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THE TENSOR PRODUCT OF RIGHT GROUPS

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Let A, B and C be semigroups. A mapping $\alpha: A \times B \to C$ of the cartesian product $A \times B$ into the semigroup C is called a bilinear mapping (also a bihomomorphism) if $\alpha(a_1a_2, b) = \alpha(a_1, b)\alpha(a_2, b)$ and $\alpha(a, b_1b_2) = \alpha(a, b_1)\alpha(a, b_2)$ for every $a_1, a_2, a \in A$ and $b_1, b_2, b \in B$.

P. A. Grillet defined in [1] a (noncommutative) tensor product $A \otimes B$ of semigroups A, B as the maximal bilinear image of the cartesian product $A \times B$. Maximal in the sense that there exists a bilinear mapping $\omega: A \times B \to A \otimes B$ with the following property: For every bilinear mapping $\alpha: A \times B \to C$ of the cartesian product $A \times B$ into any semigroup C, there exists a unique homomorphism $\varphi: A \otimes B \to C$ such that $\alpha = \varphi \circ \omega$ (Fig. 1).

The existence theorem, which is equivalent to this definition says: The tensor product $A \otimes B$ of semigroups A, B is the factor semigroup $\mathscr{F}(A \times B)/\tau$, where $\mathscr{F}(A \times B)$ is the free semigroup on the cartesian product $A \times B$ and τ is the smallest congruence on $\mathscr{F}(A \times B)$ such that $(a_1a_2, b)\tau(a_1, b)(a_2, b)$ and $(a, b_1b_2)\tau(a, b_1)(a, b_2)$ for every $a_1, a_2, a \in A$ and $b_1, b_2, b \in B$. We shall denote by $a \otimes b$ the class of the factor semigroup $\mathscr{F}(A \times B)/\tau$ which contains the element $(a, b) \in A \times B$. The mapping $\omega: A \times B \to A \otimes B = \mathscr{F}(A \times B)/\tau$ defined by $\omega(a, b) = a \otimes b$ for every $(a, b) \in A \times B$ has the universal property required in the definition of the tensor product.

In this paper we shall treat the tensor product of: I. Right simple semigroups; II. Right groups.

I. A semigroup S is called right simple if it contains no proper right ideal. A semigroup S is right simple if and only if for every $a, b \in S$ there exists $x \in S$ such that $ax = b$.

Theorem 1. *The tensor product $S \otimes T$ of right simple semigroups S and T is a right simple semigroup.*

Proof. 1. Let S and T be right simple semigroups. For every $a, b \in S$ and $c, d \in T$ there exist $x \in S$ and $y \in T$ such that $ax = b$ and $cy = d$. This implies: For $a \otimes c, b \otimes d \in S \otimes T$ there exists an element $z = (x \otimes c)(b \otimes y) \in S \otimes T$ such that $(a \otimes c)z = [(a \otimes c)(x \otimes c)](b \otimes y) = (b \otimes c)(b \otimes y) = b \otimes d$.

2. Let $(s_1 \otimes t_1)(s_2 \otimes t_2) \in S \otimes T$. Then there exists $x \in S$ such that $s_2x = s_1$.

Hence $(s_1 \otimes t_1)[(s_2 \otimes t_2)(x \otimes t_2)] = (s_1 \otimes t_1)(s_1 \otimes t_2) = (s_1 \otimes t_1 t_2)$. Repeating this procedure we obtain: Let $k = (s_1 \otimes t_1)(s_2 \otimes t_2) \dots (s_n \otimes t_n)$ be an element of $S \otimes T$. Then there exists an element $q \in S \otimes T$ such that $kq = s \times t$ for some $s \in S$ and $t \in T$.

