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ON DISTRIBUTIONS OF ORDER STATISTICS FOR ABSOLUTELY CONTINUOUS COPULAS WITH APPLICATIONS TO RELIABILITY

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Performance of coherent reliability systems is strongly connected with distributions of order statistics of failure times of components. A crucial assumption here is that the distributions of possibly mutually dependent lifetimes of components are exchangeable and jointly absolutely continuous. Assuming absolute continuity of marginals, we focus on properties of respective copulas and characterize the marginal distribution functions of order statistics that may correspond to absolute continuous and possibly exchangeable copulas. One characterization is based on the vector of distribution functions of all order statistics, and the other concerns the distribution of a single order statistic.

Keywords: coherent system, order statistic, copula, exchangeable distribution, absolute continuous distribution, absolute continuous copula

AMS Subject Classification: 60E05, 62G30, 62H05, 62N05

1. INTRODUCTION

The paper has two main goals. One is to formulate the necessary and sufficient conditions for a sequence of distribution functions to be a family of distribution functions of order statistics corresponding to an absolutely continuous copula. The other consists in characterizing a single distribution function as the distribution function of a given order statistic corresponding to an absolutely continuous copula. The motivation for dealing with this problem is closely connected with the ongoing research in the area of coherent systems which are considered in reliability theory and other branches of applied probability.

In the reliability theory, coherent systems are used for describing complex technical systems composed of simple elements whose working status affects the performance of the system. For the comprehensive theory, we refer to books by Barlow and Proschan [1, 2], and Lai and Xie [15]. Coherent systems were also considered in other branches of applied probability, e. g. they represent pension plans for groups of people at the same age or the prices of the synthetic CDOs (collateralized debt obligations). The so called k-out-of-n systems play a central role in the theory. The k-out-of-n coherent system is one that works iff do so at least k of its components. In other words, if $X_1, \dots, X_n$ are the random lifetimes of the components and

$X_{1:n}, \dots, X_{n:n}$ denote the respective order statistics, then the failure time of the k-out-of-n system coincides with the kth greatest order statistic $X_{n+1-k:n}$. Samaniego [22] determined a convenient representation for the distribution function of the failure time T of an arbitrary coherent system composed of n independent components that have identically continuously distributed lifetimes $X_1, \dots, X_n$ with a common distribution function F as a convex combination of distributions of order statistics

$$\Pr(T \le x) = \sum_{i=1}^{n} p_i \Pr(X_{i:n} \le x) \tag{1.1}$$

$$= \sum_{i=1}^{n} p_i \sum_{j=i}^{n} \binom{n}{j} F^j(x)[1 - F(x)]^{n-j}, \tag{1.2}$$

where

$$p_i = \Pr(T = X_{i:n}), \qquad i = 1, \dots, n. \tag{1.3}$$

The vector $\mathbf{p} = (p_1, \dots, p_n)$ is called the Samaniego signature of the system. The signature depends on the structure function of the system, but is independent of the distribution function F of the components. Evidently, there is a finite number of coherent systems of fixed size n, and so there is a finite number of Samaniego signatures $(p_1, \dots, p_n)$ of size n. Boland and Samaniego [4] introduced a convenient notion of mixed systems with representation (1.1) and arbitrary mixture coefficients $p_i \ge 0$, $i = 1, \dots, n$, summing up to one. The mixed system can be interpreted in the following way: one of the k-out-of-n systems, $k = 1, \dots, n$, is randomly chosen, and the probability of choosing the kth one is p_{n+1-k}. In particular, every coherent system can by represented as a mixture of k-out-of-n systems.

In many practical problems, the assumptions of independence of system components is unrealistic, and is replaced by exchangeability. This means that the components are actually identical, but their lifetimes are dependent, e. g., the failure of a component causes an increased burden on the others and results in shortening their lifetimes. Navarro and Rychlik [17] formally proved that the representations (1.1) and (1.3) are also valid in the case of mutually dependent components with exchangeable absolutely continuous joint distributions (such a possibility was earlier noticed by Kochar et al. [14]). Clearly, in this case the representation of the order statistics distributions presented in (1.2) should be replaced by more general formulae depending on the joint distribution of $X_1, \dots, X_n$. A further generalization is due to Navarro et al. [18], where the assumption of continuity of the distribution is dropped. The representation (1.1) still holds, but the coefficients cannot be interpreted by means of (1.3) in general.

In the paper we study the question of characterizing the marginal distributions of order statistics which are based on dependent samples with absolutely continuous exchangeable joint distributions, or more generally, with identical marginal distributions. Following David and Nagaraja ([5], Section 5.3, p. 99), and Galambos [9], we write the distribution function of the mth order statistic $X_{m:n}$ from the sample $X_1, \dots, X_n$ as

$$\Pr(X_{m:n} \le x) = \sum_{j=m}^{n} (-1)^{j-m} \binom{j-1}{m-1} \sum_{1 \le i_1 < \dots < i_j \le n} \Pr(X_{i_1} \le x, \dots, X_{i_j} \le x). \tag{1.4}$$

If $X_1, \ldots, X_n$ are exchangeable, (1.4) simplifies to

$$\Pr(X_{m:n} \leq x) = \sum_{j=m}^{n} (-1)^{j-m} \binom{j-1}{m-1} \binom{n}{j} \Pr(X_1 \leq x, \ldots, X_j \leq x). \tag{1.5}$$

Obviously both (1.4) and (1.5) can be represented by means of the joint distribution function H, and so by means of the common marginal distribution F and the copula C, which describes all the dependencies among the variables. The distribution function H of identically distributed random variables is absolutely continuous if the common marginal distribution F is absolutely continuous and so is the respective copula C. The first condition is usually easy to verify and in many practical situations can be simply assumed. Here we concentrate on the problem of absolute continuity of copula C which in the case of identical marginals can be determined by

$$H(x_1, \ldots, x_n) = C(F(x_1), \ldots, F(x_n)). \tag{1.6}$$

For convenience, here and later on, we consider the copula as a cumulative distribution function defined on $\mathbb{R}^n$ and supported on the cube $[0,1]^n$ with all the standard uniform one-dimensional marginals. Representation (1.6), due to Sklar [23], and many other properties of copula functions can be found in the monographs by Joe [13] and Nelsen [19]. Note that in the identically distributed case with the continuous marginal distribution function F, the order statistics from the copula C and the original distribution function H are simply related. If $U_1, \ldots, U_n$ and $X_1, \ldots, X_n$ have the joint distribution functions C and H, respectively, then $U_{1:n}, \ldots, U_{n:n}$ and $F(X_{1:n}), \ldots, F(X_{n:n})$ have identical distributions. Therefore we further analyze distributions of order statistics based on dependent standard uniform samples with a common distribution being a copula.

In the present paper, we answer two questions. One is to give necessary and sufficient conditions for a vector of distribution functions $(G_1, \ldots, G_n)$ to be the marginal distribution functions of consecutive order statistics based on dependent standard uniform random variables which have a joint absolutely continuous distribution. The other is an analogous characterization of the marginal distribution function of a single order statistic. It occurs that the same conditions characterize the narrower class of absolutely continuous exchangeable copulas.

