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A CONCEPT OF MEASURABILITY FOR THE DANIELL INTEGRAL

IVAN DOBRAKOV

Introduction

The m in purpose of this paper is to give a new concept of measurability for the Daniell scheme of integration. This concept seems to be more natural and effective than the commonly used Stone concept (for the latter see [6], [4], point 13., and [7, sections 6—5 and 6—7]). Moreover, this concept will be needed in the non additive generalization of the Daniell integral, see [2].

To explain the main ideas, let $(T, \mathscr{F}, I)$ be an elementary Daniell integral and let $\mathscr{L}^*$ denote the class of all summable functions f: $T \rightarrow R^* = \langle -\infty, +\infty \rangle$. A function f: $T \rightarrow R^*$ is measurable in the sense of Stone iff $g \vee (f \wedge h) \in \mathscr{L}^*$ for every choice of $g, h \in \mathscr{L}^*$ such that $g \leqq 0 \leqq h$, see [7, section 6—5]. Thus to decide a given function f is measurable, we must first determine $\mathscr{L}^*$ and then prove the summability of all $g \vee (f \wedge h)$, $-g$, $h \in \mathscr{L}^{*+}$. Roughly speaking, the measurability depends on and comes after summability, and this strongly reduces its importances for the theory.

For our concept of measurability we first determine the class $\mathbf{N}$ of all I-null sets, i.e., those sets $N \subset T$ for which $\bar{I}(\chi_N) = 0$. Then we define the class $\mathscr{B}'_{\mathbf{N}}(\mathscr{F})$ of R' — valued measurable functions ($R' = R$ or R^*) as the smallest class of R' — valued functions on T which contains $\mathscr{F}$ and which is closed under the formation of pointwise limits a.e. $\mathbf{N}$ of sequences. In this way our measurability depends only on $\mathscr{F}$ and $\mathbf{N}$, and is before summability. Moreover, using our concept of measurability, in Theorem 24 we give many necessary and sufficient conditions for $\mathscr{L}^* = \mathscr{L}^*(T, \sigma(\mathbf{P}), \mu)$, where $(T, \sigma(\mathbf{P}), \mu)$ is the measure space induced by I. Out of them let us mention the following: 1) $f \cdot g \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ $\forall f, g \in \mathscr{F}$, 2) $f^{+n} \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ $\forall f \in \mathscr{F}$, $n = 2, 3, \ldots$, 3) $1 \wedge \mathscr{F}^+ \subset \mathscr{B}_{\mathbf{N}}(\mathscr{F})$, 4) $1 \wedge \mathscr{L}^+ \subset \mathscr{L}$, and 5) $1 \wedge \mathscr{F}^+ \subset \mathscr{L}$. In Stone's concept only condition 4) was known, see [7, section 6—7].

The material of the paper is divided into four parts. § 0 contains the basic notations. In § 1 $\mathscr{B}'_{\mathbf{N}}(\mathscr{F})$ and the measurable sets $\mathbf{B}'_{\mathbf{N}}(\mathscr{F}) = \{E : E \subset T, \chi_E \in \mathscr{B}'_{\mathbf{N}}(\mathscr{F})\}$ are investigated for abstract and concrete $\mathscr{F}$ and $\mathbf{N}$. Particularly, measurability with respect to a σ-ring and Baire measurability are obtained as special cases. The main Theorem 18 in § 2 give five necessary and sufficient conditions when there are enough measurable sets. Its proof essentially uses the Stone—Weierstrass theorem. In § 3 we apply the results of §§ 1 and 2 to the Daniell integral.

§ 0. Basic notations

Throughout this paper $R = (-\infty, +\infty)$ will denote the set of real numbers with the well-known topological, lattice and algebraic structure. $R^* = \langle -\infty, +\infty \rangle$ will be the set of extended real numbers. We shall use R' and R'' to denote either R or R^*, and we always suppose that $R' \subset R''$. From the topological point of view R^* will be considered as the two-point compactification of R. Particularly the sets $(c, +\infty\rangle$, and $\langle -\infty, c)$, $c \in R$, will form a base of neighbourhoods of the points $\{+\infty\}$ and $\{-\infty\}$, respectively. Thus, R^* will be a separable compact metric space.

