\tau=t+1}^{T} f_{j\tau} \prod_{h=t}^{\tau-1} (1 + r_h)^{-1} \tag{29}$$

where T is greater or equal to the time to maturity.

The future interest rates are not known with certainty and are modeled as random. We assume that (as a result of a scenario generation procedure) their discrete probability distribution P is carried by a finite number of scenarios – T-dimensional vectors r^s of the short rates $r_t^s, t = 0, \ldots, T-1,\ s = 1, \ldots, S$, with probabilities $p_s > 0, s = 1, \ldots, S, \sum_s p_s = 1$; r_0 (the rate valid in the first period) is known.

The problem is formulated below as a multiperiod two-stage scenario-based SLP with complete random recourse (e. g. [4] or Chapter 6 of [15]).

We denote

$j = 1, \ldots, J$, indices of the considered bonds and T_j the dates of their maturities; $T = \max_j T_j$.

$t = 0, \ldots, T_0 < T$ the considered discretization of the planning horizon T_0;

$a_j \geq 0$ the initial known holdings (in face value) of bond j;

$a_0 \geq 0$ the initial known holding in riskless asset;

f_{jt}^s cash flow generated from bond j at time t under scenario s expressed as a fraction of its face value;

ξ_{jt}^s and η_{jt}^s are the selling and purchasing prices of bond j at time t for scenario s obtained from the corresponding fair prices (29) by adding the accrued interest A_{jt}^s and by subtracting or adding scenario independent transaction costs and spread; the initial prices ξ_{j0} and η_{j0} are known, i. e. scenario independent;

x_j/u_j are face values of bond j purchased/sold at the beginning of the planning period, at $t = 0$; x_{jt}^s/u_{jt}^s are the corresponding values for period t under scenario s.

h_{j0} is the face value of bond j held in portfolio after the initial decisions x_j, u_j have been made; h_{jt}^s are the corresponding holdings for period t under scenario s.

The first-stage decision variables x_j, u_j, h_{j0} are nonnegative,

$$u_j + h_{j0} = a_j + x_j \quad \forall\, j, \tag{30}$$

$$h_0^+ + \sum_j \eta_{j0} x_j = a_0 + \sum_j \xi_{j0} u_j \tag{31}$$

where the nonnegative variable h_0^+ denotes the surplus. We assume a *positive market value of the initial portfolio*, $W_0 = a_0 + \sum_j \xi_{j0} a_j > 0$, which implies that the *set of the feasible first-stage solutions is nonempty and bounded.*

Provided that an initial trading strategy determined by feasible scenario independent first-stage decision variables x_j, u_j, h_0^+ and h_{j0} for all j has been accepted, the subsequent second-stage scenario-dependent decisions have to be made in an optimal way regarding the goal of the model, i. e. to maximize the final value of the portfolio subject to linear constraints on conservation of holdings and rebalancing the portfolio

$$\text{maximize} \quad W_{T_0}^s := \sum_j \xi_{jT_0}^s h_{jT_0}^s + h_{T_0}^{+s} \tag{32}$$

subject to

$$h_{jt}^s + u_{jt}^s = h_{j,t-1}^s + x_{jt}^s \,\forall\, j,\, 1 \le t \le T_0, \tag{33}$$

$$\sum_j \xi_{jt}^s u_{jt}^s + \sum_j f_{jt} h_{j,t-1}^s + (1 - \delta + r_{t-1}^s) h_{t-1}^{+s} = \sum_j \eta_{jt}^s x_{jt}^s + h_t^{+s},\ 1 \le t \le T_0, \tag{34}$$

$$x_{jt}^s \ge 0, u_{jt}^s \ge 0, h_{jt}^s \ge 0, h_t^{+s} \ge 0 \,\forall\, j,\, 1 \le t \le T_0, \tag{35}$$

with $h_0^{+s} = h_0^+, h_{j0}^s = h_{j0} \,\forall\, j$; the auxiliary variables h_t^{+s} describe investments in the riskless asset (cash) for period t under scenario s. Positive values of parameter δ account for difference between the returns for bonds and for cash.

