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ON THE EXTENSION OF MEASURES

RASTISLAV POTOCKÝ

The purpose of this paper is to extend a measure defined on an algebra $A$ and having values in a vector lattice to a measure on the smallest $\sigma$-algebra containing $A$. We present two classes of spaces in which the extension is possible. At the end of the paper we derive several results from the main theorem; some of them are known, the rest seem to be new.

We recall some notions and definitions which will be used throughout the paper. The vector lattice $X$ is called

a) Dedekind $\sigma$-complete if every non-empty at most countable subset of $X$ which is bounded from above has a supremum.

b) $\sigma$-separable if every non-empty subset $Y \subset X$ possessing a supremum contains an at most countable subset possessing the same supremum as $Y$.

We shall say that the sequence $x_n$ in a Dedekind $\sigma$-complete vector lattice $X$ is order convergent to an element $x$ in $X$, if $\lim \sup x_n = \lim \inf x_n = x$. The above definitions as well as many interesting results on vector lattices can be found in [1], [2].

A set function $m$ defined on an algebra $A$ and having values in a Dedekind $\sigma$-complete vector lattice $X$ is said to be a (vector) measure if

1) $m(\emptyset) = 0$;
2) $m(E) \geq 0$ for every $E \in A$;
3) $m(E) = \sum_{i=1}^{\infty} m(E_i)$ for every disjoint sequence $(E_i)$ of sets in $A$ whose union is $E$.

There is another definition of measure. A set function $m$ on an algebra $A$ with values in a Dedekind $\sigma$-complete vector lattice $X$ is a measure if

1) $m(\emptyset) = 0$;
2) $m(E) \geq 0$ for every $E \in A$;
3) $m(E) + m(F) = m(E \cup F) + m(E \cap F)$ for every $E, F \in A$;
4) $m(E) = \lim m(E_n)$ (in $\sigma$-sense) for every increasing sequence $(E_n)$ of sets in $A$ such that $E = \cup E_n \in A$.

It is easy to prove that both definitions are, in fact, the same.
A linear functional on $X$ is called
a) positive (monotone) if $Tx \geq 0$ for all $x \geq 0$;
b) order continuous if for each sequence $(x_n)$ in $X$ with the order limit $x$, $Tx_n$ converges to $Tx$;
c) $o$-bounded if it maps $o$-bounded sets into bounded sets.

In what follows the set of all $o$-bounded linear functionals and the set of all linear functionals continuous with respect to a topology on $X$ will be denoted by $X^+$ and $X^*$, respectively.

**Theorem 1.** If $m$ is a (vector) measure on an algebra $A$ with values in a Dedekind $\sigma$-complete $o$-separable vector lattice such that the set of all $o$-continuous linear functionals on $X$ separates points of $X$, then there is a unique (vector) measure $\tilde{m}$ on the $\sigma$-algebra $S(A)$ such that for $E \in A$ $\tilde{m}(E) = m(E)$.

**Proof.** The measure $m$ is an operator on $A$ with the following properties:
1) $E \subseteq F \Rightarrow m(E) \leq m(F)$ for every $E, F \in A$;
2) $m(E) + m(F) = m(E \cup F) + m(E \cap F)$ for every $E, F \in A$;
3) $E \subseteq F \Rightarrow m(F) = m(E) + m(F \setminus E)$ for every $E, F \in A$;
4) $m(E \cup F) \leq m(E) + m(F)$ for every $E, F \in A$;
5) $E_n \uparrow E, E_n, E \in A \Rightarrow m(E) = \lim m(E_n)$ for every sequence $(E_n)$ of sets in $A$.

Let $S$ denote the set of all subsets of the basic space $\Omega$. Put $B = \{ E \in S ; \exists (E_n) \in A ; E_n \uparrow E \}$ and define $m_1(E) = \lim m(E_n)$ for every $E$ in $B$. The definition does not depend on the choice of the sequence $(E_n)$.

