Mathematica Slovaca

Štefan Schwarz

Semigroups containing maximal ideals

Mathematica Slovaca, Vol. 28 (1978), No. 2, 157--168

Persistent URL: http://dml.cz/dmlcz/136171

Terms of use:

© Mathematical Institute of the Slovak Academy of Sciences, 1978

Institute of Mathematics of the Academy of Sciences of the Czech Republic provides access to digitized documents strictly for personal use. Each copy of any part of this document must contain these *Terms of use*.

This paper has been digitized, optimized for electronic delivery and stamped with digital signature within the project *DML-CZ: The Czech Digital Mathematics Library* http://project.dml.cz

Math. Slovaca 28, 1978, No. 2, 157—168

SEMIGROUPS CONTAINING MAXIMAL IDEALS

ŠTEFAN SCHWARZ

A left ideal L of a semigroup S is called maximal if $L \neq S$ and no proper left ideal of S properly contains L. Analogously maximal right and two-sided ideals are defined.

Denote by L^*, R^*, M^* the intersection of all maximal left, maximal right and maximal two-sided ideals of S, respectively.

The purpose of this paper is to clarify the interdependence of the sets L^*, R^* and M^*. Necessary and sufficient conditions are given for the validity of $L^* = M^*$ and $L^* = R^*$. Conditions for inclusions like $L^* \subsetneqq M^*$ or $L^* \supsetneqq M^*$ are obtained.

A semigroup need not contain maximal left (right, two-sided) ideals. The non-existence of, e.g., maximal two-sided ideals has two sources. *i*) The semigroup S is a simple semigroup (without zero), so that there are no two-sided ideals except S itself. *ii*) To any two-sided ideal $A_\alpha \neq S$ there is a two-sided ideal $A_\beta \neq S$ such that $A_\alpha \subsetneqq A_\beta$. Analogously for one-sided ideals.

To get some results we shall impose, where needed, some of the following weakest possible conditions:

M_L: S contains at least one maximal left ideal.

M_R: S contains at least one maximal right ideal.

M_J: S contains at least one maximal two-sided ideal.

The questions concerning maximal ideals can be treated by means of $\mathscr{L}$, $\mathscr{R}$, $\mathscr{J}$-classes using the usual ordering of these classes.

The following result will be used: A left ideal L of S is a maximal left ideal of S iff $S - L$ is a maximal $\mathscr{L}$-class of S. Analogously for maximal two-sided ideals. (See, e.g., [3], [5].)

We shall use the following special notation: If L_α is a maximal left ideal of S, then the maximal $\mathscr{L}$-class $S - L_\alpha$ will be denoted by L^α. Hence $L^\alpha = S - L_\alpha$ and $L_\alpha = S - L^\alpha$.

Note finally for further purposes: If any $\mathscr{L}$-class L^α (not necessarily a maximal $\mathscr{L}$-class) meets a left ideal L, then $L^\alpha \subset L$.

The conditions M_L, M_R and M_J are independent. This is shown on the following examples.

Example 1. If S satisfies M_L and M_R, it need not satisfy M_J. Let B be the bicyclic semigroup, i.e. the semigroup with two generators p, q submitted to the relation $pq=1$. Then $L=S-\{1, q, q^2, \ldots\}$ is the unique maximal left ideal of B, $R=S-\{1, p, p^2, \ldots\}$ is the unique maximal right ideal of B, while B (being simple) has no maximal two-sided ideal.

A much more instructive example in which M_L and M_R hold, there is an increasing chain of two-sided ideals, but not a maximal two-sided ideal is given in [5] (Example 5,2).

Example 2. We next show that M_L and M_J do not imply M_R. Let S be a simple semigroup containing at least two minimal left ideals, which is not completely simple. It is known that such semigroups exist and S is the union of its minimal left ideals, $S=\bigcup_v l_v$. Further any $\mathscr{R}$-class in S is a one-point set. (See [1], section 8,2.)

Suppose, for an indirect proof, that S contains a maximal right ideal R. Then $R=S-\{a\}$ for some $\mathscr{R}$-class $\{a\}$. The element a is contained in some minimal left ideal, say $a \in l_\alpha \subset S$. We have $(S-\{a\})\cdot S \supset \left(\bigcup_{v\neq\alpha} l_v\right) S=S$. Hence R is not a right ideal (even less a maximal right ideal). S does not contain maximal right ideals.