With $W_{T_0}(x, u, h_0, h_0^+; r^s)$ the corresponding *maximal value* of the second-stage scenario r^s subproblem (32) – (35) for fixed feasible first-stage decisions h_0^+, x_j, u_j, h_{j0} for $j = 1, \dots, J$, the full stochastic program can be written in the form (1):

- The vector of decision variables $x \longleftrightarrow [x, u, h_0, h_0^+]$,
- the set of feasible solutions $\mathcal{X}$ is defined by nonnegativity constraints on all first-stage variables and by constraints (30) – (31),
- the random objective function $f(x, \omega) \longleftrightarrow -W_{T_0}(x, u, h_0, h_0^+; r)$.

(The symbol $\longleftrightarrow$ relates the notation used in Section 1 to that used in the application.)

The function $W_{T_0}(\bullet; r)$ is piece-wise linear, concave in $[x, u, h_0, h_0^+]$ for any interest rate scenario $r \in \mathbb{R}^T$. The considered stochastic program $\min_{x \in \mathcal{X}} \mathrm{E}_P f(x, \omega)$ is

$$\text{minimize} \quad -\sum_{s=1}^{S} p_s W_{T_0}(x, u, h_0, h_0^+; r^s) \tag{36}$$

subject to nonnegativity constraints on all variables and subject to (30) – (31).

The initial goal – to preserve the value of the portfolio over time – was formulated as maximization of the expected wealth at the horizon. Alternatively, one may relate the outcome to the (known) initial value $W_0 = a_0 + \sum_j \xi_{j0} a_j$ of the portfolio and to solve

$$\max \frac{1}{W_0} \sum_{s=1}^{S} p_s W_{T_0}(x, u, h_0, h_0^+; r^s) \text{ or } \max \frac{1}{W_0} \left[\sum_{s=1}^{S} p_s W_{T_0}(x, u, h_0, h_0^+; r^s) - W_0 \right],$$

or

$$\min -\frac{1}{W_0} \sum_{s=1}^{S} p_s W_{T_0}(x, u, h_0, h_0^+; r^s) \text{ or } \min \frac{1}{W_0} \left[W_0 - \sum_{s=1}^{S} p_s W_{T_0}(x, u, h_0, h_0^+; r^s) \right]$$

over (30) – (31) and nonnegativity constraints.

To take into account risks, a suitable utility or disutility function may be applied in (36). This traditional approach suffers from the fact that it may be difficult to assess the form of the utility function which reflects adequately decision maker's risk attitude. Moreover, it introduces nonlinearities into the objective function; see [4] for a discussion and for results of numerical experiments.

To apply in (36) the mean-variance criterion to the best scenario-dependent outcomes $-W_{T_0}(x, u, h_0, h_0^+; r^s)$ is not a satisfactory choice. Among others, it leads to problems with piece-wise convex linear-quadratic objective function which need not be convex; cf. [2]. Robust optimization objective function suggested in [22] applies the mean-variance criterion to *feasible* scenario outcomes $-W_{T_0}^s := -\sum_j \xi_{jT_0}^s h_{jT_0}^s - h_{T_0}^{+s}$; hence, it preserves convexity properties of the resulting deterministic program. These two criteria may be related to deviation functionals of [27].

Another possibility is to replace the expectation in (36) by a risk functional which preserves convexity. Polyhedral risk functionals fulfil this requirement and, similarly as for the scenario-based stochastic linear programs with an expectation-type objective, they lead to a linear program. Using CVaR criterion or a mean-CVaR objective is a special choice.

