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THE EXTREMAL CONNECTIVITY OF THE STRICTLY WEAK DIGRAPH

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The connectivity of an undirected graph (its extremal values) was considered in [2]. While for an undirected graph the connectivity is defined in a unique way, for a digraph we can us many different definitions of this concept. The author uses one way in [3] (under the connectivity of a digraph there is meant the minimal number of elements of $V(G)$, $E(G)$, respectively, by the removing of which we get either a digraph which is not strong or the trivial digraph), where the extremal connectivities of digraphs are shown. Using this way we have *many* digraphs for which the connectivity equals 0. In the presented paper the connectivity of a digraph is defined in a similar way as for an undirected graph and for the strictly weak digraphs its extremal values are deduced.

All concepts and symbols are used in the sense of the monograph [1], for example $C_0(\alpha, \beta)$ ($C_1(\alpha, \beta)$) denotes the set of disconnected (strictly weak, i. e. weak but not unilateral) digraphs with α vertices and β edges.

Definition. *The vertex (edge) connectivity $c_v(G)$ ($c_e(G)$) of a digraph G is the minimum number of vertices (edges) by the removing of which we get either a disconnected or the trivial digraph.*

In this paper we determine the extremal values $c_v(G)$ ($c_e(G)$) for $G \in C_1(\alpha, \beta)$. (For $G \in C_0(\alpha, \beta)$ we obviously have $c_v(G) = c_e(G) = 0$). We introduce now some strictly weak digraphs which we will use later. Let α be a positive integer and $V = \{v_1, v_2, \ldots, v_\alpha\}$ a set of α elements. We define digraphs

$$G^+(\alpha) = (V, E^+), E^+ = \{v_1v_i, v_2v_i, v_iv_j\}; i, j = 3, 4, \ldots, \alpha; i \neq j,$$

$$G^-(\alpha) = (V, E^-), E^- = \{v_iv_1, v_iv_2, v_iv_j\}; i, j = 3, 4, \ldots, \alpha; i \neq j,$$

$$G_1(\alpha) = (V, E_1), E_1 = \{v_2v_1, v_2v_i, v_iv_j\}; i, j = 3, 4, \ldots, \alpha; i \neq j,$$

$$G_2^n(\alpha) = (V, E_2), E_2 = \{v_{\alpha-1}v_i, v_\alpha v_j, v_jv_k\}; i = 1, 2, \ldots, n; \\ j, k = 1, 2, \ldots, \alpha - 2; j \neq k; 2 \leqq n \leqq \alpha - 2,$$

$$G_3(\alpha) = (V, E_3), E_3 = \{v_1v_i, v_2v_i\} \cup E_0; i = 3, 4, \ldots, \alpha,$$

where E_0 is a set of edges which are incident with vertices $v_3, v_4, \ldots, v_\alpha$ such that there exists a subgraph of the digraph $G_3(\alpha)$ induced by this vertex set that is a tournament with $\alpha - 2$ vertices.

Remark 1. If $G \in C_1(\alpha, \beta)$, then $\alpha \geqq 3$ and $\beta \geqq 2$. For $\alpha = 3$ there exist exactly two strictly weak digraphs: $G' = (V', E')$ with $V' = \{v_1, v_2, v_3\}$, $E' = \{v_1v_2, v_3v_2\}$ and $G'' = (V'', E'')$ with $V'' = V'$, $E'' = \{v_2v_1, v_2v_3\}$.

Lemma 1. *For $G \in C_1(\alpha, \beta)$ we have:*

$$1 \leqq c_v(G) \leqq c_e(G) \leqq \min_{v \in V(G)} \deg v,$$
$$\text{where } \deg v = od\, v + id\, v.$$

The proof of Lemma 1 is analogous to that of undirected graphs used in [1].

Remark 2. There exists a strictly weak digraph G for which $c_v(G) < c_e(G)$. As an example we take the digraph $G \in C_1(5, 6)$, $V(G) = \{v_1, v_2, v_3, v_4, v_5\}$, $E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_1v_5, v_2v_3, v_4v_5\}$, which has $c_v(G) = 1$ and $c_e(G) = 2$.