Let now $S^0=\{0\}\cup S$ be the semigroup obtained by adjoining a zero 0. Then S^0 contains a maximal two-sided ideal, namely $\{0\}$. It is clear that S contains maximal left ideals but no maximal right ideals. Hence M_J and M_L do not imply M_R.

Example 3. To show that M_J does not imply M_L or M_R consider the following example used in the literature as a counterexample for various purposes.

Let S be the set of all couples (m, n) of positive real numbers and define a multiplication by $(a, b)\cdot(c, d) = (ac, bc+d)$. This is a simple cancellable semigroup in which every $\mathscr{L}$-class and every $\mathscr{R}$-class is a one-point set. Let $(m, n)\in S$. We show that $T=S-\{(m, n)\}$ is not a left ideal (even less a maximal left ideal). For, take the element $\left(m, \frac{n}{2}\right) \in T \subset S$. Then ST contains $\left(1, \frac{n}{2m}\right)\left(m, \frac{n}{2}\right) = (m, n)$. Hence T is not a left ideal of S. An analogous argument shows that S does not contain a maximal right ideal. Consider next the semigroup $S^0=\{0\}\cup S$. Then S^0 contains a maximal two-sided ideal, namely $\{0\}$, but it does not contain maximal left or right ideals.

Example 4. To stress the weakness of the condition M_J we give a simple example of a commutative semigroup S satisfying M_J, in which there is a proper ideal of S which is not contained in a maximal ideal of S. Let S_0 be the multiplicative semigroup of real numbers from the half-open interval $\langle 0, 1)$ and $S_i=\{0, a_i\}$, $i=1, 2, \ldots, n$, where $a_i^2=a_i$, the element 0 having the usual properties of a multiplicative zero. The 0-direct union $S = S_0 \cup S_1 \cup \ldots \cup S_n$ contains exactly n maximal ideals (namely the sets $S - \{a_i\}$), while the ideal $S_1 \cup S_2 \cup \ldots \cup S_n$ is not contained in a maximal ideal of S.

1. The relation between L^* and M^*

The following has been proved in [4].

Lemma 1. *If S satisfies the condition M_J, then $M^* \neq \emptyset$.*

Several authors have noticed (see, e.g., [6]) that this need not be true for L^* without giving more precise results. The following Lemma is implicitly contained in a more general statement in [2], where unary algebras are studied.

Lemma 2. *Suppose that S satisfies the condition M_L. Then $L^* = \emptyset$ iff S is a simple semigroup (without zero) containing a minimal left ideal.*

Proof. Let $\{L_\alpha / \alpha \in H\}$ be the set of all maximal left ideals of S and $\{L^\alpha \mid \alpha \in H\}$ the corresponding set of all maximal $\mathscr{L}$-classes. The formula $\bigcap_\alpha L_\alpha = \bigcap_\alpha (S - L^\alpha) = S - \bigcup_\alpha L^\alpha$ implies that $L^* = \emptyset$ iff $S = \bigcup_\alpha L^\alpha$. M_L implies card $H \geqq 2$.

Suppose $L^* = \emptyset$. Let L^α be any of the maximal $\mathscr{L}$-classes and $a \in L^\alpha$. The principal left ideal (a, Sa) cannot contain properly a left ideal B of S. For $B \subsetneqq (a, Sa)$ and $b \in B$ would imply $(b, Sb) \subset B \subsetneqq (a, Sa)$. Hence (denoting by L^b the $\mathscr{L}$-class containing b) $L^b \subsetneqq L^\alpha$. This is a contradiction with the fact that all $\mathscr{L}$-classes in S are maximal $\mathscr{L}$-classes. Hence (a, Sa) is a minimal left ideal of S and the minimality implies also $(a, Sa) = Sa$. Now $a \in Sa$ for any $a \in L^\alpha$ implies $L^\alpha \subset SL^\alpha$. Since for any $x \in L^\alpha$ we have $Sx = Sa$, we obtain $L^\alpha \subset Sa$.

Now Sa cannot meet a class L^β, $\beta \neq \alpha$. For $b \in Sa \cap L^\beta$ would imply $Sb \subset (b, Sb) \subset Sa$, therefore $Sb = Sa$. Hence $b \in L^\alpha$, which is a contradiction with $b \in L^\beta$. We have $Sa \subset L^\alpha$, and finally $L^\alpha = Sa$.