The theory of similar characterizations is most developed for the greatest order statistics whose distribution function has the simple form

$$\Pr(U_{n:n} \leq t) = \delta(t) = C(t, \ldots, t),$$

and is called the diagonal section of copula C. The characterization of all possible diagonal sections of copulas can be found in Nelsen [19] (Subsection 3.2.6) for the case $n = 2$. Various constructions of copulas with prescribed diagonals were presented in literature (see, for example, Nelsen [19], Durante et al. [7, 8]), but most of them are singular. Genest et al. [10] constructed absolutely continuous copulas whose diagonal sections satisfy $\delta(t) = t$ at a finite number of points $0 < t < 1$ at most. Jaworski [12] showed that such a construction is possible iff the Lebesgue measure of points where the relation holds is zero (the special case $n = 2$ was examined in Durante and Jaworski [6]).

Distribution functions of other order statistics $U_{m:n}$, $m < n$, can be expressed by means of copula in more complicated ways. Rychlik [20] presented characterizing conditions on a vector of distribution functions for being the marginal distribution functions of order statistics of dependent random variables with a given common marginal distribution function F. Similar conditions for the distribution of a single order statistic were presented in Rychlik [21]. These results are presented in Theorems 1 and 3, respectively, of Section 2. Their proofs were constructive, and the constructions led to singular distributions of the samples. In Theorems 2 and 4 of Section 2 we describe some additional conditions which allow us to construct absolutely continuous joint distributions with prescribed distributions of order statistics. The respective characterizations of distributions are proved in Sections 4 and 5 which is preceded by a section containing auxiliary results.

We finally note that there are also other examples of applications of copula approach to reliability theory, which are not directly connected with coherent systems, see, e. g., Bassan and Spizzichino [3].

2. CHARACTERIZATION RESULTS

Suppose that standard uniform random variables $U_1, \ldots, U_n$ have a joint distribution function C. Let $U_{1:n} \leq \cdots \leq U_{n:n}$ stand for the respective order statistics. The distribution function of $U_{m:n}$ can be written as the linear combination of diagonal sections of multidimensional marginals of C.

$$G_m(s) = \Pr(U_{m:n} \leq s) = \sum_{j=m}^{n} (-1)^{j-m} \binom{j-1}{m-1} \sum_{1 \leq i_1 < \cdots < i_j \leq n} C(\mathbf{s}_{i_1,\ldots,i_j}), \qquad (2.7)$$

$m = 1, \ldots, n$, where $\mathbf{s}_{i_1,\ldots,i_j} = (s_1, \ldots, s_n) \in \mathbb{R}^n$ are such that $s_i = s$ for $i = i_1, \ldots, i_j$, and $s_i = 1$ otherwise (cf. (1.4)) i. e.

$$C(\mathbf{s}_{i_1,\ldots,i_j}) = C(1, \ldots, 1, s, \ldots, s, 1, \ldots).$$

If $U_1, \ldots, U_n$ are exchangeable, then (2.7) have simpler forms

$$G_m(s) = \sum_{j=m}^{n} (-1)^{j-m} \binom{j-1}{m-1} \binom{n}{j} C(\mathbf{s}_{1,\ldots,j}), \quad m = 1, \ldots, n, \qquad (2.8)$$

(cf. (1.5)). We say that (2.7) (and (2.8) in the exchangeable case) are the distribution functions of order statistics corresponding to copula C. Rychlik [20] characterized the distributions $G_1, \ldots, G_n$ as follows:

Theorem 1. (Rychlik [20], Lemma 1) Distribution functions $G_1, \ldots, G_n$ are the distribution functions of consecutive order statistics corresponding to a copula iff they satisfy two conditions

$$G_k(s) \geq G_{k+1}(s), \qquad 0 \leq s \leq 1, \quad k = 1, \ldots, n-1, \qquad (2.9)$$

$$\sum_{k=1}^{n} G_k(s) = ns, \qquad 0 \leq s \leq 1. \qquad (2.10)$$

Here we prove the following.

Theorem 2. Distribution functions $G_1, \ldots, G_n$ satisfying (2.9) and (2.10) are the distribution functions of consecutive order statistics corresponding to an absolutely continuous copula iff

$$\mu(\Sigma) = 0, \tag{2.11}$$

where μ denotes the Lebesgue measure on the real line and

$$\Sigma = \{0 < t < 1 : \ t = G_k(s) = G_{k-1}(s) \text{ for some } 0 < s < 1 \text{ and some } k = 2, \ldots, n\}. \tag{2.12}$$

Now we proceed to characterizations based on single order statistics. In the general case, Rychlik [21] characterized the distribution G_m as follows:

Theorem 3. (Rychlik [21], Theorem 1) A distribution function G is the distribution function of the mth order statistics, $1 \le m \le n$, corresponding to a copula iff the following two conditions hold

$$\max\left(0, \frac{ns - m + 1}{n - m + 1}\right) \le G(s) \le \min\left(1, \frac{ns}{m}\right), \quad 0 \le s \le 1, \tag{2.13}$$

$$0 \le G(t) - G(s) \le n(t - s), \qquad 0 \le s < t \le 1. \tag{2.14}$$

A more subtle characterization in the absolutely continuous case is as follows.

Theorem 4. A distribution function G satisfying (2.13) and (2.14) is the distribution function of the mth order statistic, $1 \le m \le n$, corresponding to an absolutely continuous copula C iff

$$\mu(\Sigma_-^0 \cup \Sigma_+^0) = 0 \tag{2.15}$$

for

$$\Sigma_-^0 = \begin{cases} \left\{0 \le s \le 1 : \quad G(s) = \frac{ns-m+1}{n-m+1}\right\}, & \text{if} \quad m < n, \\ \emptyset, & \text{if} \quad m = n, \end{cases} \tag{2.16}$$

$$\Sigma_+^0 = \begin{cases} \emptyset, & \text{if} \quad m = 1, \\ \left\{0 \le s \le 1 : G(t) = \frac{ns}{m}\right\}, & \text{if} \quad m > 1. \end{cases} \tag{2.17}$$

Remark 1. We first point out that Theorems 1 to 4 characterize distributions of order statistics corresponding to exchangeable copulas as well. It is clear that in each case it is possible to construct an exchangeable copula for which corresponding distributions of order statistics satisfy the characterization conditions. Indeed, renumbering the random variables, we obtain a sequence with the same distribution of order statistics. Taking the uniform mixture of all $n!$ rearrangements of the elements of the random sequence, we preserve the distribution of order statistics, and get an exchangeable joint distribution of the vector. Similar arguments can be found, e. g. in Spizzichino [24], p. 184 and [25].

Note that (2.14) implies that all the marginal distributions of order statistics are absolutely continuous with the densities bounded by the sample size n even if the joint distribution is singular. Condition (2.11) guarantees that all the order statistics are different with probability one. Condition (2.15) says that the distribution function of order statistics cannot coincide with the increasing parts of the bounds in (2.13) on a set of positive measure. It can be shown that $G_m(s) = \frac{ns+1-m}{n+1-m}$ ($\frac{ns}{m}$, respectively) implies that $G_m(s) = \cdots = G_n(s)$ ($G_1(s) = \cdots = G_m(s)$, respectively). If either of the relations holds on a set of positive measure, the respective order statistics would necessarily be identical with a positive probability. Formal justifications of these facts can be found in the proofs below.

3. AUXILIARY RESULTS

We first present two propositions that can be of independent interest.

We denote by Π the family of all permutations of the set $\{1,\dots,n\}$, and by $\pi(j)$ the jth coordinate of π.