$\mathbf{B}_0'$ will denote the σ-algebra of all Borel (= Baire) measurable subsets of R', i.e., the smallest σ-ring containing all compact (= compact G_δ) subsets of R'. It is well known that $\mathbf{B}_0$ ($\mathbf{B}_0^*$) is the smallest σ-ring containing all sets of form $\langle c, +\infty)$, $c \in R$ ($\langle c, +\infty\rangle$, $c \in R^*$).

R^* with the usual lattice operations and order relations is a complete lattice and a totally ordered space, respectively. Further, $c_n \to c$, c_n, $c \in R'$, $n = 1, 2, \ldots$ if and only if $\liminf_n c_n = \limsup_n c_n$.

Multiplication, addition and subtraction in R^* define as in [7, section 4—1]. Particularly, $0 \cdot c = c \cdot 0 = 0$ for each $c \in R^*$, and $(+\infty) + (-\infty) = (-\infty) + (+\infty) = 0$. In this way the addition in R^* is commutative, however, it is not always associative ($[(-\infty) + (+\infty)] + (+\infty) = +\infty$, and $(-\infty) + [(+\infty) + (+\infty)] = 0$).

In what follows T will denote a non empty set, 2^T the collection of all subsets of T and R'^T the class of all R' — valued functions defined on T. Convergence, order, lattice and algebraic operations in R'^T are defined pointwise. For $f \in R'^T$ and $c \in R'$ we define $(c \vee f)(t) = c \vee f(t)$ and $(c \wedge f)(t) = c \wedge f(t)$, $t \in T$. For $f \in R'^T$ we put $f^+ = f \vee 0$ and $f^- = (-f) \vee 0$.

If $\mathscr{F} \subset R'^T$, then $\mathscr{F}^+ = \{f : f \in \mathscr{F}, f \geqq 0\}$, $\mathscr{F}^- = \{f : f \in \mathscr{F}, f \leqq 0\}$, $\mathscr{F}^o = \{f : f \in R^{*T}$, there are $f_n \in \mathscr{F}$, $n = 1, 2, \ldots$, such that $f_n \nearrow f\}$, $\mathscr{F}_u = \{f : f \in R^{*T}$, there are $f_n \in \mathscr{F}$, $n = 1, 2, \ldots$, such that $f_n \searrow f\}$, and $\tilde{\mathscr{F}} = \{f : f \in R^{*T}$, there are $g \in \mathscr{F}_u$ and $h \in \mathscr{F}^o$ such that $g \leqq f \leqq h\}$.

Definition 1. *We say that $\mathscr{F} \subset R'^T$ is an R' — linear function lattice (on T) if $f \vee g$, $f \wedge g$ and $af + bg \in \mathscr{F}$ for each f, $g \in \mathscr{F}$ and each a, $b \in R$.*

Clearly R — linear function lattices are real vector lattices, however, R^* — linear function lattices are not, since the addition in R^* is not always associative.

The next theorem follows immediately from the definitions (for the proof of 4. see the proof of Theorem 6-2III in [7]).