Write $L^\alpha = Sa_\alpha$, $a_\alpha \in L^\alpha$. Then S can be written as a union of minimal left ideals of S in the form $S = \bigcup_\alpha Sa_\alpha$. The end of the proof is now a well-known routine. For any $x \in S$ we have $Sxa_\alpha \subset Sa_\alpha$ and since Sa_α is minimal $Sxa_\alpha = Sa_\alpha$. Hence $S = \bigcup_\alpha Sxa_\alpha \subset SxS$. Therefore $S = SxS$ for any $x \in S$, which proves that S is a simple semigroup.

Conversely, if S is a simple semigroup containg a minimal left ideal, it is well known that S can be written in the form $S = \bigcup_\alpha l_\alpha$, where each $l_\alpha (\alpha \in H)$ is a minimal left ideal. Every maximal left ideal is of the form $S - l_\beta$ $(\beta \in H)$, so that $L^* = \emptyset$. (Note that M_L implies card$H \geqq 2$.)

Before introducing Definition 1 below consider the following example (see Example 5,1 in [5]):

Example 5. Let $S=\{0, e_\alpha, e_\beta, u, v, e\}$ be a semigroup with the multiplication table

	e_α	e_β	u	v	e
e_α	e_α	0	0	v	e
e_β	0	e_β	u	0	0
u	u	0	0	e_β	u
v	0	v	e	0	0
e	e	0	0	v	e

This semigroup contains two maximal left ideals $L_\alpha=\{0, e_\beta, u, v, e\}$, $L_\beta=\{0, e_\alpha, u, e\}$ and two maximal right ideals $R_\alpha=\{0, e_\beta, u, v, e\}$ $R_\beta=\{0, e_\alpha, v, e\}$. We have $L^*=\{0, u, e\}$, $R^*=\{0, v, e\}$. There is a unique maximal two-sided ideal $M^*=L_\alpha=R_\alpha$. We have $L^*=M^*-\{v, e_\beta\}=M^*-L^\beta$ and $R^*=M^*-\{u, e_\beta\}$ $=M^*-R^\beta$. Note that L_β and R_β do not contain maximal two-sided ideals of S.

This example shows that even in the finite case a maximal left ideal of S need not contain a maximal two-sided ideal of S.

The next theorem shows under what conditions this cannot take place.

Theorem 1. *Suppose that S satisfies the conditions M_L and M_J. Then a maximal left ideal L_α of S contains a maximal two-sided ideal of S iff $L^\alpha \cap M^*=\emptyset$.*

Proof. *i*) Suppose that $L^\alpha\cap M^*=\emptyset$. Then there is at least one maximal two-sided ideal of S, say M_α, which does not contain L^α (and does not meet L^α). Hence $M_\alpha \subset S-L^\alpha=L_\alpha$, q.e.d.

[Note, by the way, that M_α is uniquely determined. For, if M_α, M_β were two different maximal two-sided ideals contained in L_α, we would have $M_\alpha\cup M_\beta\subset L_\alpha$. On the other hand the maximality implies $M_\alpha\cup M_\beta=S$, which is a contradiction.]

ii) Suppose conversely that L_α is a maximal left ideal of S and $S-L_\alpha=L^\alpha\subset M^*$. L_α cannot contain a maximal two-sided ideal of S, say M_β. For, $M_\beta\subset L_\alpha$ would imply $M_\beta\cap L^\alpha=\emptyset$, hence L^α is not contained in M^*, contrary to the assumption.

Definition 1. *Let S be a semigroup satisfying M_L and M_J. We shall say that S satisfies the condition A_l if every maximal left ideal of S contains a maximal two-sided ideal of S.*

Theorem 1 implies:

Theorem 2. *A semigroup S satisfies condition A_l iff none of the maximal $\mathscr{L}$-classes of S is contained in M^*.*

Let $\{M_l | l\in\Lambda\}$ be the set of all maximal two-sided ideals of S. Denote $J^l=S-M_l$. Then $\{J^l | l\in\Lambda\}$ is the set of all maximal $\mathscr{J}$-classes. It is known ([4]) that $S=M^*\cup\left[\bigcup_{l\in\Lambda}J^l\right]$, where $J^{l_1}\cdot J^{l_2}\subset M^*$ for $l_1\neq l_2$.

The condition of Theorem 2 can be therefore formulated as follows: Every maximal $\mathscr{L}$-class is contained in some maximal $\mathscr{J}$-class.

In Example 3 we have seen that a maximal two-sided ideal of S need not be contained in a maximal left ideal of S.