Proposition 1. Suppose that distribution functions $G_1,\dots,G_n$ satisfy assumption (2.10) and C^* is a copula. Then

$$C(s_1,\dots,s_n) = \frac{1}{n!}\sum_{\pi\in\Pi} C^*(G_1(s_{\pi(1)}),\dots,G_n(s_{\pi(n)})), \tag{3.18}$$

is an exchangeable copula. Moreover, absolute continuity of C^* implies the same for C.

P r o o f. Each summand of (3.18) is a distribution function, and so is the convex combination. Clearly, this is exchangeable. By (2.10), each G_i is supported on $[0,1]$, and $[0,1]^n$ is the support of (3.18) in consequence. We now show that its marginals are uniform. To this end, we consider arguments such that $0 < s_i < 1$ for some $i = 1,\dots,n$, and $s_j = 1$ for $j \neq i$. Then, by (2.10), we have

$$\begin{aligned}
& C(1,\dots,1,s_i,1,\dots,1) \\
= \; & \frac{1}{n!}\sum_{j=1}^{n}\sum_{\pi(i)=j} C^*(G_1(1),\dots,G_{j-1}(1),G_j(s_i),G_{j+1}(1),\dots,G_n(1)) \\
= \; & \frac{1}{n!}\sum_{j=1}^{n}\sum_{\pi(i)=j} C^*(1,\dots,1,G_j(s_i),1,\dots,1) \\
= \; & \frac{1}{n!}\sum_{j=1}^{n}(n-1)!G_j(s_i) = s_i.
\end{aligned}$$

Assume finally that C^* is absolutely continuous. By (2.14), which is a consequence of (2.10), the joint density function

$$\frac{\partial C(s_1,\dots,s_n)}{\partial s_1\dots\partial s_n} = \frac{1}{n!}\sum_{\pi\in\Pi}\frac{\partial C^*(G_1(s_{\pi(1)}),\dots,G_n(s_{\pi(n)}))}{\partial s_1\dots\partial s_n}\prod_{j=1}^{n}\frac{dG(s_{\pi(j)})}{ds_{\pi(j)}}$$

is well defined almost everywhere. □

The formula describing copula (3.18) is quite simple for $n = 2$. Taking into account the fact that $G_1(s) + G_2(s) = 2s$, $0 \le s \le 1$, we obtain:

Corollary 1. Suppose that distribution functions G, supported on $[0, 1]$, satisfies assumption (2.14) and C^* is a copula. Then the function

$$C(s,t) = \frac{1}{2}(C^*(G(s), 2t - G(t)) + C^*(G(t), 2s - G(s))),$$

is an exchangeable copula. Moreover, absolute continuity of C^* implies the same for C.

Evidently distribution functions of the order statistics corresponding to copula (3.18) may differ from $G_1,\dots,G_n$. Below we show that by employing the copula (3.18) we can describe however a construction which leads to a copula with the desired vector of distribution functions of order statistics. We first introduce notions of left-side- and right-side-continuous inverses of non-decreasing functions

$$\begin{aligned} g^{\leftarrow}(t) &= \inf\{s:\ g(s)\ge t\},\\ g^{\rightarrow}(t) &= \sup\{s:\ g(s)\le t\}. \end{aligned}$$

Note that for continuous $g : [0,1] \overset{\text{onto}}{\rightarrow} [0,1]$ both $g^{\leftarrow}$ and $g^{\rightarrow}$ are right-side inverses of g so that

$$g(g^{\rightarrow}(t)) = t = g(g^{\leftarrow}(t)), \qquad 0 \le t \le 1.$$

Proposition 2. Let distribution functions $G_1,\dots,G_n$ satisfy (2.9) and (2.10). Suppose that a copula C_0 is a distribution functions of a random vector $V_1,\dots,V_n$, and the respective order statistics $V_{1:n},\dots,V_{n:n}$ have the joint distribution function

$$H(s_1,\dots,s_n) = C^*(H_1(s_1),\dots,H_n(s_n)),$$

where $H_1,\dots,H_n$ stand for the respective marginals of order statistics and C^* is the copula describing the interdependencies among them. If

$$H_1^{\leftarrow}(G_1(t)) \ge \cdots \ge H_n^{\leftarrow}(G_n(t)), \quad 0 \le t \le 1, \tag{3.19}$$

then $G_1,\dots,G_n$ are the distribution functions of order statistics corresponding to the exchangeable copula (3.18). Moreover, if C_0 is absolutely continuous, then so is (3.18).

P r o o f. Define monotone transformations of order statistics

$$Z_i = G_i^{\leftarrow}(H_i(V_{i:n})), \qquad i = 1, \ldots, n,$$

which have marginal distribution functions G_i, $i = 1, \ldots, n$, and copula C^*, identical with that of $V_{1:n}, \ldots, V_{n:n}$ (cf. Jaworski [11], Proposition 1). Let $(U_1, \ldots, U_n)$ be a random permutation of $(Z_1, \ldots, Z_n)$. By definition, it has the common distribution presented in (3.18). Relations (3.19) imply that Z_i are almost surely ordered in the ascending order, and $U_{i:n} = Z_i$ a.s., $i = 1, \ldots, n$, by definition. This ends the proof of the first claim. We finally observe that absolute continuity of $V_1, \ldots, V_n$ provides the same for $V_{1:n}, \ldots, V_{n:n}$ and for its copula. The reference to the last claim of Proposition 1 gives the final statement. □

Remark 2. We can also consider other constructions of copulas with desired marginals of order statistics, that are alternative and simpler than (3.18). For instance, we can use

$$C_\pi(s_1, \ldots, s_n) = \frac{1}{n} \sum_{i=1}^{n} C^*(G_1(s_{\tau^i(\pi(1))}), \ldots, G_n(s_{\tau^i(\pi(n))})) \tag{3.20}$$

where τ is the translation operator defined by $\tau(s_1, \ldots, s_n) = (s_2, \ldots, s_n, s_1)$, τ^i is its i-fold composition, and $\pi \in \Pi$ is arbitrary. One can also take an arbitrary convex combination of functions (3.20). The only requirement here is that for each pair $1 \leq i, j \leq n$, coordinate s_i of the left-hand side of (3.20) should be the argument of G_j at the right-hand side with the same probability $1/n$. Usually, distribution functions (3.20) are not exchangeable, but so is (3.18).

Remark 3. Suppose now that

$$C_0(s_1, \ldots, s_n) = \min(s_1, \ldots, s_n). \tag{3.21}$$

Then $V_1 = \cdots = V_n$ and $V_{1:n} = \cdots = V_{n:n}$ have identical marginals $H_1(s) = \cdots = H_n(s) = s$ and copula (3.21). Condition (3.19) is naturally satisfied, and Proposition 2 holds with

$$C(s_1, \ldots, s_n) = \frac{1}{n!} \sum_{\pi \in \Pi} \min(G_1(s_{\pi(1)}), \ldots, G_n(s_{\pi(n)})). \tag{3.22}$$

Rychlik [20] used (3.20) with $\pi = \pi_0 = (1, \ldots, n)$ and $C^* = \min$ for proving Theorem 1. The trouble here is that (3.20), (3.21) and (3.22) are not absolutely continuous.

