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THE HADWIGER NUMBER OF AN INFINITE GRAPH

BOHDAN ZELINKA

In the papers [3] and [4] the Hadwiger numbers of finite graphs were studied. Here we would like to remark on the Hadwiger numbers of infinite graphs.

Let $G$ and $G'$ be two undirected graphs. We say [2] that $G$ can be contracted onto $G'$, if and only if $G'$ can be obtained from $G$ by a finite number of the following operations:

(a) deleting an edge;
(b) deleting an isolated vertex;
(c) contracting an edge.

To contract an edge $e$ of $G$ with the end vertices $u$ and $v$ means to delete the vertices $u$ and $v$ and the edge $e$ from $G$ and to add a new vertex $w$ to $G$ and to join $w$ with all vertices of $G$ different from $u$ and $v$ which were adjacent to $u$ or $v$.

For finite graphs the following definition of the Hadwiger number was used [2].

The Hadwiger number $\eta(G)$ of an undirected graph $G$ is the maximal number of vertices of a complete graph onto which $G$ can be contracted.

In [4] it was shown that for finite connected graphs this definition is equivalent to the following one.

The Hadwiger number $\eta(G)$ of a connected undirected graph $G$ is the maximal number of the classes of a partition $\mathcal{P}$ of the vertex set of $G$ with the property that each class of $\mathcal{P}$ induces a connected subgraph of $G$ and to any two of these classes there exists an edge of $G$ joining a vertex of one of them with a vertex of the other.

For infinite graphs these definitions are not equivalent. If for some connected graph there exists the Hadwiger number according to the first definition, there exists also the Hadwiger number of it according to the second definition and it is equal to the Hadwiger number according to the first definition. If $G$ is contracted onto some complete graph $K$, then there exists a connected homomorphism [1] of $G$ onto $K$; the partition $\mathcal{P}$ is obtained so that two vertices belong to the same class of $\mathcal{P}$ if and only if their images in this homomorphism are equal. On the other hand, if we have such a partition $\mathcal{P}$ and if a class of $\mathcal{P}$
is finite, then the subgraph induced by it can be contracted onto a graph consisting of one vertex by a finite number of operations \((a), (b), (c)\). If it is infinite, this is not possible. Therefore there exist infinite connected graphs, for which there exists the Hadwiger number according to the second definition, but not according to the first definition. An example is a graph consisting of one (one-way or two-way) infinite path. The Hadwiger number of this graph according to the first definition is undefined, according to the second definition it is equal to 2.

Thus if \(G\) is a connected undirected graph, we shall consider three numbers \(\eta_1(G), \eta_2(G), \eta(G)\) for it. The number \(\eta_1(G)\) is the maximal number of vertices of a complete graph onto which \(G\) can be contracted. The number \(\eta_2(G)\) is the maximal number of classes of a partition \(\mathcal{P}\) of the vertex set of \(G\) with the property that each class of \(\mathcal{P}\) induces a connected subgraph of \(G\) and to any two of these classes there exists an edge of \(G\) joining a vertex of one of them with a vertex of the other. Using the concept of connected homomorphism, we can express the definition of \(\eta_2(G)\) in other words. The number \(\eta_2(G)\) is the maximal cardinality of the vertex set of a complete graph which is an image of \(G\) in a connected homomorphism. Finally, \(\eta(G)\) will denote the supremum of cardinalities of vertex sets of complete graphs onto which the graph \(G\) can be mapped by connected homomorphisms. This number \(\eta(G)\) will be called the Hadwiger number of the graph \(G\).

It is easy to see that \(\eta(G)\) exists for every connected undirected graph \(G\). If \(G\) is finite, it is equal to the Hadwiger number of \(G\), as it was defined for finite graphs. If \(G\) is infinite, we have seen that \(\eta_1(G)\) need not exist. We shall show that \(\eta_2(G)\) need not exist, either.