Definition 2. *Let S be a semigroup satisfying the conditions M_L and M_J. We shall say that S satisfies the condition B_l if every maximal two-sided ideal of S is contained in a maximal left ideal of S.*

In orther words: If every maximal $\mathscr{J}$-class contains a maximal $\mathscr{L}$-class of S.

Consider now the set of all maximal $\mathscr{L}$-classes. Such an $\mathscr{L}$-class is contained either in M^* or in one of the J^l, $l \in \Lambda$.

Denote by $\{L^j \mid j \in I\}$ the set of all maximal $\mathscr{L}$-classes contained in M^* and put $Z_l = \bigcup_{j \in I} L^j$.

Denote by $\{J^k \mid k \in K\}$ the set of those maximal $\mathscr{J}$-classes each of which contains at least one maximal $\mathscr{L}$-class of S. Then

$$S = M^* \cup \Big[\bigcup_{k \in K} J^k\Big] \cup T_l, \tag{1}$$

where $T_l = \bigcup_{h \in \Lambda - K} J^h$. Here K or $\Lambda - K$ may be empty. The $\mathscr{J}$-class J^h, $h \in \Lambda - K$, is characterized by the fact that no $\mathscr{L}$-class contained in J^h is maximal.

Let $\{L^{k,\alpha} \mid \alpha \in \Lambda_k\}$ be the set of all maximal $\mathscr{L}$-classes contained in J^k, $k \in K$. Then $S - L^{k,\alpha}$ is a maximal left ideal containing the maximal two-sided ideal $S - J^k = M_k$.

The intersection of all maximal left ideals of S,

$$L^* = \bigcap_{\beta \in H} L_\beta = \bigcap_{\beta} (S - L^\beta) = S - \bigcup_{\beta \in H} L^\beta,$$

is given by

$$L^* = S - Z_l - \bigcup_k \bigcup_\alpha L^{k,\alpha}.$$

Using the expression (1) we have

$$L^* = (M^* - Z_l) \cup T_l \bigcup_{k \in K} \Big[J^k - \bigcup_{\alpha \in \Lambda_k} L^{k,\alpha}\Big].$$

For a fixed $k \in K$ we have

$$C_k = J^k - \bigcup_{\alpha \in \Lambda_k} L^{k,\alpha} = S - M_k - \bigcup_\alpha L^{k,\alpha} = \Big(S - \bigcup_\alpha L^{k,\alpha}\Big) - M_k = \bigcap_{\alpha \in \Lambda_k} L_{k,\alpha} - M_k.$$

Here the first term $\bigcap_{\alpha} L_{k,\alpha}$ is the intersection of all maximal left ideals containing M_k.

The formula

$$L^* = (M^* - Z_l) \cup T_l \cup \Big[\bigcup_{k \in K} C_k\Big] \tag{2}$$

will allow us to give very definite results concerning the relation between L^* and M^*.

To understand well the meaning of the set C_k consider the factor semigroup $\bar{S} = S/M_k$ and the corresponding homomorphism φ: $S \to \bar{S}$, which sends M_k into a new zero $\bar{0}$ while retaining in essential the meaning of all the elements $\in S - M_k = J^k$. The semigroup $\bar{S}$ is a 0-simple semigroup (with zero $\bar{0}$). If L is a maximal left ideal of S containing M_k, then $\varphi(L)$ is a maximal left ideal of $\bar{S}$ and $C_k \cup \{\bar{0}\}$ is the intersection of all maximal left ideals of $\bar{S}$. (All up to a trivial isomorphism.)

It should be remarked that we shall use several times the following: If A is a two-sided ideal of S, then the $\mathscr{L}$-classes contained in $S - A$ are just the non-zero $\mathscr{L}$-classes of S/A. Analogously for $\mathscr{R}$ and $\mathscr{J}$-classes.