So we face the following problem: given a fixed sequence of distribution functions $G_1, \ldots, G_n$ satisfying (2.9), (2.10), and the additional assumption (2.11), we should construct an absolutely continuous copula C_0 such that the respective marginal distribution functions of order statistics $H_1, \ldots, H_n$ satisfy (3.19). Note that the following holds

$$H_1^{\leftarrow}(t) \leq \cdots \leq H_n^{\leftarrow}(t), \quad 0 < t < 1.$$

It means that all H_i, $i=1,\dots,n$, should differ from the identity functions so little that the compositions $H_1^{\leftarrow}(G_i(t))$, $i=1,\dots,n$, do not disturb the original ordering (2.9) of functions G_i, $i=1,\dots,n$.

Below in Lemma 1 we present some sufficient conditions guarantying that for given $G_1,\dots,G_n$ functions $H_1,\dots,H_n$ fulfill (3.19).

Lemma 1. For $G_1,\dots,G_n$ satisfying (2.9) and (2.10), define

$$\begin{aligned}\varphi_{1,i}(t) &= \frac{3}{4}t+\frac{1}{4}G_{i+1}(G_i^{\rightarrow}(t)),\\ \varphi_{2,i}(t) &= \frac{3}{4}t+\frac{1}{4}G_i(G_{i+1}^{\leftarrow}(t)), \qquad i=1,\dots,n-1,\end{aligned}$$

and

$$\Phi_1(t) = \min\{\varphi_{1,i}^{\leftarrow}(t):\ i=1,\dots,n-1\}, \tag{3.23}$$

$$\Phi_2(t) = \max\{\varphi_{2,i}^{\rightarrow}(t):\ i=1,\dots,n-1\}. \tag{3.24}$$

Then $\Phi_2(t)\le t\le \Phi_1(t)$, $0\le t\le 1$, and both the inequalities are sharp iff $t\notin\Sigma$.

Moreover, if $H_1,\dots,H_n$ satisfy

$$\Phi_2(t)\le H_n(t)\le\cdots\le H_1(t)\le\Phi_1(t), \quad 0\le t\le 1, \tag{3.25}$$

then (3.19) holds.

P r o o f. Let $0\le t\le 1$ be arbitrary. Note that

$$G_{i+1}(G_i^{\rightarrow}(t))\ge t \ge G_i(G_{i+1}^{\leftarrow}(t)), \tag{3.26}$$

$$\varphi_{1,i}(t)\ge t\ge\varphi_{2,i}(t), \qquad i=1,\dots,n-1, \tag{3.27}$$

and

$$\Phi_2(t)\le t\le\Phi_1(t). \tag{3.28}$$

If $t\notin\Sigma$, then all the inequalities in (3.26) to (3.28) are sharp. On the other hand, $t\in\Sigma$ implies that at least one of inequalities of (3.26) for some $i=1,\dots,n-1$ becomes the equality. In consequence, we have the same in (3.27) and (3.28).

It follows from (3.25) that for all $i=1,\dots,n-1$, we have

$$\varphi_{2,i}(t)\ge\Phi_2^{\leftarrow}(t)\ge H_n^{\leftarrow}(t)\ge\cdots\ge H_1^{\leftarrow}(t)\ge\Phi_1^{\leftarrow}(t)\ge\varphi_{1,i}(t),$$

and

$$\begin{aligned}& H_i^{\leftarrow}(G_i(t))-H_{i+1}^{\leftarrow}(G_{i+1}(t))\ge\varphi_{1,i}(G_i(t))-\varphi_{2,i}(G_{i+1}(t))\\ =\ & \frac{3}{4}G_i(t)+\frac{1}{4}G_{i+1}(G_i^{\rightarrow}(G_i(t)))-\frac{3}{4}G_{i+1}(t)-\frac{1}{4}G_i(G_{i+1}^{\leftarrow}(G_{i+1}(t)))\\ \ge\ & \frac{3}{4}G_i(t)+\frac{1}{4}G_{i+1}(t)-\frac{3}{4}G_{i+1}(t)-\frac{1}{4}G_i(t)\\ =\ & \frac{1}{2}(G_i(t)-G_{i+1}(t))\ge 0,\end{aligned}$$

where the middle inequality follows from

$$G_{i+1}^{\leftarrow}(G_{i+1}(t))\le t\le G_i^{\rightarrow}(G_i(t)).$$

This ends the proof of lemma. □

4. PROOF OF THEOREM 2

Necessity proof. Let $U_1, \ldots, U_n$ be standard uniform random variables with the distribution determined by an absolutely continuous copula. Let $G_1, \ldots, G_n$ denote the marginal distribution functions of order statistics $U_{1:n}, \ldots, U_{n:n}$, respectively, and Σ be defined in (2.12). Contrary to our claim, assume that $\mu(\Sigma) > 0$. It follows that at least one of the sets

$$\Sigma_k = \{0 \le t \le 1 : \ t = G_k(s) = G_{k-1}(s) \text{ for some } 0 \le s \le 1\}, \quad k = 2, \ldots, n,$$

has a strictly positive Lebesgue measure. We prove that for each k

$$\Pr(U_{k:n} = U_{k-1:n}) \ge \mu(\Sigma_k), \tag{4.29}$$

which, together with the above, contradicts absolute continuity of the joint probability measure.

We first show that for any $0 \le s_1 < s_2 \le 1$ the following holds

$$\Pr(s_1 < U_{k-1:n} \le U_{k:n} \le s_2) \ge \mu([G_k(s_1), G_k(s_2)] \cap \Sigma_k). \tag{4.30}$$

If the intersection $[G_k(s_1), G_k(s_2)] \cap \Sigma_k$ consists of one point or is empty then its measure is equal to zero, and the inequality is valid. Otherwise we can select the smallest and the greatest numbers $t_1 < t_2$ satisfying

$$t_1, t_2 \in [G_k(s_1), G_k(s_2)] \cap \Sigma_k \subset [t_1, t_2].$$

Given t_1, t_2, we take $s_1 \le s_1' < s_2' \le s_2$ such that

$$G_k(s_1') = G_{k-1}(s_1') = t_1 < G_k(s_2') = G_{k-1}(s_2') = t_2,$$

Then we obtain

$$\begin{aligned} & \Pr(s_1 < U_{k-1:n} \le U_{k:n} \le s_2) \ge \Pr(s_1' < U_{k-1:n} \le U_{k:n} \le s_2') \\ = \; & \Pr(\{U_{k:n} \le s_2'\} \setminus \{U_{k-1:n} \le s_1'\} \ge G_k(s_2') - G_{k-1}(s_1') \\ = \; & t_2 - t_1 = \mu([t_1, t_2]) \ge \mu([G_k(s_1), G_k(s_2)] \cap \Sigma_k), \end{aligned}$$

as desired.

Now we consider

$$E_m = \bigcup_{i=1}^{2^m} \left(\frac{i-1}{2^m}, \frac{i}{2^m} \right]^2, \quad m = 1, 2, \ldots.$$

By (4.30), for all m we get

$$\begin{aligned} \Pr((U_{k-1:n}, U_{k:n}) \in E_m) &= \sum_{i=1}^{2^m} \Pr\left(\frac{i-1}{2^m} < U_{k-1:n} \le U_{k:m} \le \frac{i}{2^m} \right) \\ &\ge \sum_{i=1}^{2^m} \mu\left(\left[G_k\left(\frac{i-1}{2^m}\right), G_k\left(\frac{i}{2^m}\right) \right] \cap \Sigma_k \right) \ge \mu(\Sigma_k). \end{aligned}$$

Since

$$\lim_{m \to +\infty} \Pr((U_{k-1:n}, U_{k:n}) \in E_m) = \Pr(U_{k-1:n} = U_{k:n}),$$

relation (4.29) holds, and the statement is concluded. □

Sufficiency proof. We assume that distribution functions $G_1, \ldots, G_n$ satisfy conditions (2.9) to (2.11), and aim at constructing an absolutely continuous copula such that $G_1, \ldots, G_n$ are the distribution functions of order statistics corresponding to the copula.