The minimum risk rebalancing strategy $[x, u, h_0, h_0^+]$ of the bond portfolio management problem is then obtained by solving the extended two-stage stochastic linear program

$$\text{minimize } \{d_1^\top y_1 + \sum_{s=1}^{S} p_s d_2^\top y_2^s\}$$

subject to (30), (31), (33) – (35), $y_1 \in \mathcal{Y}_1$, $y_2^s \in \mathcal{Y}_2 \, \forall \, s$ and the coupling constraints

$$w_1^\top y_1 + w_2^\top y_2^s = -\sum_j \xi_{jT_0}^s h_{jT_0}^s - h_{T_0}^{+s} \quad \forall \, s.$$

The last model was applied to the historical input data from the Italian bond market for equiprobable interest rates scenarios from scenario set Part 8, cf. [5], and with coefficients (9) corresponding to CVaR risk functional for various values of α. The planning horizon was set to 1 year and monthly time discretization was used. The considered date October 3, 1994 allows to include also puttable bonds in the portfolio; moreover, none of the bonds expires within the investment horizon. The initial portfolio market value equals 10485.

With the CVaR objective function for $\alpha \geq 0.9$, the first-stage optimal decision shows clearly the tendency to keep cash, contrary to investing all to puttable bond CTO13212 obtained for expected wealth maximization; the minimal CVaR_α value equals -11436 and the expected wealth decreased for 1.15 %. With an additional upper limit on cash holdings, the decrease of the expected wealth was lower.

To evaluate CVaR of the portfolio rebalanced according to the optimal first-stage decision $[x(P), u(P), h_0(P), h_0^+(P)]$ of (30) – (31), (36), the optimal scenario-dependent outcomes $-W_{T_0}(x(P), u(P), h_0(P), h_0^+(P); r^s)$ are used as the right-hand sides z^s in (15) or in (9).

In our example, we use equiprobable values -11909, -11778, -11640, -11426, -11419, -11386, -11354, -11336 of $-W_{T_0}(x(P), \; u(P), \; h_0(P), \; h_0^+(P); r^s)$ at the place of z^s, $s = 1, \dots, 8$. The $\mathrm{CVaR}_{0.9}(P)$ of the rebalanced portfolio equals -11336 and is attained at $y_1(P) = -11336$.

Stress testing of this CVaR_α value with respect to an out-of-sample scenario r^* and the related loss z^* means

1. to get CVaR_α for the degenerated probability distribution $Q = \delta\{r^*\}$, i. e. to use $\mathrm{CVaR}_\alpha(Q) = z^*$, $y_{1R}(Q) = z^*$ in the lower bound $(1-\lambda)y_1(P) + \lambda z^* = -(1-\lambda)11336 + \lambda z^*$,

2. to evaluate the performance of the rebalanced portfolio $[h_0(P), h_0^+(P)]$ along scenario r^* which appears in the derivative (27). The result is the left upper bound $-11336 + 10\lambda[11336 + z^*]^+$.

For the right upper bound, one starts with the optimal solution $y_1(Q) = z^*$ of the deterministic CVaR problem (14) – (15) with only one coupling constraint. The next step is to get

$$\vartheta(z^*, z^s) = \min \left\{ \frac{1}{1-\alpha} y_{21}^s \,|\, z^* + y_{21}^s - y_{22}^s = z^s, \, y_{21}^s, \, y_{22}^s \geq 0 \right\} = \frac{1}{1-\alpha}[z^s - z^*]^+$$

for all initial scenarios z^s and their average $\sum_s \frac{p_s}{(1-\alpha)}[z^s - z^*]^+$. Finally,

$$G(z^*, P) = z^* + \sum_s \frac{p_s}{(1-\alpha)}[z^s - z^*]^+$$

enters the formula for the directional derivative at Q in the direction of $Q - P$ similarly as done in (27).

Evidently, the value of the directional derivative depends on the position of z^* with respect to the initial z^s values. In our simple case, the discussion based on Proposition 8 and Corollary 9 of [26] is straightforward.

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