Theorem 1. *Let $\mathscr{F} \subset R'^T$ be an R' — linear function lattice. Then:*

1. *$\mathscr{F}^o$ and $\mathscr{F}_u$ are lattices closed under multiplication by $c \in \langle 0, +\infty)$, and $\mathscr{F}^{o+}$ and $\mathscr{F}_u^-$ are closed under addition. If $R' = R$, then $\mathscr{F}^o$ and $\mathscr{F}_u$ are closed under addition,*
2. *$f \in \mathscr{F}^o \Leftrightarrow -f \in \mathscr{F}_u$,*
3. *$f \in \mathscr{F}^o \Rightarrow f^+ \in \mathscr{F}^o$ and $f^- \in \mathscr{F}_u$. If $R' = R$, then the converse is also true.*
4. *If $f_n \in \mathscr{F}^o$, $n = 1, 2, \ldots$, and $f_n \nearrow f$, then there are $u_n \in \mathscr{F}$, $n = 1, 2, \ldots$, such that $u_n \leqq f_n$ for each n and $u_n \nearrow f$. Thus $f \in \mathscr{F}^o$.*
5. *If $f_n \in \mathscr{F}_u$, $n = 1, 2, \ldots$, and $f_n \searrow f$, then there are $u_n \in \mathscr{F}$, $n = 1, 2, \ldots$, such that $u_n \geqq f_n$ for each n and $u_n \searrow f$. Thus $f \in \mathscr{F}_u$,*
6. *$\tilde{\mathscr{F}} = \{f : f \in R^{*T}$, there exists $h \in \mathscr{F}^{o+}$ such that $|f| \leqq h\}$, and*
7. *$\tilde{\mathscr{F}}$ is a σ-complete R^* — linear function lattice closed under the formation of pointwise limits of sequences. If $f \in \tilde{\mathscr{F}}$, $g \in R^{*T}$ and $\{t : t \in T, g(t) \neq 0\} \subset \{t : t \in T, f(t) \neq 0\}$, then $g \in \tilde{\mathscr{F}}$.*

In the sequel **R**, **D** and **S** will be used to denote a ring, a δ-ring (a ring closed under the formation of countable intersections) and a σ-ring of subsets of T, respectively. If $\mathbf{E} \subset 2^T$, then $\varrho(\mathbf{E})$, $\delta(\mathbf{E})$ and $\sigma(\mathbf{E})$ will denote the smallest ring, δ-ring and σ-ring containing **E**, respectively.

We shall say that a class $\mathbf{E} \subset 2^T$ is hereditary if $A \cap E \in \mathbf{E}$ for each $A \in 2^T$ and each $E \in \mathbf{E}$. If $A \in 2^T$, then χ_A denotes its characteristic function on T. For a ring $\mathbf{R} \subset 2^T$ $\mathscr{S}(\mathbf{R})$ will denote the R — linear function lattice of all **R** — simple functions on T, i.e., the class of all functions f of form $f = \sum_{i=1}^{r} a_i \cdot \chi_{A_i}$, where $a_i \in R$, $A_i \in \mathbf{R}$ and $A_i \cap A_j = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, r < +\infty$. If $\mathbf{E} \subset 2^T$, then clearly $\mathscr{S}(\varrho(\mathbf{E}))$ is the smallest R — linear function lattice containing all χ_E, $E \in \mathbf{E}$.

If T is a locally compact Hausdorff topological space, then **F**, **C** and $\mathbf{C}_0$ will denote the class of all closed, compact and compact G_δ subsets of T, respectively. Further, **U** denotes the class of all open subsets of T and $\mathbf{U}_0 = \mathbf{U} \cap \sigma(\mathbf{C}_0)$. We shall say that $\sigma(\mathbf{F}) = \sigma(\mathbf{U})$, $\sigma(\mathbf{C})$ and $\sigma(\mathbf{C}_0)$ are the weakly Borel (see [1, p. 181]), Borel and Baire subsets of T, respectively. $C_{00}(T)$ denotes the R — linear function lattice of all R — valued continuous functions on T with compact supports, and $C_0(T)$ is its closure in the sup norm in the Banach space of all bounded functions on T. Clearly $C_0(T)$ is again an R — linear function lattice.