This part of the proof of Theorem 2 consists in constructing an absolutely continuous copula with the corresponding distribution function $H_1, \ldots, H_n$ of consecutive order statistics which satisfy assumption (3.25). For this purpose we recall the notion of ordinal sum of copulas (cf. Nelsen [19], Section 3.2.2 for $n = 2$ and Mesiar and Sempi [16] for $n \geq 2$). The ordinal sum of copulas $\{C_i\}_{i=1}^N$ $(1 \leq N \leq +\infty)$ with respect to non-overlapping intervals $\{(a_i, b_i)\}_{i=1}^N$ $(0 \leq a_i < b_i \leq 1)$ is a function defined by

$$\begin{aligned} C(s_1, \ldots, s_n) &= \sum_{i=1}^{N} (b_i - a_i) C_i \left(\frac{s_1 - a_i}{b_i - a_i}, \ldots, \frac{s_n - a_i}{b_i - a_i} \right) \\ &+ \mu \left([0, \min(s_1, \ldots, s_n)] \setminus \bigcup_{i=1}^{N} [a_i, b_i] \right). \end{aligned} \tag{4.31}$$

It can be shown that (4.31) is a copula. One can generate random variables $V_1, \ldots, V_n$ with the common distribution function (4.31) using the following procedure (which can be easily adapted for a sampling). We first take a standard uniform variable V. If $V \in (a_i, b_i)$ for some $1 \leq i \leq N$, then we generate n random variables $V_1, \ldots, V_n$ with the uniform marginal distributions on $[a_i, b_i]$ and copula C_i. If $V \notin \bigcup_{i=1}^N (a_i, b_i)$, we simply put $V_1 = \cdots V_n = V$. Clearly $a_i \leq V \leq b_i$ implies that $a_i \leq V_{j:n} \leq b_i$, $j = 1, \ldots, n$, and we have $V_{1:n} = \cdots = V_{n:n} = V$ when $V \notin \bigcup_{i=1}^N (a_i, b_i)$. Therefore the graphs of distribution functions $H_1, \ldots, H_n$ of the order statistics satisfy

$$\{(s, H_j(s)) : \ 0 \leq s \leq 1\} \subset \bigcup_{i=1}^{N} [a_i, b_i]^2 \cup \{(s, s) : \ 0 \leq s \leq 1\}, \quad j = 1, \ldots, n.$$

If all the copulas C_i, $1 \leq i \leq N$, are absolute continuous with respective density functions c_i, $1 \leq i \leq N$, and $\sum_{i=1}^N (b_i - a_i) = 1$, then (4.31) has the density

$$c(s_1, \ldots, s_n) = \begin{cases} \frac{1}{(b_i - a_i)^{n-1}} c_i \left(\frac{s_1 - a_i}{b_i - a_i}, \ldots, \frac{s_n - a_i}{b_i - a_i} \right), & \text{if } s_j \in (a_i, b_i), \ j = 1, \ldots, n, \\ 0, & \text{otherwise}. \end{cases}$$

Now for $G_1, \ldots, G_n$ satisfying (2.9) to (2.11) and functions (3.23) and (3.24), we define

$$Z = \{(t, x) : \ 0 \leq t \leq 1, \ \Phi_2(t) \leq x \leq \Phi_1(t)\} \subset [0, 1]^2.$$

Note that $\Phi_2(t) < t < \Phi_1(t)$ iff $t \in [0, 1] \setminus \Sigma$ which is an open set consisting of at most countably many disjoint open intervals whose Lebesgue measures sum up to one. We aim at constructing another at most countable family of disjoint open intervals (a_i, b_i), $1 \leq i \leq N$ contained in $[0, 1]$, with the same total measure and such that $\bigcup_{i=1}^N [a_i, b_i]^2 \subset Z$. To this end we apply a recurrent procedure which at each step generates a finite number of intervals. We set first $\mathcal{U}_0 = \{(0,1)\}$ and $\mathcal{V}_0 = \emptyset$, and given $\mathcal{U}_n$, $\mathcal{V}_n$, $n \geq 0$, we define

$$\begin{aligned} \mathcal{U}_{n+1} &= \left\{ \left(a, \frac{a+b}{2}\right), \left(\frac{a+b}{2}, b\right) : (a,b) \in \mathcal{U}_n \setminus \mathcal{V}_n \right\}, \\ \mathcal{V}_{n+1} &= \{(a,b) \in \mathcal{U}_{n+1} : \ [a,b]^2 \subset Z\}. \end{aligned}$$

Let W_n, $n \geq 1$, denote the sum of all disjoint open intervals contained in $\mathcal{V}_1, \dots, \mathcal{V}_n$. Then

$$W = \lim_{n\to\infty} W_n = \bigcup_{i=1}^{N} (a_i, b_i), \quad 1 \leq N \leq \infty,$$

is an at most countable sum of disjoint open intervals (a_i, b_i), is contained in $[0,1]\backslash\Sigma$, has the same measure and $\bigcup_{i=1}^{N}[a_i,b_i]^2 \subset Z$, as desired.

It remains to take the ordinal sum of arbitrary sequence of absolutely continuous copulas (we can simply take all the identical product copulas) with respect to the family of intervals $\{(a_i,b_i)\}_{i=1}^N$. The corresponding distribution functions of order statistics $H_1, \dots, H_n$ satisfy relation (3.25), and so (3.19) in consequence. Applying the construction of Proposition 2, we obtain an absolutely continuous copula with desired marginal distributions of order statistics. □

5. PROOF OF THEOREM 4

Observe first that condition (2.15) is equivalent to saying that $\mu(\Sigma_- \cup \Sigma_+) = 0$, where

$$\Sigma_- = \begin{cases} \{0 \leq t \leq 1 : \ t = G(s) = \frac{ns-m+1}{n-m+1} \text{ for some } 0 \leq s \leq 1\}, & \text{if } \ m < n, \\ \emptyset, & \text{if } \ m = n, \end{cases} \tag{5.32}$$

$$\Sigma_+ = s \begin{cases} \emptyset, & \text{if } \ m = 1, \\ \{0 \leq t \leq 1 : \ t = G(s) = \frac{ns}{m} \text{ for some } 0 \leq s \leq 1\}, & \text{if } \ m > 1. \end{cases} \tag{5.33}$$

because $\mu(\Sigma_-) = \frac{n}{n-m+1}\mu(\Sigma_-^0)$ and $\mu(\Sigma_+) = \frac{n}{m}\mu(\Sigma_+^0)$. The former condition is more natural, but below we consider the latter one, because this is directly connected with condition (2.11) of Theorem 2 which is applied in the proof.