In order to find conditions under which C_k is empty we first prove

Lemma 3. *Let S be a semigroup with 0 satisfying the condition M_L. Then $L^* = 0$ iff S is a 0-disjoint union of 0-minimal left ideals.*

Proof. Let $\{L_\alpha | \alpha \in H\}$ be the set of all maximal left ideals of S. Then $L^* = \bigcap_{\alpha \in H} (S - L^\alpha) = S - \bigcup_{\alpha} L^\alpha$. Hence $L^* = 0$ iff $S = \{0\} \cup \{\bigcup_{\alpha \in H} L^\alpha\}$, where each L^α is a maximal $\mathscr{L}$-class of S. The proof is now analogous to that of Lemma 2 but we must be careful, since nilpotent elements may occur.

i) Suppose $L^* = 0$ and let $a \in L^\alpha$. The left ideal (a, Sa) cannot contain properly a non-zero left ideal B of S. For, suppose $0 \neq B \subsetneqq (a, Sa)$. Choose $b \in B$, $b \neq 0$. Then $(b, Sb) \subset B \subsetneqq (a, Sa)$, hence $L^b \subsetneqq L^a$, a contradiction. Therefore (a, Sa) is a 0-minimal left ideal of S. [Note explicitly that there may happen that $Sa = 0$, in which case $(0, a)$ is nilpotent.]

For any $x \in L^\alpha$ we have $(x, Sx) = (a, Sa)$, hence $L^\alpha \subset (a, Sa)$. Next (a, Sa) cannot meet L^β, $\beta \neq \alpha$. For, $b \in (a, Sa) \cap L^\beta$, $b \neq 0$, would imply $(b, Sb) \subset (a, Sa)$, and (with respect to the minimality) $(b, Sb) = (a, Sa)$ and $b \in L^\alpha$, a contradiction. Therefore $(a, Sa) - \{0\} \subset L^\alpha$. Finally $L^\alpha = (a, Sa) - \{0\}$. Hence S is a 0-disjoint union of 0-minimal left ideals:

$$S = \bigcup_{\alpha \in H} l_\alpha . \tag{3}$$

Hereby $l_\alpha = L^\alpha \cup \{0\}$.

ii) If, conversely, S is of the form (3), then any maximal left ideal of S is of the form $L_\alpha = \bigcup_\beta l_\beta$, $\beta \in H$, $\beta \neq \alpha$, so that $L^* = 0$.

Corollary 3. *Let S be a 0-simple semigroup satisfying condition M_L. Then $L^* = 0$ iff S contains a 0-minimal left ideal.*

Corollary 3 implies:

Lemma 4. *The set $C_k (k \in K)$ is empty iff the semigroup $\bar{S} = S/M_k$ is a 0-simple semigroup containing a 0-minimal left ideal of $\bar{S}$.*

For brevity in formulations we introduce the following notion:

Definition 3. *A 0-simple semigroup is called a G_l-semigroup if it contains a 0-minimal left ideal.*[1]

The decomposition (1) implies that S/M^* is a 0-direct union of 0-simple semigroups

$$S/M^* \cong \left[\bigcup_{k \in K} \bar{J}^k\right] \cup \left[\bigcup_{h \in \Lambda - K} \bar{J}^h\right],$$

where $\bar{J}^l \cong S/M_l$.

The set $\bigcup_{k \in K} C_k$ is empty iff each $\bar{J}^k (k \in K)$ is a G_l-semigroup.

Recall that $T_l = \emptyset$ iff S satisfies condition B_l and $Z_l = \emptyset$ iff S satisfies condition A_l. The decomposition (2) implies the following results:

Theorem 3. *Let S be a semigroup satisfying M_L, M_J, and the condition B_l. Then $L^* = M^* - Z_l$ iff S/M^* is either a G_l-semigroup or a 0-direct union of G_l-semigroups.*

If S is finite, the condition B_l is satisfied, S/M^* is always either a G_l-semigroup or a 0-direct union of G_l-semigroups. Further M_L and M_J are satisfied, unless S is a simple semigroup. Hence we have:

Theorem 4. *Let S be a finite semigroup which is not simple. Then $L^* = M^* - Z_l$. We have $L^* = M^*$ iff S satisfies the condition A_l.*

Note that in this case if the condition A_l is not satisfied, we have strictly $L^* \subsetneq M^*$.

In the most general case we have:

Theorem 5. *Let S be a semigroup satisfying the conditions M_L and M_J. Then $L^* = M^*$ iff*

i) *S satisfies the conditions A_l and B_l;*

ii) *S/M^* is either a G_l-semigroup or a 0-direct union of G_l-semigroups.*

---

[1] This includes the case of a null semigroup of order two.

Note that in this case if S satisfies A_l, we have $L^* = M^* \cup T_l \cup \left[\bigcup_{k \in K} C_k \right]$. Hence L^* may be strictly larger than M^*. [This is the case, e.g., for the bicyclic semigroup B with a zero adjoined.]

2. The relation between L^* and R^*

We now take into account the intersection of all maximal right ideals R^*.