3. Let $k = (s_1 \otimes t_1)(s_2 \otimes t_2) \dots (s_n \otimes t_n)$ and $l = (p_1 \otimes r_1)(p_2 \times r_2) \dots \dots (p_m \otimes r_m)$ be elements of $S \otimes T$. According to 2. there exists $q \in S \times T$ such that $kq = s \otimes t$ for some $s \in S$ and $t \in T$. By 1 to the couple $s \otimes t$, $p_1 \otimes r_1$ there exists a $z \in S \otimes T$ such that $(s \otimes t)z = p_1 \otimes r_1$. Now the element $u = qz(p_2 \otimes r_2) \dots (p_m \otimes r_m)$ has the property that $ku = l$. This proves Theorem 1.

Corollary. *The tensor product of groups is a group.*

II. A semigroup S is called a right group if it is right simple and left cancellable. Equivalently: To any elements $a, b \in S$ there exists a unique element $x \in S$ such that $ax = b$.

Lemma 1. (Clifford, Preston [2] p. 38) *The following assertions concerning a semigroup S are equivalent:*

a) S is a right group;

b) S is right simple and contains an idempotent;

c) S is isomorphic to the direct product $[G \times E]$ of a group G and a right zero semigroup E.

Remark. If S is a right group, its set of idempotents E is not empty and it is a right zero semigroup. Every element $e \in E$ is a left identity element of S. Let e be a fixed chosen element of E. Then $Se = G$ is a subgroup of S and e is the identity element of G. It is known that $S = GE$. When considering the direct product $[G \times E]$, the mapping $\vartheta\colon GE \to [G \times E]$ defined by $\vartheta(ge) = [g, e]$ is an isomorphism of the semigroups S and $[G \times E]$. In this case the group G and the semigroup E are a special subgroup and a special subsemigroup of S. But it is easy to prove the next lemma.

Lemma 2. *Suppose that $S = GE$, where G is an arbitrary subgroup of S and E is a right zero subsemigroup of S. Let the identity element 1_G of the group G be an element of E. Then S is a right group and S is isomorphic to the direct product $[G \times E]$.*

Note further: If A contains an idempotent e and B is any semigroup, then $e \otimes b$ is an idempotent in $A \otimes B$ for an arbitrary $b \in B$. Hence $A \otimes B$ certainly contains an idempotent if one of the „factors" contains an idempotent.

Theorem 1 and Lemma 1 b) imply:

Theorem 2. *The tensor product of a right group and a right simple semigroup is a right group.*

Let $A = GE$, $B = HF$ be right groups. By Theorem 2 $A \otimes B$ is also a right group. By Lemma 1 $A \otimes B = KJ$, where K is a subgroup of the tensor product $A \otimes B$ and J is the set of all idempotents of $A \otimes B$. In the following we shall describe the group K and the right zero semigroup J by means of G, H, E and F.

Lemma 3. *If E, F are right zero semigroups, then the tensor product $E \otimes F$ is a right zero semigroup which is isomorphic to the direct product $[E \times F]$.*

Proof. 1) The direct product $[E \times F]$ of right zero semigroups E, F is a right zero semigroup. For $[e_1, f_1][e_2, f_2] = [e_1e_2, f_1f_2] = [e_2, f_2]$.

2) Let $E \times F$ be the cartesian product of right zero semigroups E, F. The mapping $i: E \times F \to [E \times F]$ defined by $i(e,f) = [e,f]$ is a bilinear mapping, since $i(e_1e_2, f) = i(e_2, f) = [e_2, f] = [e_1, f][e_2, f] = i(e_1, f)i(e_2, f)$ for every $e_1, e_2, e \in E$ and $f_1, f_2, f \in F$. Similarly $i(e, f_1f_2) = i(e, f_1)i(e, f_2)$.