Necessity proof. Suppose that $G_1, \dots, G_n$ are the distribution functions of order statistics corresponding to an absolutely continuous copula and $G_m = G$. If $m < n$ and $t = G(s) = \frac{ns+1-m}{n+1-m} \in \Sigma_-$, then, using notation (2.16) and (5.32), and applying conditions (2.9) and (2.10), we obtain

$$ns = \sum_{i=1}^{n} G_i(s) \leq m - 1 + (n+1-m)G_m(s) = ns,$$

which implies

$$t = G_m(s) = \cdots = G_n(s) = \frac{ns+1-m}{n+1-m},$$

and so $t \in \Sigma$. Similarly, for $m > 1$ and $t = G(s) = \frac{ns}{m} \in \Sigma_+$ relations

$$ns = \sum_{i=1}^{n} G_i(s) \geq mG_m(s) = ns,$$

yield

$$t = G_1(s) = \cdots = G_m(s) = \frac{ns + 1 - m}{n + 1 - m} \in \Sigma.$$

Relation $\mu(\Sigma_- \cup \Sigma_+) > 0$ implies $\mu(\Sigma) > 0$, and by Theorem 2 contradicts continuity of the joint distribution. □

Sufficiency proof. Suppose that G satisfying (2.13) and (2.14) is such that the sets Σ_- and Σ_+ defined in (5.32) and (5.33) have measures zero. Here we construct a sequence of functions $G_1, \ldots, G_n$ satisfying (2.9), (2.10) and $G_m = G$, and such that the respective set Σ, defined in (2.12) coincides with $\Sigma_- \cup \Sigma_+$. If its Lebesgue measure were zero, it would enable us to determine an absolute continuous copula with assigned distribution functions of order statistics $G_1, \ldots, G_n$ by use of the method presented in Theorem 2. The construction is based on two following lemmas.

Lemma 2. If $m > 1$, then there exists a continuous function $A : [0, n-1] \to [0, m-1]$ such that

(i) A is zero on $[m-1, n-1]$,

(ii) functions $x \mapsto x - A(x)$ and $x \mapsto x + A(x)$ are non-decreasing,

(iii) for all $s \in [0, 1]$

$$ns - G(s) - A(ns - G(s)) \geq (m-1)G(s), \tag{5.34}$$

(iv) the equality holds in (5.34) iff $G(s) \in \Sigma_+$.

Lemma 3. If $m < n$, then there exists a continuous function $B : [0, n-1] \to [0, n-m]$ such that

(i) B is zero on $[0, m-1]$,

(ii) functions $x \mapsto x - B(x)$ and $x \mapsto x + B(x)$ are non-decreasing,

(iii) for all $s \in [0, 1]$

$$ns - G(s) + B(ns - G(s)) - (m-1) \leq (n-m)G(s), \tag{5.35}$$

(iv) the equality holds in (5.35) iff $G(s) \in \Sigma_-$.

Proof of Lemma 2. If $\Sigma_+ = [0, 1]$ then we simply put $A = 0$. Therefore we further assume that Σ_+ is a proper subset of $[0, 1]$. We consider an auxiliary function $G_+ : [0, 1] \mapsto [0, 1]$ defined as

$$G_+(s) = \min\left(1, \frac{ns - G(s)}{m-1}\right).$$

Due to (2.14), it is non-negative and non-decreasing. By (2.13), we have

$$\begin{aligned} G_+(s) &= \min\left(1, \frac{m}{m-1}\frac{ns}{m} - \frac{1}{m-1}G(s)\right) \\ &\geq \min\left(1, \frac{m}{m-1}G(s) - \frac{1}{m-1}G(s)\right) = G(s), \end{aligned}$$

and the equality holds iff $G(s) \in \Sigma_+ \cup \{1\}$. Hence the graph of G_+ is located above the graph of G, and $\Sigma_+ \cup \{1\}$ is the projection of the intersection of the graphs onto the ordinate axis. Note that $0 \in \Sigma_+$, and $1 \in \Sigma_+$ iff $G(\frac{m}{n}) = 1$.

Below we transform the graphs changing the scales in both the axes by means of the continuous mapping $\Phi_+ : [0,1]^2 \mapsto [0, n-1] \times [0, m-1]$ defined as

$$\Phi_+(s,t) = (ns - G(s), (m-1)t).$$

The image of the graph of G_+,

$$\begin{aligned} \Phi_+(\text{Graph } G_+) &= \{(ns - G(s), \min(ns - G(s), m-1)) : \ s \in [0,1]\} \\ &= \{(x,x) : x \in [0, m-1]\} \cup \{(x, m-1) : x \in [m-1, n-1]\}, \end{aligned}$$

coincides with the graph of the function $x \mapsto \min(x, m-1)$, $0 \leq x \leq n-1$, and

$$\Phi_+(\text{Graph } G) = \{(ns - G(s), (m-1)G(s)) : \ s \in [0,1]\}.$$

Therefore

$$\begin{aligned} &\Phi_+(\text{Graph } G) \cap \Phi_+(\text{Graph } G_+) = \Phi_+(\text{Graph } G \cap \text{Graph } G_+) \\ &= \left\{(x,x) : \frac{x}{m-1} \in \Sigma_+\right\} \cup \{(ns-1, m-1) : G(s) = 1\}. \end{aligned}$$

If $t \in (0,1)\setminus\Sigma_+$ then $((m-1)t, (m-1)t) \in \Phi_+(\text{Graph } G_+)$ lies above $\Phi_+(\text{Graph} G)$. Since $\Phi_+(\text{Graph} G)$ is closed, there exists $0 < \varepsilon_t^+ < \min(t, 1-t)$ such that the square $[(m-1)(t-\varepsilon_t^+), (m-1)(t+\varepsilon_t^+)]^2$ does not intersect the image of the graph of G either. The family of all the intervals $\{(t-\varepsilon_t^+, t+\varepsilon_t^+)\}$, $t \in (0,1)\setminus\Sigma_+$, is a covering of the open set $(0,1)\setminus\Sigma_+$. We can select a countable subcovering $\{(t_i^+ - \varepsilon_i^+, t_i^+ + \varepsilon_i^+)\}$, $i = 1, 2, \ldots$ (where $\varepsilon_i^+ = \varepsilon_{t_i}^+$ for convenience) such that

$$\bigcup_{i=1}^{\infty}(t_i^+ - \varepsilon_i^+, t_i^+ + \varepsilon_i^+) = (0,1)\setminus\Sigma_+,$$

$$\bigcup_{i=1}^{\infty}[t_i^+ - \varepsilon_i^+, t_i^+ + \varepsilon_i^+]^2 \cap \Phi_+(\text{Graph } G) = \emptyset.$$

In contrast to the construction of Section 3, some intervals from the countable subcovering overlap so that the whole set $(0,1)\setminus\Sigma_+$ is actually covered.

Set

$$A_i(x) = \begin{cases} \frac{1}{2}[x - (m-1)(t_i^+ - \varepsilon_i^+)], & \text{if } (m-1)(t_i^+ - \varepsilon_i^+) \leq x \leq (m-1)t_i^+, \\ \frac{1}{2}[-x + (m-1)(t_i^+ + \varepsilon_i^+)], & \text{if } (m-1)t_i^+ \leq x \leq (m-1)(t_i^+ + \varepsilon_i^+), \\ 0, & \text{otherwise.} \end{cases}$$

for $i = 1, 2, \ldots$, and
$$A(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} A_i(x), \quad 0 \le x \le n-1.$$
Since all A_i are bounded, non-negative and continuous, the sum converges uniformly and A is non-negative continuous. By definition, A vanishes on $[m-1, n-1]$. Moreover, since for each i both $x \mapsto x - A_i(x)$ and $x \mapsto x + A_i(x)$ are non-decreasing, so are the functions
$$x \mapsto x \pm A(x) = \sum_{i=1}^{\infty} \frac{1}{2^i}(x \pm A_i(x)).$$

It remains to prove that inequality (5.34) becomes the equality if $G(s) \in \Sigma_+$, and is sharp otherwise. If $1 > t = G(s) = \frac{ns - G(s)}{m-1} \in \Sigma_+$, then $A((m-1)t) = A(ns - G(s)) = 0$, by definition, and so
$$\frac{ns - G(s) - A(ns - G(s))}{m-1} = G(s). \tag{5.36}$$
If $t = G(\frac{m}{n}) = 1 \in \Sigma_+$, then $s = \frac{m}{n}$ and $ns - G(s) = m - 1$, $A(ns - G(s)) = A(m-1) = 0$. Therefore both the sides of (5.36) amount to 1. This proves the former claim.