Definition 4. *Suppose that S satisfies M_R and M_J. We shall say that S satisfies the condition A_r if every maximal right ideal of S contains a maximal two-sided ideal of S. Further we shall say that S satisfies the condition B_r if every maximal two-sided ideal of S is contained in a maximal right ideal of S.*

In order to get a formula analogous to (2) we denote by $Z_r = \bigcup_{j \in I_1} R^j$ the union of all maximal $\mathcal{R}$-classes of S contained in M^*. Next we denote by T_r the union of all maximal $\mathcal{J}$-classes each of which does not contain a maximal $\mathcal{R}$-class of S. Let finally $\{M_k | k \in K_1\}$ be the set of all maximal two-sided ideals of S which are contained in a maximal right ideal of S. For a fixed M_k, $k \in K_1$, denote by $M_k \cup D_k [M_k \cap D_k = \emptyset]$ the intersection of all maximal right ideals of S containing M_k.

With these notations we have

$$R^* = (M^* - Z_r) \cup T_r \cup \left[\bigcup_{k \in K_1} D_k \right]. \tag{4}$$

We first clarify under what conditions $Z_r = Z_l$ and $C_k = D_k$, $k \in K \cap K_1$.

Lemma 5. *Let S be a semigroup with 0, satisfying M_L and M_R. Suppose that $L^* = R^* = 0$. Then S is a 0-direct union of a null semigroup A and of completely 0-simple semigroups $K_j (j \in \Lambda')$:*

$$S = A \cup \left[\bigcup_{j \in \Lambda'} K_j \right].$$

Hereby A or the K_j may reduce to $\{0\}$.

Proof. By Lemma 3 and its right dual, S is a 0-disjoint union of 0-minimal left ideals

$$S = \bigcup_{\alpha \in \Lambda_1} l_\alpha, \tag{5}$$

as well as a 0-direct union of 0-minimal right ideals

$$S = \bigcup_{\alpha \in \Lambda_2} r_\alpha. \tag{6}$$

We now use Theorem 6,37 of [1] by which a semigroup having the properties (5) and (6) is a 0-direct union of a null semigroup and of completely 0-simple semigroups.

More precisely: Denote by $N_0[N_0']$ the union of all summands in (5) [in (6)] which are nilpotent and by $N_1[N_1']$ the union of all summands in (5) [(6)] which are non-nilpotent. Hence $S = N_0 \cup N_1 = N_0' \cup N_1'$. Then $N_1 = N_1'$ is a two-sided ideal and if $N_1 \neq 0$, N_1 is a 0-direct union of all the completely 0-simple ideals of S. Further $N_0 = N_0'$ is a two-sided ideal of S and $N_0^2 = 0$.

Remark. It follows from the proof of Lemma 3 that the conditions of Lemma 5 are satisfied iff all $\mathscr{L}$-classes and $\mathscr{R}$-classes contained in $S - \{0\}$ are maximal $\mathscr{L}$-classes and maximal $\mathscr{R}$-classes of S.

Corollary 5. *Let S be a semigroup with zero satisfying M_L and M_R. Suppose that $L^* = R^* = 0$. Then any non-zero $\mathscr{L}$-class [$\mathscr{R}$-class] of S is contained in a maximal $\mathscr{J}$-class of S.*

Proof. Write in accordance with the last Lemma

$$S = A \cup \left[\bigcup_{j \in \Lambda'} K_j \right],$$

where the K_j are completely 0-simple and all unions are 0-direct.

If $\Lambda' \neq \emptyset$, then $M_j = A \cup \left[\bigcup_i K_i \middle| i \in \Lambda', i \neq j \right]$ is clearly a maximal two-sided ideal of S and $M^j = S - M_j$ is a maximal $\mathscr{J}$-class of S. Each non-zero $\mathscr{L}$-class contained in $\bigcup_{j \in \Lambda'} K_j$ is contained in some K_j, hence in some M^j.

If $A \neq \{0\}$, then A is a 0-disjoint union of the form $A = \bigcup_{j \in \Lambda''} \{a_j, 0\}$ with $a_j^2 = 0$, $j \in \Lambda''$, and each $\{a_j\}$ itself is a maximal $\mathscr{J}$-class, since $S - \{a_j\}$ is clearly a maximal two-sided ideal of S. This proves our statement.