If $\mathscr{F}_1, \mathscr{F}_2 \subset R'^T$ and o is an algebraic or lattice operation, then $\mathscr{F}_1 \underset{\cdot}{o} \mathscr{F}_2 = \{f : f = f_1 o f_2, f_1 \in \mathscr{F}_1$ and $f_2 \in \mathscr{F}_2\}$. Similarly we define $\mathbf{E}_1 \underset{\cdot}{o} \mathbf{E}_2$, when $\mathbf{E}_1, \mathbf{E}_2 \subset 2^T$ and o is a set operation.

§ 1. $(\mathscr{F}, \mathbf{N})$ — measurable functions and sets

Definition 2. *Let $\mathbf{N} \subset 2^T$. We say that a property P defined on T is valid almost everywhere $\mathbf{N}$, shortly a.e. $\mathbf{N}$, if $\{t : t \in T,\ P(t) \text{ is not true}\} \in \mathbf{N}$.*

It is clear that under suitable assumptions on $\mathbf{N}$ the analogs of the results of §§ 18 and 19 in [1] are valid for R' — valued functions and the a.e. $\mathbf{N}$ concept.

Definition 3. *Let $\mathbf{N} \subset 2^T$ and let $\mathscr{F} \subset R'^T$. By $\mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ we denote the smallest class of R'' — valued functions on T which contains $\mathscr{F}$ and which is closed under the formation of pointwise limits a e $\mathbf{N}$ of its sequences. Elements of $\mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ are called R'' — valued $(\mathscr{F}, \mathbf{N})$ — measurable functions. By $\mathbf{B}''_{\mathbf{N}}(\mathscr{F})$ we denote the class of all subsets $E \subset T$ such that $\chi_E \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$. Elements of $\mathbf{B}''_{\mathbf{N}}(\mathscr{F})$ are called R'' — $(\mathscr{F}, \mathbf{N})$ — measurable sets. If $\mathbf{N} = \emptyset$, then we write simply $\mathscr{B}''(\mathscr{F})$ and $\mathbf{B}''(\mathscr{F})$.*

Let $\mathbf{N} \subset 2^T$ and let $\mathscr{F} \subset R^T$. Then clearly $\mathscr{B}_{\mathbf{N}}(\mathscr{F}) \subset \mathscr{B}^*_{\mathbf{N}}(\mathscr{F}) \cap R^T$, hence $\mathbf{B}_{\mathbf{N}}(\mathscr{F}) \subset \mathbf{B}^*_{\mathbf{N}}(\mathscr{F})$. Further $\mathscr{B}^*_{\mathbf{N}}(\mathscr{B}_{\mathbf{N}}(\mathscr{F})) = \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$. If $\mathbf{M} \subset \mathbf{N} \subset 2^T$ and $\mathscr{F} \subset R'^T$, then $\mathscr{B}''_{\mathbf{M}}(\mathscr{F}) \subset \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ and $\mathbf{B}''_{\mathbf{M}}(\mathscr{F}) \subset \mathbf{B}''_{\mathbf{N}}(\mathscr{F})$. Particularly, $\mathscr{B}''(\mathscr{F}) \subset \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ and $\mathbf{B}''(\mathscr{F}) \subset \mathbf{B}''_{\mathbf{N}}(\mathscr{F})$ for any $\mathbf{N} \subset 2^T$ and any $\mathscr{F} \subset R'^T$.

Theorem 2. *Let $\mathbf{N} \subset 2^T$, $\mathscr{F} \subset R'^T$, let $n \geqq 1$ and let $\varphi : R''^n \to R''$ be a separately continuous function. Suppose that $\varphi(f_1, \ldots, f_n) \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ for each $f_1, \ldots, f_n \in \mathscr{F}$. Then $\varphi(f_1, \ldots, f_n) \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ for each $f_1, \ldots, f_n \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$.*

