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CONDITIONS FOR INDEPENDENCE OF VARIETIES

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Preliminaries. Varieties K_0, K_1 of the same type are said to be *independent* (cf. [4]) if there exists a binary polynomial symbol p such that $p(x_0, x_1) = x_i$ holds in K_i, $i = 0,1$. The smallest variety K containing K_0 and K_1 is denoted by $K = K_0 \vee K_1$. The class of all algebras $\mathfrak{A} = \langle A; F \rangle$ which are isomorphic to an algebra of the form $\mathfrak{A}_0 \times \mathfrak{A}_1$ (direct product), $\mathfrak{A}_0 \in K_0$, $\mathfrak{A}_1 \in K_1$ is denoted by $K_0 \times K_1$. $\mathscr{C}(\mathfrak{A})$ denotes the lattice of all congruence relations of the algebra $\mathfrak{A}$. In [4] among others the following assertion is proved.

Theorem A [4, Theorem 2]. *Let the varieties K_0, K_1 (of the same type) satisfy the following conditions.*

(1) *The variety $K_0 \wedge K_1$ consists of one-element algebras only.*

(2) $K_0 \vee K_1 = K_0 \times K_1$.

(3) *Every algebra $\mathfrak{A} \in K_0 \vee K_1$ has a modular congruence lattice.*

Then K_0, K_1 are independent.

J. Płonka put a question (oral communication) whether the condition (3) in Theorem A can be omitted. Example 2 of [2] gives independent varieties K_0 (of all groups) and K_1 (of all skew-lattices), where the congruence lattice of the algebras of K_1 is not modular exept in trivial cases. The following Theorem 1 shows that in Theorem A the condition (3) can be replaced by a weaker condition (4). (In the above quoted example the condition (4) is satisfied — by Theorem 1 — but not the condition (3).) Theorem 2 and Corollary show that (3) can be omitted in some special cases.

We shall use the following assertion.

Theorem B [3, Theorem 3]. *Let K_0, K_1 be varieties of the same type. The following conditions are equivalent.*

(1) *The variety $K_0 \wedge K_1$ consists of one-element algebras only.*

(2) *There exist binary polynomial symbols p_k, $k = 0, 1, \ldots, n$ such that*

(i) $p_0(x, y) = x$ *and* $p_n(x, y) = y$,

(ii) $p_k(x, y) = p_{k+1}(x, y)$ *holds in K_0 for k even,*

(iii) $p_k(x, y) = p_{k+1}(x, y)$ *holds in K_1 for k odd.*

Statement of the results

Theorem 1. *Varieties K_0, K_1 (of the same type) are independent if and only if the conditions* (1), (2) *and the following condition* (4) *are satisfied.*

(4) *For each $\mathfrak{A} \in K_0 \vee K_1$ and every α_0, α_1, $\beta \in \mathscr{C}(\mathfrak{A})$ such that $\mathfrak{A}/\alpha_i \in K_i$, $i=0,1$, and either $\alpha_0 \leqq \beta$ or $\alpha_1 \leqq \beta$, the equality $(\alpha_0 \vee \alpha_1) \wedge \beta = (\alpha_0 \wedge \beta) \vee (\alpha_1 \wedge \beta)$ holds.*

Theorem 2. *Let K_0, K_1 be varieties (of the same type) satisfying the conditions* (1), (2) *of Theorem A and let there exist binary polynomial symbols $p_0, p_1, \ldots, p_n$ from Theorem B having the following property* (6).

(6) *For every algebra $\mathfrak{A} \in K_i$, $i=0,1$, for each element $a \in \mathfrak{A}$ and for each $k \in \{1, 2, \ldots, n-1\}$ there exist elements b, $c \in \mathfrak{A}$ such that $a = p_k(b, c)$.*

Then K_0, K_1 are independent.

Corollary. *Let K_0, K_1 be varieties (of the same type) having idempotent operations only and let the conditions* (1) *and* (2) *be satisfied. Then K_0, K_1 are* independent.