After this diversion we now return to the formulae (2) and (4). These formulae imply that

$$L^* \cap M^* = M^* - Z_l,$$

$$R^* \cap M^* = M^* - Z_r.$$

Hence we have $L^* \neq R^*$ if $Z_r \neq Z_l$. We now prove that $Z_r = Z_l$ holds iff $Z_r = Z_l = \emptyset$.

Lemma 6. *Suppose that S satisfies M_L, M_R and M_J. If $Z_l = Z_r$, then S satisfies both conditions A_l and A_r so that both sets Z_l and Z_r are empty.*

Proof. Suppose for an indirect proof that $Z_l = Z_r \neq \emptyset$. Consider the sets

$$S - Z_l = S - \bigcup_{j \in I} L^j = \bigcap_{j \in I} (S - L^j) = \bigcap_{j \in I} L_j, \tag{7}$$

$$S - Z_r = S - \bigcup_{j \in I_1} R^j = \bigcap_{j \in I_1} (S - R^j) = \bigcap_{j \in I_1} R_j . \qquad (8)$$

Denote $M = S - Z_l = S - Z_r$. It follows from (7) and (8) that M is a two-sided ideal of S. By (7) and (8) the factor semigroup S/M is a semigroup with zero $\bar{0}$ in which the intersection of all maximal left ideals and the intersection of all maximal right ideals is $\bar{0}$. By Corollary 5 any non-zero $\mathscr{L}$-class contained in S/M is contained in a maximal $\mathscr{J}$-class of S/M. For the semigroup S itself this implies that every $\mathscr{L}$-class contained in $S - M = Z_l$ is contained in a maximal $\mathscr{J}$-class of S. This is a contradiction, since Z_l has been defined as the union of those maximal $\mathscr{L}$-classes of S none of which is contained in a maximal $\mathscr{J}$-class of S. This proves Lemma 6.

Lemma 7. *Suppose that $k \in K \cap K_1 \neq \emptyset$. Then $C_k = D_k$ iff $C_k = D_k = \emptyset$. In this case S/M_k is a completely 0-simple semigroup or a null semigroup of order two.*

Proof. If $C_k = D_k$, then $M_k \cup C_k = M_k \cup D_k$ is a two-sided ideal of S containing M_k and different from S. With respect to the maximality of M_k we have $C_k = D_k = \emptyset$. If $C_k = D_k = \emptyset$, then by Lemma 3 and its right dual, S/M_k is a 0-simple semigroup containing a 0-minimal left and a 0-minimal right ideal. Hence S/M_k is completely 0-simple or a null semigroup of order two.

Suppose now that S satisfies the conditions A_l and A_r, i.e., $Z_l = Z_r = \emptyset$. Then

$$L^* = M^* \cup \left[\bigcup_{l \in \Lambda - K} J^l \right] \cup \left[\bigcup_{k \in K} C_k \right],$$

$$R^* = M^* \cup \left[\bigcup_{l \in \Lambda - K_1} J^l \right] \cup \left[\bigcup_{k \in K_1} D_k \right].$$

If $\alpha \in (\Lambda - K) \cap K_1$, then L^* contains the whole class J^α, while R^* contains only a proper subset D_α of J^α (D_α may be, eventually, empty), so that $R^* \cap J^\alpha \subsetneqq L^* \cap J^\alpha$. Analogously if $\beta \in (\Lambda - K_1) \cap K$, we have $R^* \cap J^\beta \supsetneqq L^* \cap J^\beta$. Therefore a further necessary condition for the validity of $L^* = R^*$ is $\Lambda - K = \Lambda - K_1$, hence $T_r = T_l$.

Finally, for $L^* = R^*$ we must have $\bigcup_{k \in K} C_k = \bigcup_{k \in K} D_k$, i.e., $C_k = D_k$ for any $k \in K$. By Lemma 7 we then have $C_k = D_k = \emptyset$ for all $k \in K$.

If, conversely, $Z_l = Z_r = \emptyset$, $K = K_1$, and $C_k = D_k$ for every $k \in K$, then $L^* = R^* = M^* \cup T_r = M^* \cup T_l$.

The condition $T_l = T_r \neq \emptyset$ says that the maximal $\mathscr{J}$-classes which constitute $T_r = T_l$ contain neither a maximal $\mathscr{L}$-class nor a maximal $\mathscr{R}$-class.