Let $\alpha: E \times F \to S$ be a bilinear mapping of the cartesian product $E \times F$ into an arbitrary semigroup S. Define the mapping $\varphi: [E \times F] \to S$ in the following way: $\varphi([e,f]) = \alpha(e,f)$ for every $[e,f] \in [E \times F]$. (See Fig. 2.) We have: $\varphi([e_1, f_1][e_2, f_2]) = \varphi([e_2, f_2]) = \alpha(e_2, f_2) = \alpha(e_1e_2, f_2) = \alpha(e_1, f_2)\alpha(e_2, f_2) = \alpha(e_1, f_1f_2)\alpha(e_2, f_2) = \alpha(e_1, f_1)\alpha(e_1, f_2)\alpha(e_2, f_2) = \alpha(e_1, f_1)\alpha(e_2, f_2) = \varphi([e_1, f_1])\varphi([e_2, f_2])$. Hence φ is a homomorphism of the direct product $[E \times F]$ into the semigroup S such that $\varphi \circ i(e,f) = \varphi([e,f]) = \alpha(e,f)$. Thus $\varphi \circ i = \alpha$.

3) The mapping i and the direct product $[E \times F]$ have the universal property required in the definition of the tensor product $E \otimes F$. Hence $[E \times F]$ is the tensor product of the semigroups E, F (which is determined up to an isomorphism). This proves Lemma 2.

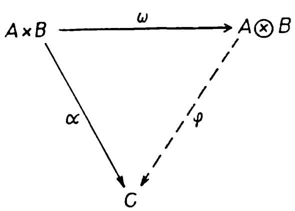

Fig. 1

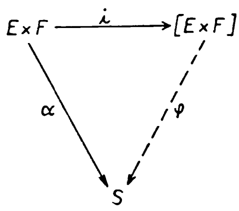

Fig. 2

Theorem 3. *Let $A = GE$ and $B = HF$ be right groups. Then $A \otimes B$ is isomorphic to the direct product of the tensor product $G \otimes H$ and the direct product $[E \times F]$. In formula: $A \otimes B \cong [(G \otimes H) \times [E \times F]]$.*

Proof. 1) In order to distinguish between $g \otimes h$ in $A \otimes B$ and $g \otimes h$ in $G \otimes H$, we denote in the following by $G \mathbin{\overline{\otimes}} H$ the tensor product of the semigroups G and H and by $g \otimes h$ its generating elements. The elements $g \otimes h \in A \otimes B$ with $g \in G$, $h \in H$ generate in $A \otimes B$ a subsemigroup. We denote this subsemigroup by $\otimes (G, H)$. Similarly $\otimes (E, F)$ will be the subsemigroup of $A \otimes B$, the elements of which are of the form $(e_1 \otimes f_1)(e_2 \otimes f_2) \ldots \ldots (e_n \otimes f_n)$, where $e_i \in E$, $f_i \in F$. It is known that the semigroup $\otimes (G, H)$ is a subgroup of $A \otimes B$. It is further easy to see that the semigroup $\otimes (E, F)$ is a right zero subsemigroup of the tensor product $A \otimes B$.

2) The tensor product $A \otimes B$ is a right group. An arbitrary element $x \in A \otimes B$ is of the form $x = (g_1e_1 \otimes h_1f_1) \ldots (g_ne_n \otimes h_nf_n)$, where $g_i \in G$, $h_i \in H$, $e_i \in E$, $f_i \in F$. Further we can write $x = (g_1 \otimes h_1)(g_2 \otimes h_2) \ldots \ldots (g_n \otimes h_n)(e_n \otimes f_n)$ since the idempotents of the right group $A \otimes B$ are left identity elements in $A \otimes B$. It is clear that $A \otimes B = (\otimes(G, H))(\otimes(E, F))$ and by Lemma 2 the semigroup $A \otimes B$ is isomorphic to the direct product of $\otimes(G, H)$ and $\otimes(E, F)$.

We shall now prove that $\otimes(G, H)$ is isomorphic to $G \otimes H$ and $\otimes(E, F)$ is isomorphic to $E \overline{\otimes} F$.