Proof. Put $\mathscr{B}_1 = \{f_1 : f_1 \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F}),\ \varphi(f_1, f_2, \ldots, f_n) \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F}) \text{ for each } f_2, \ldots, f_n \in \mathscr{F}\}$. Then $\mathscr{F} \subset \mathscr{B}_1$ and if $f_{1,k} \in \mathscr{B}_1$, $k = 1, 2, \ldots$, and $f_{1,k} \to f \in R''^T$ a.e. $\mathbf{N}$, then $f \in \mathscr{B}_1$ by the definition of $\mathscr{B}''_{\mathbf{N}}(\mathscr{F})$. Thus $\mathscr{B}_1 = \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$. Put $\mathscr{B}_2 = \{f_2 : f_2 \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F}),\ \varphi(f_1, f_2, \ldots, f_n) \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ for each $f_1 \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ and each $f_3, \ldots, f_n \in \mathscr{F}\}$. Then similarly as above $\mathscr{B}_2 = \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$. Continuing in this way we obtain that $\mathscr{B}_n = \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$, i.e., the assertion of the theorem.

Corollary 1. *Let $\mathbf{N} \subset 2^T$ and let $\mathscr{F} \subset R'^T$ be a lattice. Then:*

1. *$\mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ is the smallest lattice of R'' — valued functions on T satisfying both a): $\mathscr{F} \subset \mathscr{B}''(\mathscr{F}) \subset \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$, and b): if $f_n \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$, $n = 1, 2, \ldots$, is a monotone sequence and $f_n \to f \in R''^T$ a.e. $\mathbf{N}$, then $f \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$,*
2. *$\mathbf{B}''_{\mathbf{N}}(\mathscr{F})$ is a σ-complete lattice of subsets of T, and*
3. *If moreover $\mathscr{F}$ is an R' — linear function lattice, then $|f| \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ for each $f \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ and $\mathbf{B}''_{\mathbf{N}}(\mathscr{F})$ is a σ-ring.*

Proof. 1) Since $(x, y) \to x \vee y$, $x \wedge y$ are separately continuous functions from R''^2 to R'', $\mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ is a lattice by the theorem. That $\mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ is the smallest lattice satisfying both a) and b) follows from the definition of $\mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ and from the fact that $f_n \to f$ a.e. $\mathbf{N}$ if and only if $f = \liminf_n f_n$ a.e. $\mathbf{N}$.

2) is a consequence of 1).

3) $x \to |x|$ is a continuous function from R'' to R'', and $(x, y) \to 1 \wedge (x \vee 0) - 1 \wedge (x \vee 0) \wedge (y \vee 0)$ is a separately continuous function from R''^2 to R''.

Corollary 2. *Let* $\mathbf{N} \subset 2^T$ *and let* $\mathscr{F} \subset R^T$ *be an* R *— linear function lattice. Then:*
1) $\mathscr{B}_{\mathbf{N}}(\mathscr{F})$ *is an* R *— linear function lattice, and*
2) $\mathbf{B}^*_{\mathbf{N}}(\mathscr{F}) \supset \mathbf{B}_{\mathbf{N}}(\mathscr{F})$ *are σ-rings containing* $\mathbf{B}(\mathscr{F})$ *and* $\mathbf{N}$.

Proof. 1). Since $(x, y) \to x \vee y$, $x \wedge y$, $ax + by$, $a, b \in R$, are separately continuous functions from R^2 to R, 1) follows from the theorem.

2) is a direct consequence of Corollary 1—3) and the fact that $0 \in \mathscr{F} \Rightarrow \mathbf{N} \subset \mathbf{B}_{\mathbf{N}}(\mathscr{F})$.

Similarly as Theorem 2 one can prove

Theorem 3. *Let* $\mathbf{N} \subset 2^T$, *let* $\mathscr{F} \subset R^T$ *and let* $\varphi: R \times R^* \to R^*$ *be a separately continuous function. Suppose that* $\varphi(f, g) \in \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$ *for each* $f, g \in \mathscr{F}$. *Then* $\varphi(f, g) \in \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$ *for each* $f \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ *and each* $g \in \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$.

