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MOVES WITHOUT FORBIDDEN TRANSITIONS
IN A GRAPH

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Throughout the paper we understand by a graph a non-oriented finite
graph without loops. Let \( G \) be a graph with the vertex set \( V(G) = \{v_1, v_2, \ldots, v_n\} \). Denote by \( E_i \) the set of all edges of \( G \) incident at \( v_i \) and by \( d_i \) the
dergree of \( v_i \) (i.e. \( d_i = |E_i| \)).

A sequence \( P = \{x_1, y_1, x_2, y_2, \ldots, y_n, x_{n+1}\} \) of elements of \( G \) (where
\( n > 0; \ x_i \) are vertices; \( y_j \) are edges and \( y_j \) joins in \( G \) the vertices \( x_j \neq x_{j+1} \))
is called a move of \( G \) if any edge of \( G \) occurs in \( P \) at most once. The move
is closed (or open) if \( x_{n+1} = x_1 \) (or \( x_{n+1} \neq x_1 \)). The moves obtained from
\( P \) by the reversion of its elements and by a translation of its elements (if \( P \)
is closed) are not considered here as different from \( P \).

By a transition of \( P \) through \( x_i \) we mean a triple of elements \( \{y_i-1, x_i, y_i\} \).
In the case when \( P \) is a closed move, we consider as a transition through
\( x_1 = x_{n+1} \) also the triple \( \{y_n, x_1, y_1\} \).

Under the decomposition of \( G \) into moves we understand such a set \( \bar{P} \) of
moves of \( G \), that every edge of \( G \) belongs exactly to one move from \( \bar{P} \). Obviously,
we have:

**Lemma 1.** The decomposition of \( G \) into closed moves exists if and only if every
vertex \( v_i \in V(G) \) is of an even degree.

If the decomposition of \( G \) into closed moves contains only one element,
then we call this closed move a Eulerian line of \( G \). The following lemma is
known (see, e.g. [1]):

**Lemma 2.** The Eulerian line of \( G \) exists if and only if \( G \) is connected, contains
at least two vertices and every of its vertices is of an even degree.

**Lemma 3.** Let \( \bar{P} \) be a decomposition of \( G \) into closed moves and let \( v_i \) be a vertex
of \( V(G) \). For the number \( p(i) \) of different transitions \( \{e_x, v_i, f_x\} \) (\( x = 1, 2, \ldots, 
p(i)\)) through \( v_i \) we have \( p(i) = \frac{1}{2}d_i \). Further we have: \([j \neq k] = \{e_j, f_j\} \cap
\cap \{e_k, f_k\} = \emptyset \) and \( \{e_1, f_1\} \cup \{e_2, f_2\} \cup \ldots \cup \{e_{p(i)}, f_{p(i)}\} = E_i \).

Proof. The validity of the lemma is evident from the definition of the

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closed move, of the decomposition of a graph into moves and from the definition of the transition through the vertex.

We call the decomposition \( D_t = \{\{e_1, f_1\}, \{e_2, f_2\}, \ldots, \{e_{p(t)}, f_{p(t)}\}\} \) of the set \( E_t \) the decomposition of \( G \) formed by the decomposition \( P \) and we call the system \( \bar{D} = \{D_1, D_2, \ldots, D_n\} \) the \( \delta \)-system formed in \( G \) by \( P \). We denote the fact that \( \bar{D} \) is the \( \delta \)-system formed by \( P \) thus: \( \bar{D} = \delta(P) \).

**Lemma 4.** Every decomposition of \( G \) into closed moves forms exactly one \( \delta \)-system in \( G \), and to every system \( \bar{D} = \{D_1, D_2, \ldots, D_n\} \) of the decompositions of the sets \( E_t \) into pairs of edges there exists exactly one decomposition \( \tilde{P} \) of \( G \) into closed moves such that \( \bar{D} = \delta(\tilde{P}) \).

**Proof.** The validity of the first assertion of the lemma follows directly from the definition of the \( \delta \)-system.