Then define $m_2(E) = \inf \{ m_1(F) ; E \subseteq F \in B \}$ for every set $E$ in $S$. It follows that $m_2$ is a monotone operator on $S$ with values in $X$ such that $m_2(E \cup F) \leq m_2(E) + m_2(F)$ for every $E, F \in S$. Moreover $m_2$ coincides with $m$ on $A$.

For every monotone, $o$-continuous linear functional $T$ on $X$ define now an operator $*T$ from $A$ into $R$ (the field of real numbers) as follows: $*T(E) = Tm(E)$ for every $E \in A$. $*T$ has the following properties:
1) $E \subseteq F \Rightarrow *T(E) \leq *T(F)$;
2) $*T(E) + *T(F) = *T(E \cup F) + *T(E \cap F)$;
3) $E \subseteq F \Rightarrow *T(F) = *T(E) + *T(E \setminus F)$;
4) $*T(E \cup F) \leq *T(E) + *T(F)$;
5) $E_n \uparrow E, E_n, E \in A \Rightarrow *T(E) = \lim *T(E_n)$ for every sequence $(E_n)$ of sets in $A$.

Then put $*T(E) = \sup *T(E_n) = \sup Tm(E_n)$ for every $E \in B, E_n \in A, E_n \uparrow E$. One can show that this is a correct definition. It follows that $T^*(E) = Tm_1(E)$.

Finally define $T^{**}(E) = \inf \{ T^*(F) ; E \subseteq F \in B \}$ for every $E \in S$.

Since the field of real numbers is $o$-separable, we may suppose that there exists a decreasing sequence $(F_n)$ of elements in $B$ greater than $E$ such that

$$T^{**}(E) = \inf \{ T^*(F_n) ; E \subseteq F_n \in B \}.$$  

On the other hand, since $X$ is supposed to be $o$-separable, we have $m_2(E) = \inf \{ m_1(G_n) ; E \subseteq G_n \in B \}$ and, consequently, $T^{**}(E) = \inf \{ T^*(F_n) ; E \subseteq F_n \in B \} = 360$.
inf \{ T m_1(F_n); E \subset F_n \in B \} = T \inf \{ m_1(F_n); E \subset F_n \in B \} \geq T m_2(E) \text{ for every } E \text{ in } S. \text{ The reverse inequality is immediate.}

Denote by \( L \) the set of all \( E \in S \) such that

\[
\sup \{ m_2(C); E \supset C \in D \} = \inf \{ m_2(F); E \subset F \in B \},
\]

where \( D \) is the set of all \( E \in S \) for which there exists a decreasing sequence \( (A_n) \) of elements of \( A \) such that \( A_n \downarrow E \).

We define, similarly,

\[
L^* = \{ E \in S; \sup \{ T**((C_n); E \supset C \in D \} = \inf \{ T**((F_n); E \subset F \in B \} \}.
\]

Since \( \sup \{ m_2(C_n); E \supset C_n \in D \} = \inf \{ m_2(F_n); E \subset F_n \in B \} \) with an increasing sequence \( (C_n) \) and a decreasing sequence \( (F_n) \) implies that \( \sup \{ T**((C_n); E \supset C_n \in D \} = \inf \{ T**((F_n); E \subset F_n \in B \} \), we have \( L \subset L^* \).

The next problem is to prove that if \( (E_n) \) is a monotone sequence in \( L \) which converges to a set \( E \) in \( S \), then \( E \) belongs to \( L \). Since \( L \subset L^* \), we obtain from the extension theorem for real valued measures that \( E \in L^* \), i.e. \( \sup \{ T**((C'_n); E \supset C'_n \in D \} = \inf \{ T**((F'_n); E \subset F'_n \in B \} \). It follows, since the set of all \( \sigma \)-continuous linear functionals separates points of \( X \), that \( \sup \{ m_2(C'_n); E \supset C'_n \in D \} = \inf \{ m_2(F'_n); E \subset F'_n \in B \} \), i.e. that \( E \in L \).