Proofs of the theorems

Proof of Theorem 1. In [4, Theorem 1] it is proved that the independence of K_0, K_1 implies the conditions (1), (2) and permutability of every couple of congruences Φ_i of the algebra $\mathfrak{A} \in K_0 \vee K_1$ such that $\mathfrak{A}/\Phi_i \in K_i$, $i=0,1$. Hence using [1, Chap. IV, Theorem 13] we get the validity of (4). Conversely, by analysis of the proof of Theorem 2 in [4] it can be seen that besides (1), (2) there was used only the modularity of such triples of congruences as there are in (4).

Proof of Theorem 2. A short analysis of the proof of Theorem 2 [4] shows that it suffices to prove the following assertion (a).

(a) Let $\mathfrak{A} = \mathfrak{A}_0 \times \mathfrak{A}_1$, $\mathfrak{A}_i \in K_i$, $i=0,1$ and let Φ_0, $\Phi_1 \in \mathscr{C}(\mathfrak{A})$ such that $(a_0, a_1)\Phi_i(b_0, b_1)$ if and only if $a_i = b_i$. Suppose $\Theta_i \in \mathscr{C}(\mathfrak{A})$, $i \in \{0,1\}$, $\Theta_i \leqq \Phi_i$ and $\mathfrak{A}/\Theta_i \in K_i$. Then $\Theta_i = \Phi_i$.

Proof of (a). Let Θ_0 be as in (a). Then $(a_0, a_1)\Theta_0(b_0, b_1)$ implies $a_0 = b_0$. Define the following relation Σ on $\mathfrak{A}_1$. $a\Sigma b$ if and only if for each $u \in \mathfrak{A}_0$, $(u, a)\Theta_0(u, b)$. It is clear that Σ is an equivalence relation on $\mathfrak{A}_1$. We shall show that for every x, $y \in \mathfrak{A}_1$, $x\Sigma y$ holds by showing $p_k(x, y)\Sigma p_{k+1}(x, y)$ for $k \in \{0,1, \ldots, n-1\}$. Let $u \in \mathfrak{A}_0$ and x, $y \in \mathfrak{A}_1$. If k is even, then for some a, $b \in \mathfrak{A}_0$ $u = p_k(a, b) = p_{k+1}(a, b)$ by (6) and (ii). Because of $\mathfrak{A}/\Theta_0 \in K_0$ and (ii) we get $(u, p_k(x, y)) = (p_k(a, b), p_k(x, y)) = p_k((a, x), (b, y))\Theta_0 p_{k+1}((a, x), (b, y)) = (p_{k+1}(a, b), p_{k+1}(x, y)) = (u, p_{k+1}(x, y))$, hence $p_k(x, y)\Sigma p_{k+1}(x, y)$. If k is odd, then according to (iii) $p_k(x, y) = p_{k+1}(x, y)$. Hence Σ is the greatest congruence relation on $\mathfrak{A}_1$. This implies $\Theta_0 = \Phi_0$. $\Theta_1 = \Phi_1$ can be shown analogously.

Proof of Corollary. It suffices to use Theorem 2 and Theorem B.

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УСЛОВИЯ НЕЗАВИСИМОСТИ МНОГООБРАЗИЙ

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Резюме

Пусть K_0, K_1 многообразия алгебр одинакового типа. K_0 и K_1 независимые тогда и только тогда, когда выполнены следующие условия:

(1) $K_0 \wedge K_1$ состоит только из одноэлементных алгебр,

(2) всякая алгебра $\mathfrak{A} \in K_0 \vee K_1$ является прямым произведением алгебр $\mathfrak{A}_i \in K_i$ $(i=0, 1)$,

(3) для каждой алгебры $\mathfrak{A} \in K_0 \vee K_1$ и всяких конгруэнций α_0, α_1, β на $\mathfrak{A}$ таких, что $\mathfrak{A}/\alpha_i \in K_i$, имеет место

$$(\alpha_0 \vee \alpha_1) \wedge \beta = (\alpha_0 \wedge \beta) \vee (\alpha_1 \wedge \beta).$$

Ю. Плонка ставил вопрос, достаточны-ли условия (1), (2) для независимости K_0 и K_1. Показано, что это верно в некоторых частных случаях (например когда все операции на K_0 и K_1 идемпотентны).