Now let \( \bar{D} = \{D_1, D_2, \ldots, D_n\} \) be a system of decompositions of the sets \( E_t \) into pairs of edges. Let us travel along the elements of \( G \) according to the following rules:

1. If in a travelling we arrive along an edge \( e \) to its endpoint (= vertex \( v_x \)), then we proceed along that edge which forms with \( e \) a pair in the decomposition \( D_x \in \bar{D} \) (in other words: the system \( \bar{D} \) determines all transitions through the vertices from \( V(G) \)).

2. We can continue to travel only along such an edge along which we have not been travelling yet; if we should according (1) continue to travel along an edge which we passed already, our travelling is finished.

Let us begin our travelling along an edge \( e_1 \) from one of its endpoints. It is clear that after having passed a finite number of edges our travelling must end and it ends in the vertex from which our travelling started.

We can describe our whole travelling by a certain closed move (denote it by \( P_1 \)). If \( P_1 \) contains all the edges of \( G \), then for \( \tilde{P} = \{P_1\} \) we have: \( \delta(\tilde{P}) = \bar{D} \).

If in \( G \) there exists an edge \( e_2 \), not belonging to \( P_1 \), we can begin our analogical travelling along the edge \( e_2 \). We obtain then a closed move \( P_2 \). Then either \( P_1 \cup P_2 \) contains all the edges of \( G \) and for \( \tilde{P} = \{P_1, P_2\} \) we have \( \delta(\tilde{P}) = \bar{D} \) or there exists an edge \( e_3 \), not belonging to \( P_1 \cup P_2 \). This consideration can be repeated and after a finite number of steps we obtain such a set \( \tilde{P} = \{P_1, P_2, \ldots, P_m\} \) of closed moves of \( G \), where every edge from \( G \) belongs at least to one of the moves from \( \tilde{P} \). From the construction of these moves (the departure from the vertex after the given arrival in the vertex is determined uniquely!) it follows that every edge of \( G \) belongs only to one move from \( \tilde{P} \). From this it follows that \( \delta(\tilde{P}) = \bar{D} \), q. e. d.

Let for every vertex \( v_i \in V(G) \) a certain decomposition \( Q_i \) of the set \( E_i \) into classes of edges (incident to \( v_i \)) be given. We shall call a move \( P_x \) in \( G \) admissible with respect to the system \( \bar{Q} = \{Q_1, Q_2, \ldots, Q_n\} \) if and only if for
every \( v_l \in V(G) \) the following holds: two edges of an arbitrary transition of \( P_x \) through \( v_l \) belong to different classes of \( Q_i \). If all moves of a certain decomposition \( \overline{P} \) of \( G \) into moves are admissible with respect to \( \overline{Q} \), we call \( \overline{P} \) an admissible decomposition into moves with respect to \( \overline{Q} \).

Note 1. To the system \( \overline{Q} \) there corresponds a set of „forbidden“ transitions through the vertices. It is clear that we can construct the set of forbidden transitions also in another way. Some related problems have already been solved [2], [3], [4]). The first who formulated the problem to find the necessary and sufficient conditions for the existence of a Eulerian line not-containing the forbidden transitions was Nash — Williams (at Tihany, in September 1966). In this paper we deal with the solution of this problem, and also with that of a more general problem.

**Theorem 1.** Let \( G \) be a connected graph, and let its every vertex \( v_l \) be of an even degree (i.e. let \( d_l = 2c_l \), where \( c_l \) is a natural number). Let \( \overline{Q} = \{Q_1, Q_2, \ldots, Q_n\} \) be a given system of decompositions of the sets \( (E_i \ i = 1, 2, \ldots, n) \) into classes of edges. A Eulerian line of \( G \), admissible with respect to \( \overline{Q} \) exists if and only if for all \( i \in \{1, 2, \ldots, n\} \) we have: the number of elements of an arbitrary class of \( Q_i \) is not greater than \( c_i \).

**Proof.** If a certain class of a decomposition \( Q_l \in \overline{Q} \) has more edges than \( c_l \), then \( E_i \) cannot be decomposed into pairs of edges so that two edges of every pair belong to different classes of the decomposition \( Q_i \). Hence the condition for the existence of the Eulerian line admissible with respect to \( \overline{Q} \) mentioned in the theorem is a necessary condition.