Lemma 2. *Let $G \in C_1(\alpha, \beta)$, $\beta \geqq (\alpha - 2)^2 + 2$. Then G is isomorphic either with a subgraph of the digraph $G^+(\alpha)$ or with a subgraph of the digraph $G^-(\alpha)$.*

Proof. Let $G \in C_1(\alpha, \beta)$. Then there exist the vertices $a, b \in V(G)$ such that the paths connecting them do not exist. Let us denote $V_0 = V(G) - \{a, b\}$, $|V_0| = \alpha - 2$. In order to prove Lemma 2 we have to prove that either od a = od $b = 0$ or id a = id $b = 0$. We do it indirectly. We suppose that *od a* $= k \geqq 1$, id $b = h \geqq 1$ and let us denote

$$V_0(a) = \{v \in V_0 |\ av \in E(G)\},$$
$$V_0(b) = \{u \in V_0 |\ ub \in E(G)\},$$

where $V_0(a) \cap V_0(b) = \emptyset$, $|V_0(a)| = k$, $|V_0(b)| = h$, $2 \leqq k + h \leqq \alpha - 2$. The digraph G does not contain ab, ba and also the following edges:

vb for $\forall\, v \in V_0 - V_0(b)$	(in number $\alpha - 2 - h$)
aw for $\forall\, w \in V_0 - V_0(a)$	$(\alpha - 2 - k)$
xy for $\forall\, x \in V_0(a),\ y \in V_0(b)$	$(k \,.\, h)$

Finally, the digraph G contains at most one of every pair of the edges bu, ua, $u \in V_0$. We obtain

$$\beta \leqq \alpha(\alpha - 1) - (\alpha - 2 - h + \alpha - 2 - k + k \,.\, h + \alpha - 2 + 2)$$
$$= \alpha^2 - 4\alpha + 4 + h + k - k \,.\, h \leqq \alpha^2 - 4\alpha + 5 = (\alpha - 2)^2 + 1,$$

which is a contradiction. Analogously, the inequalities *id a* $\geqq 1$, od $b \geq 1$ cannot hold simultaneously either. From these facts and from the inequalities od a + id $a \geqq 1$, od b + id $b \geqq 1$ it follows that od a = od $b = 0$ or id a = id $b = 0$.

Lemma 3. *Let* $G \in C_1(\alpha, \beta)$, $\bar{G} = (V(G), E(G) \cup \{e\})$, $\bar{G} \in C_1(\alpha, \beta + 1)$, *where* $e \notin E(G)$. *Then we have:*

$$c_v(\bar{G}) \leqq c_v(G) + 1,$$
$$c_e(\bar{G}) \leqq c_e(G) + 1.$$

The proof of Lemma 3 is trivial and analogous to that of undirected graphs, we do not state here.

Theorem 1. *Let* $C_1(\alpha, \beta) \neq \emptyset$. *Then*

$$\min_{G \in C_1(\alpha,\beta)} c_v(G) = \min_{G \in C_1(\alpha,\beta)} c_e(G) = \max\{1, \beta - (\alpha - 2)^2\}.$$

Proof. If $C_1(\alpha, \beta) \neq \emptyset$, then according to [4] we have $\alpha - 1 \leqq \beta \leqq (\alpha - 1)(\alpha - 2)$.

I. Let $\alpha - 1 \leqq \beta \leqq (\alpha - 2)^2 + 1$. Then there exists a digraph $G \in C_1(\alpha, \beta)$ with $c_v(G) = 1$. We can take for G a connected subgraph of $G_1(\alpha)$ with β edges.