Corollary. *Let* $\mathbf{N} \subset 2^T$, *let* $\mathscr{F} \subset R^T$ *and let* $f + g \in \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$ *for each* $f, g \in \mathscr{F}$. *Then* $\mathscr{B}^*_{\mathbf{N}}(\mathscr{F}) = \mathscr{B}_{\mathbf{N}}(\mathscr{F}) + \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$.

Theorem 4. *Let* $\mathbf{N} \subset 2^T$, $\mathscr{F} \subset R'^T$, $n \geqq 1$ *and let* $\Phi \subset C''_s(R''^n) =$ *the set of all separately continuous* R'' *— valued functions on* R''^n. *Suppose that* $\varphi(f_1, \ldots, f_n) \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ *for each* $f_1, \ldots, f_n \in \mathscr{F}$ *and each* $\varphi \in \Phi$. *Then* $\varphi(f_1, \ldots, f_n) \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ *for each* $f_1, \ldots, f_n \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ *and each* $\varphi \in \mathscr{B}''(\Phi)$.

Proof. Put $\mathscr{B} = \{\varphi: \varphi \in \mathscr{B}''(\Phi), \ \varphi(f_1, \ldots, f_n) \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ for each $f_1, \ldots, f_n \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})\}$. Then $\Phi \subset \mathscr{B}$ by Theorem 2. Let $\varphi_k \in \mathscr{B}$, $k = 1, 2, \ldots$, and let $\varphi_k \to \varphi \in R''^{R''^n}$, i.e., let $\varphi_k(x_1, \ldots, x_n) \to \varphi(x_1, \ldots, x_n) \in R''$ for each $(x_1, \ldots, x_n) \in R''^n$. Then $\varphi(f_1, \ldots, f_n) = \lim \varphi_k(f_1, \ldots, f_n) \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ for each $f_1, \ldots, f_n \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$, hence $\varphi \in \mathscr{B}$. Thus $\mathscr{B} = \mathscr{B}''(\Phi)$, and the theorem is proved.

Definition 4. *For* $\mathbf{N} \subset 2^T$ *put* $\mathscr{N}'(\mathbf{N}) = \{f: f \in R'^T, \{t: t \in T, f(t) \neq 0\} \in \mathbf{N}\}$, *and for* $\mathscr{N}' \subset R'^T$ *put* $\mathbf{N}(\mathscr{N}') = \{E: E \in 2^T, \chi_E \in \mathscr{N}'\}$.

Clearly $f \in \mathscr{N}'(\mathbf{N}) \Leftrightarrow -f \in \mathscr{N}'(\mathbf{N}) \Leftrightarrow |f| \in \mathscr{N}'(\mathbf{N}) \Leftrightarrow af \in \mathscr{N}'(\mathbf{N})$ for each $a \in R - \{0\}$. If $\mathscr{F} \subset R^T$, then $\mathscr{B}_{\mathbf{N}}(\mathscr{F}) \supset \mathscr{B}_{\mathbf{N}}(\mathscr{F}) \dotplus \mathscr{N}(\mathbf{N})$. Since $E \in \mathbf{N} \Leftrightarrow \chi_E \in \mathscr{N}'(\mathbf{N})$, $\mathbf{N} = \mathbf{N}(\mathscr{N}'(\mathbf{N}))$ for any $\mathbf{N} \subset 2^T$. Further, $f + g \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$ when $f \in \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$, $g \in \mathscr{N}''(\mathbf{N})$ and $f \cdot g = 0$. If $\mathbf{N} \subset 2^T$ is a hereditary class, then $\mathscr{B}''_{\mathbf{N}}(\mathscr{F}) \supset \mathscr{B}''_{\mathbf{N}}(\mathscr{F}) \dotplus \mathscr{N}''(\mathbf{N})$ and $\mathbf{B}''_{\mathbf{N}}(\mathscr{F}) \supset \mathbf{B}''_{\mathbf{N}}(\mathscr{F}) \dot{-} \mathbf{N}$.