Suppose now that $G(s) < t = \frac{ns - G(s)}{m-1} \in (0,1) \setminus \Sigma_+ = \bigcup_{i=1}^{\infty}(t_i^+ - \varepsilon_i^+, t_i^+ + \varepsilon_i^+)$. If $t \notin (t_i^+ - \varepsilon_i^+, t_i^+ + \varepsilon_i^+)$, then $A_i((m-1)t) = 0$ and
$$\frac{ns - G(s) - A_i(ns - G(s))}{m-1} = \frac{ns - G(s)}{m-1} > G(s). \tag{5.37}$$
If $t \in (t_i^+ - \varepsilon_i^+, t_i^+ + \varepsilon_i^+)$, then
$$\begin{aligned} ns - G(ns) - A_i(ns - G(s)) &= (m-1)t - A_i((m-1)t) \\ &\ge (m-1)(t_i^+ - \varepsilon_i^+) - A_i((m-1)(t_i^+ - \varepsilon_i^+)) \\ &= (m-1)(t_i^+ - \varepsilon_i^+) > (m-1)G(s), \end{aligned} \tag{5.38}$$
because $x \mapsto x + A_i(x)$ is non-decreasing and the lower edge of the square $[(m-1)(t_i^+ - \varepsilon_i^+), (m-1)(t_i^+ + \varepsilon_i^+)]^2$ lies above point $(ns - G(s), (m-1)G(s))$ belonging to $\Phi_+(\text{Graph } G)$. Therefore
$$\begin{aligned} &\frac{ns - G(s) - A(ns - G(s))}{m-1} - G(s) \\ &= \frac{1}{m-1} \sum_{i=1}^{\infty} \frac{1}{2^i}[ns - G(s) - A_i(ns - G(s)) - (m-1)G(s)] > 0, \end{aligned}$$
because, owing to (5.37) and (5.38), every term of the sum is positive. If either $\Sigma_+ \not\ni t = \frac{ns - G(s)}{m-1} = 1 > G(s)$ or $t = \frac{ns - G(s)}{m-1} > 1 \ge G(s)$, then $(m-1)t = ns - G(s) \ge m - 1$, $A(ns - G(s)) = 0$ and
$$\frac{ns - G(s) - A(ns - G(s))}{m-1} = \frac{ns - G(s)}{m-1} > G(s).$$
This ends the proof of the lemma. □

Proof of Lemma 3 is similar to the previous one. If $\Sigma_- = [0,1]$, we set $B = 0$. Otherwise we define the distribution function

$$G_-(s) = \max\left(0, \frac{ns - G(s) - (m-1)}{n-m}\right).$$

Due to (2.13),

$$\begin{aligned} G_-(s) &= \max\left(0, \frac{n-m+1}{n-m}\frac{ns-m+1}{n-m+1} - \frac{1}{n-m}G(s)\right) \\ &\leq \max\left(0, \frac{n-m+1}{n-m}G(s) - \frac{1}{n-m}G(s)\right) = G(s), \end{aligned}$$

and the equality holds iff $G(s) \in \Sigma_- \cup \{0\}$. In other words, the graph of G lies above that of G_-, and they touch each other when $G(s) \in \Sigma_- \cup \{0\}$. Here $1 \in \Sigma_-$, and $0 \in \Sigma_-$ iff $G(\frac{m-1}{n}) = 0$.

Consider the continuous transformation $\Phi_- : [0,1]^2 \mapsto [0, n-1] \times [m-1, n-1]$ defined by

$$\Phi_-(s,t) = (ns - G(s), (n-m)t + (m-1)).$$

Then

$$\Phi_-(\mathrm{Graph}\, G_-) = \{(ns - G(s), \max(m-1, ns - G(s)) : s \in [0,1]\}$$

is identical with the graph of function $x \mapsto \max(m-1, x)$, $0 \leq x \leq n-1$, and lies below

$$\Phi_-(\mathrm{Graph}\, G) = \{(ns - G(s), (n-m)G(s) + m - 1) : s \in [0,1]\}.$$

If $t \in (0,1) \setminus \Sigma_-$ then $((n-m)t + (m-1), (n-m)t + (m-1))$ lies below $\Phi_-(\mathrm{Graph}\, G)$, and for some $0 < \varepsilon_t^- < \min(t, 1-t)$ the square $[(n-m)(t - \varepsilon_t^-) + (m-1), (n-m)(t+\varepsilon_t^-) + (m-1)]^2$ is beneath $\Phi_-(\mathrm{Graph}\, G)$ as well. We replace the covering $\{(t - \varepsilon_t^-, t + \varepsilon_t^-)\}$, $t \in (0,1) \setminus \Sigma_-$, of $(0,1) \setminus \Sigma_-$ by a countable subcovering $\{(t_i^- - \varepsilon_i^-, t_i^- + \varepsilon_i^-)\}$, $i = 1, 2, \ldots$, such that

$$\bigcup_{i=1}^{\infty} [t_i^- - \varepsilon_i^-, t_i^- + \varepsilon_i^-]^2 \cap \Phi_-(\mathrm{Graph}\, G) = \emptyset,$$

and define

$$B_i(x) = \begin{cases} \frac{1}{2}[x - (n-m)(x_i^- - \varepsilon_i^-) - (m-1)], \\ \qquad \text{if } (n-m)(t_i^- - \varepsilon_i^-) + (m-1) \leq x \leq (n-m)t_i^- + (m-1), \\ \frac{1}{2}[-x + (n-m)(t_i^- + \varepsilon_i^-) + (m-1)], \\ \qquad \text{if } (n-m)t_i^- + (m-1) \leq x \leq (n-m)(t_i^- + \varepsilon_i^-) + (m-1), \\ 0, \qquad \text{otherwise}. \end{cases}$$

$i = 1, 2, \ldots$, and

$$B(x) = \sum_{i=1}^{\infty} \frac{1}{2^i} B_i(x), \qquad 0 \leq x \leq n-1.$$

We easily check that B is non-negative, continuous, equal to zero on $[0, m-1]$, and such that functions $x \mapsto x \pm B(x)$ are non-decreasing.