II. Let $(\alpha - 2)^2 + 2 \leqq \beta \leqq (\alpha - 1) . (\alpha - 2)$. We have to prove that $\min c_v(G) = \beta - (\alpha - 2)^2$. We do it indirectly. Let $\bar{G} \in C_1(\alpha, \beta)$ and $c_v(\bar{G}) < \beta - (\alpha - 2)^2$. According to Lemma 2, $\bar{G}$ is isomorphic either with a subgraph of $G^+(\alpha)$ or with a subgraph of $G^-(\alpha)$. By adding $(\alpha - 1) . (\alpha - 2) - \beta$ edges to the digraph $\bar{G}$ we obtain a digraph which is isomorphic with $G^+(\alpha)$ (or $G^-(\alpha)$). By using Lemma 3 repeatedly we have:

$$c_v(G^{\pm}(\alpha)) \leqq c_v(\bar{G}) + (\alpha - 1) . (\alpha - 2) - \beta <$$
$$< \beta - (\alpha - 2)^2 + (\alpha - 1) . (\alpha - 2) - \beta = \alpha - 2,$$

which is a contradiction. We have proved that $c_v(G) \geqq \beta - (\alpha - 2)^2$. Finally, there exists a digraph $G \in C_1(\alpha, \beta)$ with $c_v(G) = \beta - (\alpha - 2)^2$; e. g. $G = G_2^n(\alpha)$ for $n = \beta - (\alpha - 2)^2$. Thus, one part of Theorem 1 is proved.

By Lemma 1, $\min c_e(G) \geqq \min c_v(G)$ for $G \in C_1(\alpha, \beta)$. For the minimum edge connectivity the same extremal digraphs can be chosen as those used in the proof above. It means that the equality $\min c_e(G) = \min c_v(G)$ holds. Thus, Theorem 1 is proved.

Theorem 2. *Let* $C_1(\alpha, \beta) \neq \emptyset$. Then

$$\max_{G \in C_1(\alpha,\beta)} c_v(G) = \max_{G \in C_1(\alpha,\beta)} c_e(G) = \min\left\{\alpha - 2, \left[\frac{2\beta}{\alpha}\right]\right\}.$$

Proof. I. Let $\frac{\alpha(\alpha - 1)}{2} - 1 \leqq \beta \leqq (\alpha - 1) . (\alpha - 2)$, then there exist digraphs $G \in C_1(\alpha, \beta)$ with $c_e(G) = \alpha - 2$, e. g. the digraph $G_3(\alpha)$ with arbitrarily added $\beta - \frac{\alpha(\alpha - 1)}{2} + 1$ edges incident with vertices $v_3, v_4, \ldots, v_\alpha$. Moreover, $c_e(H) \leqq \alpha - 2$ for every $H \in C_1(\alpha, \beta)$.

II. Let $\alpha - 1 \leqq \beta < \frac{\alpha(\alpha - 1)}{2} - 1$. For $\beta < \frac{\alpha(\alpha - 1)}{2}$ we take an undirected graph H with α vertices and β edges with the connectivity $\left[\frac{2\beta}{\alpha}\right]$. This construction is described in [2]. (The vertex (edge) connectivity of a graph is the minimum number of elements of $V(H)$ $(E(H))$ after the removing of which we obtain either a disconnected or a trivial graph). In H there exist vertices a, b for which $e = ab \notin E(H)$. We direct all edges of H in such a way the direction of all edges incident with verices a, b are outgoing from a or b and the others are directed arbitrarily. We obtain a strictly weak digraph $\bar{H}$ for which $c_e(\bar{H}) = \left[\frac{2\beta}{\alpha}\right]$. Now we prove that $c_e(G) \leqq \left[\frac{2\beta}{\alpha}\right]$ for $G \in C_1(\alpha, \beta)$

Let $\deg v > \frac{2\beta}{\alpha}$ for all $v \in V(G)$. Then

$$2\beta = \sum_{v \in V(G)} \deg v > \alpha \frac{2\beta}{\alpha} = 2\beta,$$

gives a contradiction. We have at least one vertex $u \in V(G)$ with $\deg u \leq \frac{2\beta}{\alpha}$.

By Lemma 1, $c_e(G) \leqq \left[\frac{2\beta}{\alpha}\right]$ for all $G \in C_1(\alpha, \beta)$. It completes the proof for the edge connectivity.

III. Using Lemma 1, $\max c_v(G) \leqq \max c_e(G)$ for $G \in C_1(\alpha, \beta)$. Extremal digraphs for the maximum vertex connectivity can be chosen the same as those used in the preceding parts of the proof.

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