Let \( \overline{Q} \) have the requested property, i.e. for all \( i \in \{1, 2, \ldots, n\} \) the following holds: the number of elements of an arbitrary class from \( Q_i \) is less than or equal to \( c_i \). Let us denote successively the edges of the set \( E_i \) by the symbols \( f_1(1), f_1(2), \ldots, f_1(2c_l) \) so that first of all we denote all edges of one class from \( Q_l \) and then all edges of the other classes of \( Q_i \). We then construct the decomposition \( D_i \) of the set \( E_i \) into \( c_i \) pairs of edges as follows: \( D_i = \{\{f_i(1), f_i(1 + c_l)\}, \{f_i(2), f_i(2 + c_l)\}, \ldots, \{f_i(c_l), f_i(2c_l)\}\} \). It is clear that every pair of \( D_i \) contains edges belonging to different classes of \( Q_i \). Let us put \( \overline{D} = \{D_1, D_2, \ldots, D_n\} \). According to Lemma 4 there exists exactly one decomposition \( \overline{P} = \{P_1, P_2, \ldots, P_m\} \) of \( G \) into closed moves so that \( \overline{D} = \delta(\overline{P}) \). From the construction of the system \( \overline{D} \) it follows that \( \overline{P} \) is an admissible decomposition into moves with respect to \( \overline{Q} \). If \( m = 1 \), then \( P_1 \) is the requested Eulerian line.

Suppose that \( m > 1 \). As \( G \) is a connected graph, there exists at least one such a vertex \( v_k \), which is the common vertex of certain two moves \( P_r, P_s \) from \( \overline{P} \). Let \( \{g_r, v_k, h_r\} \) be an arbitrary transition of \( P_r \) through \( v_k \) and let \( \{g_s, v_k, h_s\} \) be an arbitrary transition of \( P_s \) through \( v_k \).

From the construction of \( D_k \) it follows that the edges \( g_r, h_r \) (and also the
edges $g_r, h_s$) belong to different classes of $Q_k$. We distinguish the following
three cases: the edges of the set $H = \{g_r, h_r, g_s, h_s\}$ belong: (I) to 4 different
classes; (II) to 3 different classes; (III) to 2 different classes of $Q_k$. Without
loss of generality it can be supposed that in the case (II) the edges $g_r, g_s$ be-
long to the same class of $Q_k$ and in the case (III) the edges $g_r, g_s$ belong to
one and the edges $h_r, h_s$ belong to another class of $Q_k$.

Assertion. The moves $P_r, P_s$ can be unified into one closed move $P_t$ so that two
edges of an arbitrary transition through an arbitrary vertex $v_x$ belong again to
different classes of $Q_x$. Let us prove it. Consider that we can travel along all
the elements of a closed move (we are talking about a travelling mentioned in
the proof of Lemma 4) so that the transition of the move does not change for
any of the vertices even if we begin our travelling in an arbitrary chosen vertex
and if we are travelling in one of the two possible senses.

It is possible to travel through both the moves $P_r, P_s$ in one closed move in
such a way that we begin at the vertex $v_k$ along $h_r$, we travel along the whole
move $P_r$, and after we return into $v_k$ along the last edge of $P_r$ (it is clearly the
edge $g_r$) we continue to travel along the whole move $P_s$ beginning along the
edge $h_s$. If we have finished the whole move $P_t$ our travel comes to an end in
$v_k$, and the last edge along which we have been will be $g_s$. Instead of the
transition $\{g_r, v_k, h_r\}$ from $P_r$ and of the transition $\{g_s, v_k, h_s\}$ from $P_s$, there
will appear in $P_t$ the following two new transitions: $\{g_r, v_k, h_s\}, \{g_s, v_k, h_r\}$
and the rest of the transitions through all the vertices of the moves $P_r, P_s$
remain without changes in $P_t$. Since the edges $g_r, h_s$ (and also the edges $g_s, h_r$)
belong according to the supposition in all three possible cases to different classes
of $Q_k$, $P_t$ has the required properties and the validity of our assertion is proved.