If either $0 < t = G(s) = \frac{ns-G(s)-(m-1)}{n-m} \in \Sigma_-$ or $t = G(\frac{m-1}{n}) = 0 \in \Sigma_-$, then $B((n-m)t + (m-1)) = B(ns - G(s)) = 0$, and

$$\frac{ns - G(s) + B(ns - G(s)) - (m-1)}{n-m} = \frac{ns - G(s) - (m-1)}{n-m} = G(s).$$

Suppose that $G(s) > t = \frac{ns-G(s)-(m-1)}{n-m} \in (0,1) \setminus \Sigma_-$. If $t \notin (t_i^- - \varepsilon_i^-, t_i^- + \varepsilon_i^-)$, then $B_i((n-m)t + (m-1)) = B_i(ns - G(s)) = 0$, and

$$\frac{ns - G(s) + B_i(ns - G(s)) - (m-1)}{n-m} < G(s).$$

If $t \in (t_i^- - \varepsilon_i^-, t_i^- + \varepsilon_i^-)$, then

$$\begin{aligned} & ns - G(ns) + B_i(ns - G(s)) = (n-m)t + (m-1) + B_i((n-m)t + (m-1)) \\ \leq \; & (n-m)(t_i^+ - \varepsilon_i^+) + (m-1) + B_i((n-m)(t_i^+ - \varepsilon_i^+) + (m-1)) \\ = \; & (n-m)(t_i^+ - \varepsilon_i^+) + (m-1) < (n-m)G(s) + (m-1), \end{aligned}$$

because the upper edge of the square $[(n-m)(t_i^- - \varepsilon_i^-) + (m-1), (n-m)(t_i^- + \varepsilon_i^-) + (m-1)]^2$ lies beneath $(ns - G(s), (n-m)G(s) + (m-1)) \in \Phi_-(\text{Graph } G)$. Hence

$$\begin{aligned} & G(s) - \frac{ns - G(s) + B(ns - G(s)) - (m-1)}{n-m} \\ = \; & \frac{1}{n-m} \sum_{i=1}^{\infty} \frac{1}{2^i} \{(n-m)G(s) - [ns - G(s) + B_i(ns - G(s)) - (m-1)]\} > 0. \end{aligned}$$

Finally, if either $\Sigma_- \not\ni t = \frac{ns-G(s)-(m-1)}{n-m} = 0 < G(s)$ or $t = \frac{ns-G(s)-(m-1)}{n-m} < 0 \leq G(s)$, then $(n-m)t + (m-1) = ns - G(s) \leq m-1$, $B(ns - G(s)) = 0$ and

$$\frac{ns - G(s) + B(ns - G(s)) - (m-1)}{n-m} = \frac{ns - G(s) - (m-1)}{n-m} < G(s). \qquad \square$$

We proceed to the last part of the proof of Theorem 4. To this end, we apply the functions defined in Lemmas 2 and 3 for determining the sequence of continuous functions

$$\begin{aligned} G_k^*(s) &= \min\left(1, \frac{ns - G(s) + a_k A(ns - G(s))}{m-1}\right), \quad k = 1, \ldots, m-1, &(5.39) \\ G_m^*(s) &= G(s), \\ G_k^*(s) &= \max\left(0, \frac{ns - G(s) + b_k B(ns - G(s)) - (m-1)}{n-m}\right), \\ & \qquad\qquad k = m+1, \ldots, n, &(5.40) \end{aligned}$$

where

$$a_k = \begin{cases} \frac{m-2k}{m-2}, & \text{if} \quad m > 2, \\ 0, & \text{if} \quad m = 2, \end{cases}$$

$$b_k = \begin{cases} \frac{n-m+1-2k}{n-m-1}, & \text{if} \quad m < n-1, \\ 0, & \text{if} \quad m = n-1, \end{cases}$$

are decreasing sequences ranging between 1 and −1, and summing up to 0.

Therefore

$$\begin{aligned} 1 &\geq G_1^*(s) \geq \cdots \geq G_{m-1}^*(s) = \min\left(1, \frac{ns - G(s) - A(ns - G(s))}{m-1}\right) \\ &\geq G(s) \geq \max\left(0, \frac{ns - G(s) + A(ns - G(s)) - (m-1)}{n-m}\right) \\ &= G_{m+1}^* \geq \cdots \geq G_n^*(s) \geq 0, \end{aligned} \tag{5.41}$$

where the inequalities in the second line follow from Lemmas 2(iii) and 3(iii). Since the function $s \mapsto ns - G(s)$ is continuous and non-decreasing, and due to Lemmas 2(ii) and 3(ii),

$$\begin{aligned} ns - G(s) + a_k A(ns - G(s)) &= (1 - |a_k|)(ns - G(s)) \\ &+ |a_k|(ns - G(s) + \operatorname{sgn}(a_k) A(ns - G(s))), \\ ns - G(s) + b_k B(ns - G(s)) &= (1 - |b_k|)(ns - G(s)) \\ &+ |b_k|(ns - G(s) + \operatorname{sgn}(b_k) B(ns - G(s))), \end{aligned}$$

are non-decreasing for all a_k and b_k as well. This implies the same property for (5.39) and (5.40). If $ns_0 - G(s_0) = m - 1$ for some $0 \leq s_0 \leq 1$, then

$$\begin{aligned} G_k^*(s_0) &= \frac{ns_0 - G(s_0) + a_k A(ns_0 - G(s_0))}{m-1} = 1, \qquad k = 1, \dots, m-1, \\ G_k^*(s_0) &= \frac{ns_0 - G(s_0) + b_k B(ns_0 - G(s_0)) - (m-1)}{n-m} = 0, \quad k = m+1, \dots, n. \end{aligned}$$

Accordingly, for all $s \leq s_0$ yields

$$\begin{aligned} G_k^*(s) &= \frac{ns - G(s) + a_k A(ns - G(s))}{m-1}, \quad k = 1, \dots, m-1, \\ G_k^*(s) &= 0, \qquad k = m+1, \dots, n, \end{aligned}$$

and in consequence

$$\sum_{k=1}^{n} G_k^*(s) = ns. \tag{5.42}$$

Similarly, relations

$$\begin{aligned} G_k^*(s) &= 1, \qquad k = 1, \dots, m-1, \\ G_k^*(s) &= \frac{ns - G(s) + b_k B(ns - G(s)) - (m-1)}{n-m}, \quad k = m+1, \dots, n. \end{aligned}$$

when $s \geq s_0$, give the same claim then. Since moreover all G_k^*, $k = 1, \dots, n$, are continuous non-decreasing and have values in the unit interval, they are distribution functions. Relations (5.41) and (5.42) show that they are distribution functions of order statistics corresponding to a copula.

Note finally that $G(s) = G_m(s) \notin \Sigma_- \cup \Sigma_+$ implies $A(ns - G(s)) > 0$, $B(ns - G(s)) > 0$, and so $G_1^*(s) > \cdots > G_n^*(s)$, and ultimately $G(s) \notin \Sigma$. When $G(s) \in \Sigma_+$ (Σ_-, respectively), then $G_1^*(s) = \cdots = G_m^*(s) = G(s)$ ($G(s) = G_m^*(s) = \cdots = G_n^*(s)$, respectively), and so $G(s) \in \Sigma$. Assumption $\mu(\Sigma_- \cup \Sigma_+) = 0$ guarantees existence of an absolutely continuous copula corresponding to $G_1^*, \dots, G_n^*$. □

Remark 4. Rychlik [21] proved Theorem 3 using (5.39) and (5.40) with $A(s) = B(s) = 0$. Then

$$G^*_1(s) = \cdots = G^*_{m-1}(s) = G_+(s) \geq G(s) \geq G_-(s) = G^*_{m+1}(s) = \cdots = G^*_n(s)$$

satisfy conditions of Theorem 1. However, they also imply

$$U_{1:n} = \cdots = U_{m-1:n} \leq U_{m:n} \leq U_{m+1:n} = \cdots = U_{n:n}$$

almost surely, which contradicts continuity of the parent joint distribution.

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