Theorem 5. *Let* $\mathbf{N} \subset 2^T$ *be a hereditary ring. Then* $f + g$, $h \in \mathscr{N}'(\mathbf{N})$ *for any* f, $g \in \mathscr{N}'(\mathbf{N})$ *and any* $h \in R'^T$ *satisfying* $|h| \leqq |f|$. *Moreover* $\mathscr{N}'(\mathbf{N}) \subset \mathscr{B}'(\mathscr{S}(\mathbf{N}))$. *Conversely, if* $\mathscr{N}' \subset R'^T$ *is such that* $f + g$, $h \in \mathscr{N}'$ *for any* $f, g \in \mathscr{N}'$ *and any* $h \in R'^T$ *satisfying* $|h| \leqq |f|$, *then* $\mathbf{N}(\mathscr{N}')$ *is a hereditary ring.*

Proof. Only the inclusion $\mathscr{N}'(\mathbf{N}) \subset \mathscr{B}'(\mathscr{S}(\mathbf{N}))$ is not immediate. Let $f \in \mathscr{N}'(\mathbf{N})$, and for each $n = 1, 2, \ldots$ put

$$f_n = -n\cdot\chi_{\{t:t\in T,\, f(t)<-n\}} + \sum_{k=-n^2}^{n^2-1}\frac{k}{n}\chi_{\{t:t\in T,\,(k/n)\leqq f(t)<(k+1)/n\}} + \\ + n\cdot\chi_{\{t:t\in T,\, f(t)\geqq n\}}.$$

Then obviously $f_n\in\mathcal{S}(\mathbf{N})$ for each n, and $f_n\to f$. Thus $f\in\mathcal{B}'(\mathcal{S}(\mathbf{N}))$.

Theorem 6. *Let $\mathbf{N}\subset 2^T$ be a hereditary σ-ring. Then $\mathcal{N}'(\mathbf{N}) = \mathcal{B}'(\mathcal{N}'(\mathbf{N})) = \mathcal{B}'(\mathcal{S}(\mathbf{N}))$, and $f+g$, $h\in\mathcal{N}'(\mathbf{N})$ for any f, $g\in\mathcal{N}'(\mathbf{N})$ and any $h\in R'^T$ satisfying $|h|\leqq|f|$. Conversely, if $\mathcal{N}'\subset R'^T$ is such that $\mathcal{N}'=\mathcal{B}'(\mathcal{N}')$, and $f+g$, $h\in\mathcal{N}'$ for any f, $g\in\mathcal{N}'$ and any $h\in R'^T$ satisfying $|h|\leqq|f|$, then $\mathbf{N}(\mathcal{N}')$ is a hereditary σ-ring. Moreover in this case $\mathcal{N}'(\mathbf{N}(\mathcal{N}'))=\mathcal{N}'$.*

Proof. For the first part, clearly $\mathcal{B}'(\mathcal{N}'(\mathbf{N})) = \mathcal{N}'(\mathbf{N})$. Thus by Theorem 5 $\mathcal{N}'(\mathbf{N}) \subset \mathcal{B}'(\mathcal{S}(\mathbf{N})) \subset \mathcal{B}'(\mathcal{N}'(\mathbf{N})) = \mathcal{N}'(\mathbf{N})$.

For the converse part, $\mathbf{N}(\mathcal{N}')$ is a hereditary ring by Theorem 5. Let $E_n\in\mathbf{N}(\mathcal{N}')$, $n=1, 2, \ldots$. Then for each n we have $f_n = 1\wedge\left(\sum_{k=1}^{n}\chi_{E_k}\right)\in\mathcal{N}'$. But then $\chi_{\bigcup_{n=1}^{\infty}E_n} = \lim f_n\in\mathcal{B}'(\mathcal{N}')=\mathcal{N}'$. Thus $\mathbf{N}(\mathcal{N}')$ is a hereditary σ-ring.