**Theorem 1.** There exists a connected graph \(G\) for which \(\eta_2(G)\) is undefined.

**Proof.** Let \(K_1, K_2, \ldots\) be an infinite (of the ordinal number \(\omega_0\)) sequence of pairwise vertex-disjoint complete graphs; let the number of vertices of \(K_1\) be \(n\) for each positive integer \(n\). In each \(K_n\) we choose a vertex \(u_n\). Each \(u_n\) will be joined with \(u_{n+1}\) by an edge. The graph thus obtained will be denoted by \(G\). Suppose that \(\eta_2(G)\) exists. If it is finite, then denote \(\eta_2(G) = m\), where \(m\) is some positive integer. Define \(\mathcal{P}\) so that one class of \(\mathcal{P}\) consists of \(u_n\) and of all vertices of \(G\) not belonging to \(K_{m-1}\) and each other class of \(\mathcal{P}\) consists of only one vertex which belongs to \(K_{m-1}\) and is different from \(u_m\). The partition \(\mathcal{P}\) satisfies the conditions from the definition and has \(m - 1\) classes, thus we have a contradiction. Now suppose that \(\eta_2(G)\) is infinite. Let \(\mathcal{P}\) be the corresponding partition. For any two edges joining vertices of different classes of \(\mathcal{P}\) there exists a circuit containing both of them; thus they lie in the same block of \(G\). Therefore all such edges lie in the same block of \(G\) and in this block there is at least one vertex from each class of \(\mathcal{P}\). But the classes...
The numbers \( r_{j1}(G) \) and \( r_{j2}(G) \) need not exist, but evidently \( r_j(G) \) exists for every graph \( G \). Further, if \( r_{j1}(G) \) or \( r_{j2}(G) \) exists, it is equal to \( r_j(G) \). For the graph \( G \) from the proof of Theorem 1 we have \( r_j(G) = \aleph_0 \).

**Theorem 2.** Let \( n \) be a regular transfinite cardinal number. Let \( G \) be a connected undirected graph with \( r_{j2}(G) = n \). Then \( G \) contains at least \( n \) vertices of an at least \( n \) degree.

**Proof.** The vertex set \( V \) of \( G \) can be partitioned into \( n \) pairwise disjoint subsets which satisfy the condition from the definition. Let this partition be denoted by \( \mathcal{P} \). If \( C_0 \in \mathcal{P} \), then there exist at least \( n \) edges joining a vertex of \( C_0 \) with a vertex of another class of \( \mathcal{P} \). Suppose that each vertex \( C_0 \) has a degree less than \( n \). As \( n \) is regular, there exists a transfinite cardinal number \( m < n \) such that the degree of any vertex of \( C_0 \) is at most \( m \). Let \( G_0 \) be the subgraph of \( G \) induced by \( C_0 \). The degree of any vertex of \( C_0 \) is at most \( m \) also in \( G_0 \). Let \( u \in C_0 \). As \( G_0 \) is connected, each vertex of \( C_0 \) is connected with \( u \) by some path and therefore its distance from \( u \) in \( G_0 \) is equal to some non-negative integer. The number of vertices which have the distance \( d \) from \( u \), where \( d \) is some non-negative integer, is at most \( m^d \). But \( m^d = m \), because \( m \) is a transfinite cardinal number. The number of all vertices of \( C_0 \) is at most \( m \cdot \aleph_0 = m \). From any of the vertices of \( C_0 \) at most \( m \) vertices go into other classes of \( \mathcal{P} \), therefore the total number of vertices which do not belong to \( C_0 \) and are adjacent to vertices of \( C_0 \) is at most \( m^2 \). As the number of classes of \( \mathcal{P} \) is \( n > m \), there are classes of \( \mathcal{P} \), none of whose vertices is adjacent to a vertex of \( C_0 \), which is a contradiction. Thus \( C_0 \) contains at least one vertex of degree at least \( n \). As \( C_0 \) was chosen arbitrarily, each class of \( \mathcal{P} \) contains such a vertex and the number of such vertices is at least \( n \).

**Theorem 3.** Let \( n \) be a singular transfinite cardinal number different from \( \aleph_0 \). Let \( G \) be a connected undirected graph with \( r_{j2}(G) = n \). If \( m < n \), then there exist at least \( n \) vertices of \( G \) of a degree greater than \( m \).

**Proof.** Let \( \mathcal{P} \) be the same as in the proof of Theorem 2, let \( C_0 \in \mathcal{P} \). Suppose that \( m \) is transfinite. If we suppose that the degrees of the vertices of \( C_0 \) do not exceed \( m \), we obtain the same contradiction as in the proof of Theorem 2. Thus each \( C_0 \in \mathcal{P} \) contains a vertex of a degree greater than \( m \) and the number of such vertices is at least \( n \). We have proved the assertion for each transfinite \( m \), therefore it holds evidently also for each finite cardinal number \( m \).