Therefore if $m > 1$, then there exists a decomposition of $G$ into $m - 1$
closed and also with respect to $\bar{Q}$ admissible moves. After the finite number
$m - 1$ of such unifications of two moves we obtain the decomposition with
only one move, which is a Eulerian line admissible with respect to $\bar{Q}$. This
proves the theorem.

Theorem 2. Let $F$ be a graph with following property: each of its components
contains at least one vertex of an odd degree; let $V(F) = \{v_1, v_2, ..., v_n\}$ be the
set of its vertices, $E_i$ the set of edges incident at $v_i$. Let $2p$ be the number of all
vertices of an odd degree and let $\bar{R} = \{R_1, R_2, ..., R_n\}$ be the given system of the
decompositions of the sets $E_i$ ($i = 1, 2, ..., n$). Such a decomposition of $F$ into $p$
open moves, which is admissible with respect to $\bar{R}$, exists if and only if for all
$i \in \{1, 2, ..., n\}$ the number of elements of a arbitrary class of $R_i$ is not greater
than $\frac{1}{2}(1 + |E_i|)$.

Proof. Let $X(F) = \{v_{x(1)}, v_{x(2)}, ..., v_{x(2p)}\}$ be the set of all vertices of odd
degree from $V(F)$. Let us construct from the graph $F$ the graph $G$ in such a way
that we add to it one new vertex \( v_0 \) and \( 2p \) new edges \( e_1, e_2, \ldots, e_{2p} \) so that \( e_i \) joins \( v_0 \) with \( v_{x(i)} \). The following holds: \( G \) is a connected graph and each of its vertices has an even degree; every vertex from \( G \) which has in \( F \) an odd degree is incident in \( G \) at exactly one edge of the set \( E_0 = \{ e_1, e_2, \ldots, e_{2p} \} \).

We construct from \( \bar{R} \) a system of decompositions \( \bar{Q} = \{ Q_0, Q_1, Q_2, \ldots, Q_n \} \) of the sets \( E_i (i = 0, 1, 2, \ldots, n) \) in the following manner:

1. every class from \( Q_0 \) contains exactly one edge;
2. if \( i > 0 \) and \( v_t \) does not belong to \( X(F) \), then \( Q_i = R_i \);
3. \( Q_{x(j)} (j = 1, 2, \ldots, 2p) \) contains all classes from \( R_{x(j)} \) and another class containing one element and this is \( e_j \).

According to Theorem 1 in \( G \) there exists a Eulerian line \( L \), which is admissible with respect to \( \bar{Q} \) if and only if for all \( k \in \{ 0, 1, 2, \ldots, n \} \) the following holds: none of the classes of \( Q_k \) contains more than half of the edges incident at \( v_k \). \( \bar{Q} \) has this property if and only if \( \bar{R} \) has the following property: for an arbitrary \( i \in \{ 1, 2, \ldots, n \} \) we have: the number of edges of any class of \( \bar{R} \) is not greater than \( \frac{1}{2}(1 + |E_i|) \). Hence \( L \) exists if and only if \( \bar{R} \) has the property required in the theorem.

It is clear that if we delete from \( L \) all edges belonging to \( E_0 \) and also the vertex \( v_0 \), \( L \) splits into \( p \) open moves of \( F \), which have the required properties. Conversely: if a decomposition \( P \) of \( F \) into \( p \) open moves with the required properties exists, then there exists in \( G \) a Eulerian line, which has the required property. The validity of the theorem is clear.

Note 2. It is known that \( p \) from Theorem 2 is the minimal number of open moves into which the graph with \( 2p \) vertices of odd degree can be decomposed. The reader of this paper will himself easily prove that a graph which can be decomposed in this way cannot include a component which would have none of the vertices of odd degree. So the suppositions of Theorem 2 contain the conditions which the graph must fulfill in order that there exists any decomposition into a minimal number \( p \) of open moves. (See [1], [3].)

REFERENCES


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