3) Let $1_G \in G$ and $1_H \in H$ be the identity elements of the groups G and H, respectively. We define a mapping $\alpha: A \times B \to G \otimes H$ in the following way: $\alpha(a, b) = (a1_G \overline{\otimes} b1_H)$. Then $\alpha(a_1a_2, b) = (a_1a_21_G \overline{\otimes} b1_H) = (a_11_Ga_21_G \overline{\otimes} b1_H) = (a_11_G \overline{\otimes} b1_H)(a_21_G \overline{\otimes} b1_H) = \alpha(a_1, b)\alpha(a_2, b)$. Similarly $\alpha(a, b_1b_2) = \alpha(a, b_1)\alpha(a, b_2)$ for every $a_1, a_2, a \in A$, $b_1, b_2, b \in B$. Therefore $\alpha: A \times B \to \to G \overline{\otimes} H$ is a bilinear mapping. To the bilinear mapping α there exists a unique homomorphism $\varphi: A \otimes B \to G \overline{\otimes} H$ such that $\alpha = \varphi \circ \omega$ (Fig. 3).

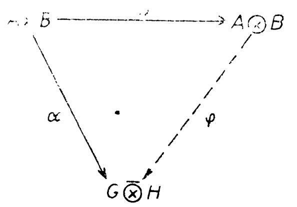

Fig. 3

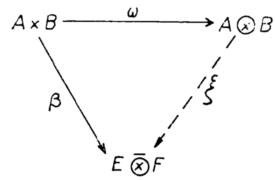

Fig. 4

Let $(g_1 \otimes h_1) \ldots (g_n \otimes h_n)$ be an arbitrary element of $\otimes(G, H)$. Then $\varphi\{(g_1 \otimes h_1) \ldots (g_n \otimes h_n)\} = \varphi(g_1 \otimes h_1) \ldots (g_n \otimes h_n) = \varphi \circ \omega(g_1, h_1) \ldots \ldots \varphi \circ \omega(g_n, h_n) = \alpha(g_1, h_1) \ldots \alpha(g_n, h_n) = (g_11_G \overline{\otimes} h_11_H) \ldots (g_n1_G \otimes h_n1_H) = (g_1 \overline{\otimes} h_1) \ldots (g_n \overline{\otimes} h_n)$. The restriction φ_1 of the mapping φ to the semigroup $\otimes(G, H)$ is a homomorphism of the group $\otimes(G, H)$ onto the group $G \otimes H$.

On the other hand it is known that $\otimes(G, H)$ is a homomorphic image of $G \overline{\otimes} H$ under the mapping ψ defined as follows: $\psi(g \overline{\otimes} h) = g \otimes h$. Hence evidently $\varphi_1 \circ \psi = i_{G \otimes H}$ and $\psi \circ \varphi_1 = i_{\otimes(G, H)}$. Hereby $i_{G \overline{\times} H}$ and $i_{\times(G, H)}$ are the identical mappings of the semigroups $G \otimes H$ and $\otimes(G, H)$ respectively. Therefore $\varphi_1 \colon \otimes (G, H) \to G \otimes H$ is an isomorphism.

4) Any element $a \in A$ can be written (in a unique way) in the form $a = ge$, with $g \in G$, $e \in E$. Similarly for any $b \in B$ we have $b = hf$, with $h \in H$, $f \in F$. Now we define a bilinear mapping $\beta \colon A \times B \to E \overline{\otimes} F$ in the following way: $\beta(a, b) \quad \beta(ge, hf) = e \otimes f$. For the mapping β there exists a unique homomorphism $\xi \colon A \otimes B \to E \otimes F$ such that $\beta = \xi \circ \omega$ (Fig. 4).

Similarly as in 3) it is easy to show that the restriction ξ_1 of ξ to the semigroup $\times(E, F)$ is an isomorphism of the semigroups $\otimes(E, F)$ and $E \overline{\otimes} F$.

5) According to Lemma 3 $E \otimes F \backsim [E \times F]$. Hence by 2), 3), 4) we obtain: $A \otimes B \backsim [(G \otimes H) \times [E \times F]]$. This proves Theorem 3.

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