Analogously we can prove a theorem concerning \( \eta(G) \).

**Theorem 4.** Let \( n \) be a singular transfinite cardinal number different from \( \aleph_0 \).
Let $G$ be a connected undirected graph such that $r_j(G) = n$. If $m < n$, then $G$ contains more than $m$ vertices of a degree greater than $m$.

Now we shall prove a theorem which differs substantially from a similar theorem for finite graphs [3].

**Theorem 5.** For each transfinite cardinal number $n$ there exists a connected undirected graph in which all vertices have the degree $n$ and whose Hadwiger number is 2.

**Proof.** Let $M$ be a set of the cardinality $n$. The vertex set of the graph $G$ is the set of all finite sequences (including the empty sequence) of elements of $M$. Two vertices are joined by an edge, if and only if one of them is obtained from the other by adding one term after its last term. The graph $G$ is a tree therefore its Hadwiger number is equal to 2.

**Theorem 6.** There exists a graph in which the degree of any vertex does not exceed 3 and whose Hadwiger number is $\aleph_0$.

**Proof.** Let us have an infinite sequence (of the ordinal number $\omega$) of vertex-disjoint simple one-way infinite paths $P_0, P_1, P_2, \ldots$. The vertices of the path $P_n$, where $n$ is some non-negative integer, are $u_n^0, u_n^1, u_n^2, \ldots$, the edges are $u_n^i u_n^{i+1}$ for $i = 0, 1, 2, \ldots$. Now we shall join each vertex $u_n^m$, where $m \neq n$, with the vertex $u_n^m$. Thus we obtain a graph $G$. A vertex $u_n^m$, where $m \neq n, m \neq 0$, has degree 3, because it is adjacent only to the vertices $u_n^{m-1}, u_n^{m+1}, u_n^m$. A vertex $u_n^0$, where $n \neq 0$, is adjacent only to the vertices $u_n^1, u_n^0$, therefore it has degree 2. A vertex $u_n^n$, where $n \neq 0$, is adjacent to the vertices $u_n^{n-1}, a_n^{n+1}$; it has degree 2. The vertex $u_0^0$ is adjacent only to $u_0$ and has degree 1. Each path $P_n$ is a connected graph. To any $m$ and $n, m \neq n$, there exists an edge $u_n^m u_n^n$ joining a vertex of $P_m$ with a vertex of $P_n$. Thus $\eta_2(G) = \eta(G) = \aleph_0$.

**Theorem 7.** Let $G$ be a connected undirected graph, let $\eta(G) = n$, where $n$ is a finite number. Then there exists a finite subgraph $G_0$ of $G$ such that $\eta(G_0) = n$.

**Proof.** If $\eta(G)$ is finite, then evidently $\eta_2(G)$ exists and $\eta_2(G) = n$. Consider again the partition $\mathcal{P}$. It has $n$ classes. If $C_1, C_2$ are two different classes of $\mathcal{P}$, we delete all edges joining vertices of $C_1$ with vertices of $C_2$, except for one. If we do this for any two classes of $\mathcal{P}$, we obtain a graph $G'$ in which for any two classes of $\mathcal{P}$ there exists only one edge joining a vertex of one with a vertex of another. Now let $C \in \mathcal{P}$. There are at most $n - 1$ vertices of $C$ which are adjacent to vertices of other classes. For any two of these vertices choose a path connecting them in the subgraph of $G$ induced by $C$ (this subgraph is connected) and delete all vertices of $C$ which do not belong to any of these paths. If we do this for each $C \in \mathcal{P}$, we obtain a graph $G_0$. Let $\mathcal{P}_0$ be the partition of the vertex set of $G_0$ such that two vertices of $G_0$ belong to the same class of $\mathcal{P}_0$, if and only if they belong to the same class of $\mathcal{P}$. Each class
of $\mathcal{D}_0$ is a finite set and induces a connected subgraph of $G_0$ and to any two of these classes an edge exists which joins a vertex of one of them with a vertex of the other. Thus $G_0$ is finite and $\eta(G_0) \geq n$. As $G_0$ is a subgraph of $G$, its Hadwiger number cannot exceed the Hadwiger number of $G$ and we have $\eta(G) = n$.

REFERENCES


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