Since \( L \) contains \( A \), we may suppose the existence of the smallest set \( N \) containing \( A \) with the following property: \( F_n \in N, F_n \uparrow F \in S(F_n \downarrow F \in S) \Rightarrow F \in N \).

Since \( N = S(A) \), we define \( \tilde{m}(E) = m_2(E) \) for \( E \in N \).

It is evident that \( \tilde{m}(\emptyset) = 0 \) and \( \tilde{m}(E) \geq 0 \) for every \( E \in S(A) \). In order to prove the continuity from below, consider arbitrary \( E_n \in S(A), E_n \uparrow E \). We have immediately that \( \tilde{m}(E) \geq \lim \tilde{m}(E_n) \) since \( \tilde{m} \) is monotone. The desired result follows then from the fact that \( T**(E) = \lim T**(E_n) \), i.e. \( T m_2(E) = \lim T m_2(E_n) \) for every \( \sigma \)-continuous linear functional under consideration and from the fact that the set of all \( \sigma \)-continuous linear functionals separates points of \( X \).

The equality \( \tilde{m}(E) + \tilde{m}(F) = \tilde{m}(E \cup F) + \tilde{m}(E \cap F) \) and the uniqueness of \( \tilde{m} \) follow without difficulty.

**Corollary 1.** (cf. [3], th. 11) If \( X \) is a regular Dedekind \( \sigma \)-complete vector lattice such that \( X^* \) separates points of \( X \), then the extension theorem holds.

**Proof.** Every regular Dedekind \( \sigma \)-complete vector lattice is \( \sigma \)-separable and every \( \sigma \)-bounded linear functional on such a space is \( \sigma \)-continuous.

**Theorem 2.** Let \( X \) be a Dedekind \( \sigma \)-complete, \( \sigma \)-separable locally convex space with an ordering given by a closed cone. Let \( x_n \rightarrow^\sigma x \) imply \( T(x_n) \rightarrow T(x) \) for every \( T \in X^* \). Then for every measure on an algebra \( A \) with values in \( X \) there exists a unique extension to \( S(A) \).

**Proof.** Analogous to that of Theorem 1.

So far we have been concerned with a set function which was a measure in the \( \sigma \)-sense. We can, however, extend our results to the case when we are primarily
interested in the topology of $X$. Substituting in the above definition of measure a topological convergence for the $o$-convergence, we shall speak about a (vector) measure in the topological sense. The following results should be compared with [4], [5], [6].

**Theorem 3.** Let $X$ be a Dedekind $\sigma$-complete, $o$-separable locally convex space ordered by normal cone and let every continuous linear functional be $o$-continuous. Then every measure (in the topological sense) on an algebra $A$ with values in $X$ can be uniquely extended to $S(A)$.

**Proof.** Since the cone is closed, the set function under consideration is a measure in the $o$-sense as well. If $\tilde{m}$ means the extension to $S(A)$ mentioned in Theorem 2, we have that $\tilde{m}(E_n) \xrightarrow{w} \tilde{m}(E)$ whenever $E_n \uparrow E, E_n, E \in S(A)$. Since the cone is normal, the result follows.

**Theorem 4.** Let $X$ be a Dedekind $\sigma$-complete, $o$-separable complete metrizable locally convex space ordered by a closed cone and let every continuous linear functional be $o$-continuous. Then for every measure on an algebra $A$ with values in $X$ there is a unique extension to $S(A)$.

**Proof.** The above assumptions imply normality of the cone.

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О ПРОДОЛЖЕНИИ МЕР

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Резюме

Пусть $m$ — векторная мера определена на алгебре $A$ с значениями в $\sigma$ — полной $\sigma$ — сепарабельной векторной решетке $X$ такой, что семейство всех $\sigma$ — непрерывных линейных форм разделяет ее точки. Тогда существует векторная мера $\tilde{m}$ на $\sigma$-алгебре $S(A)$ порожденной алгеброй $A$, являющаяся продолжением $m$. Мера $\tilde{m}$ определена однозначно.