Again, for brevity in formulations of the results, we introduce the following notion:

Definition 5. *A 0-simple semigroup S is called a G_0-semigroup if S contains neither a maximal $\mathscr{L}$-class nor a maximal $\mathscr{R}$-class of S.*

Theorem 6. *Let S be a semigroup satisfying M_L, M_R, M_J. Then $L^* = R^*$ iff*

i) S satisfies the conditions A_l and A_r;

ii) S/M^* is a 0-direct union of G_0-semigroups, completely 0-simple semigroups and null semigroups of order two.

Hereby the summands with the exception of at least one may reduce to $\{\bar{0}\}$.

If these conditions are satisfied, we have $R^* = L^* = M^* \cup T_l$.

We also have:

Theorem 7. *If S is a semigroup satisfying M_L, M_R and M_J, and S/M^* is a 0-direct union of completely 0-simple semigroups and null semigroups of order two, then $L^* = M^* - Z_l$, $R^* = M^* - Z_r$. We have $L^* = R^*$ iff S satisfies condition A_l and A_r.*

In the finite case B_l and B_r are satisfied, and $C_k = D_k$ for all $k \in K$. Hence:

Theorem 8. *Let S be a finite semigroup which is not simple. Then $L^* = M^* - Z_l$, $R^* = M^* - Z_r$. We have $L^* = R^*$ iff S satisfies the conditions A_l and A_r.*

In the last case we have $L^* = R^* = M^*$.

Finally we omit the condition M_J and prove:

Theorem 9. *Let S be a semigroup which is not completely simple. Suppose that S satisfies M_L and M_R but it does not satisfy M_J. Then $L^* \neq R^*$.*

Proof. Suppose for an indirect proof that $L^* = R^*$. Then $M = L^* = R^*$ is a two-sided ideal of S, which is $\neq S$. By Lemma 2 $L^* = R^* \neq \emptyset$. Consider the factor semigroup $\bar{S} = S/M$ (with zero $\bar{0}$). Then $\bar{S}$ is a semigroup in which the intersection of all maximal left ideals as well as the intersection of all maximal right ideals is $\bar{0}$. By Corollary 5 any maximal $\mathscr{L}$-class [$\mathscr{R}$-class] of $\bar{S}$ is contained in some maximal $\mathscr{J}$-class of $\bar{S}$. For the semigroup S itself this means that S contains a maximal $\mathscr{J}$-class, hence a maximal two-sided ideal, a contradiction with the assumption.

REFERENCES

[1] CLIFFORD, A. H.—PRESTON, G. B.: The algebraic theory of semigroups. Volume II. Amer. Math. Soc., Providence, R. I. 1967.

[2] ABRHAN, I.: О максимальных подалгебрах в унарных алгебрах. Mat. čas., 24, 1974, 113—128.

[3] GRILLET, P. A.: Intersections of maximal ideals in semigroups. Amer. Math. Monthly, 76, 1969, 503—509.

[4] SCHWARZ, Š.: Prime ideals and maximal ideals in semigroups. Czechoslovak Math. J., 19, 1969, 72—79.

[5] SCHWARZ, Š.: The ideal structure of C-semigroups. Czechoslovak Math. J., 27, 1977,313—338.

[6] SATYANARAYANA, M.: On a class of semisimple semigroups. Semigroup Forum, 10, 1975, 129—138.

Received April 14, 1976

Matematický ústav SAV
Obrancov mieru 49
886 25 Bratislava

ПОЛУГРУППЫ ИМЕЮЩИЕ МАКСИМАЛЬНЫЕ ИДЕАЛЫ

Штефан Шварц

Резюме

Пусть S – полугруппа и L^*, M^*, R^*, соответственно, пересечение всех максимальных левых, всех максимальных правых, и всех максимальных двусторонних идеалов из S.

Целью статьи является исследование взаимного отношения между множествами L^*, R^* и M^*. В частности получены необходимые и достаточные условия для равенства $L^* = M^*$ и $L^* = R^*$.

Сформулируем один из типичных результатов (Теорема 5). Пусть S – полугруппа содержащая максимальный левый и максимальный двусторонний идеал. Равенство $L^* = M^*$ имеет место тогда и только тогда, если выполняются следующие условия:

1. Каждый максимальный $\mathscr{L}$-класс из S содержится в некотором максимальном $\mathscr{I}$-классе из S, и каждый максимальный $\mathscr{I}$-класс содержит по крайней мере один максимальный $\mathscr{L}$-класс.

2. Полугруппа S/M^* либо 0 – простая полугруппа содержащая 0 – минимальный левый идеал, либо 0 – прямое объединение таких полугрупп.