Clearly $f\in\mathcal{N}' \Leftrightarrow |f|\in\mathcal{N}'$, and $1\wedge\lim n|f|\in\mathcal{N}' \Leftrightarrow \{t:t\in T, f(t)\neq 0\}\in\mathbf{N}(\mathcal{N}') \Leftrightarrow f\in\mathcal{N}'(\mathbf{N}(\mathcal{N}'))$. Thus $\mathcal{N}' = \mathcal{N}'(\mathbf{N}(\mathcal{N}'))$ if and only if $|f|\in\mathcal{N}' \Leftrightarrow 1\wedge\lim n|f|\in\mathcal{N}'$. Let $|f|\in\mathcal{N}'$. Then $1\wedge n|f|\in\mathcal{N}'$ for each n, hence $1\wedge\lim n|f|\in\mathcal{B}'(\mathcal{N}')=\mathcal{N}'$. Suppose now that $1\wedge\lim n|f|\in\mathcal{N}'$. Then $|f|\wedge k\wedge\lim n|f|\in\mathcal{N}'$ for each $k=1, 2, \ldots$, and $|f|\wedge k\wedge\lim n|f|\to|f|$ in R'. In this way $|f|\in\mathcal{B}'(\mathcal{N}')$, hence $f\in\mathcal{N}'$. The theorem is proved.

Theorem 7. *Let $\mathbf{N}\subset 2^T$ be a hereditary ring. Then:*

1) *$\mathcal{F}\subset R^T$ implies $\mathcal{B}_{\mathbf{N}}(\mathcal{F}) = \mathcal{B}(\mathcal{F}\dot{+}\mathcal{N}(\mathbf{N})) = \mathcal{B}(\mathcal{F}\dot{+}\mathcal{S}(\mathbf{N}))$ and $\mathcal{B}^*_{\mathbf{N}}(\mathcal{F}) = \mathcal{B}^*(\mathcal{F}\dot{+}\mathcal{N}(\mathbf{N})) = \mathcal{B}^*(\mathcal{B}_{\mathbf{N}}(\mathcal{F})) = \mathcal{B}^*(\mathcal{F}\dot{+}\mathcal{S}(\mathbf{N}))$, and*
2) *$\mathcal{F}\subset R^{*T}$ implies $\mathcal{B}^*_{\mathbf{N}}(\mathcal{F})=\mathcal{B}^*([\mathcal{F}\dot{+}\mathcal{N}^*(\mathbf{N})]\dot{+}\mathcal{N}^*(\mathbf{N}))$.*

Proof. 1). Clearly $\mathcal{B}_1=\mathcal{B}(\mathcal{F}\dot{+}\mathcal{N}(\mathbf{N}))\subset\mathcal{B}_{\mathbf{N}}(\mathcal{F})$. Define $\mathcal{B}=\{f:f\in\mathcal{B}_1, f\cdot\chi_{T-N} + g\cdot\chi_N\in\mathcal{B}_1$ for each $N\in\mathbf{N}$ and each $g\in R^T\}$.

Since $f\cdot\chi_{T-N}=f-f\cdot\chi_N$, $\mathcal{B}_1\supset\mathcal{B}\supset\mathcal{F}$. Let $f_n\in\mathcal{B}$, $n=1, 2, \ldots$, let $f\in R^T$ and let $f_n\to f$ a.e. $\mathbf{N}$. Then there is a set $N\in\mathbf{N}$ such that $f_n\cdot\chi_{T-N}\to f\cdot\chi_{T-N}$. Put $f'_n=f_n\cdot\chi_{T-N} + f\cdot\chi_N$. Then $f'_n\in\mathcal{B}_1$ for each n and $f'_n\to f$, hence $f\in\mathcal{B}_1$. Let $M\in\mathbf{N}$ and let $g\in R^T$. Then (1) $f'_n\cdot\chi_{T-M} + g\cdot\chi_M = (f_n\cdot\chi_{T-N} + f\cdot\chi_N)\cdot\chi_{T-M} + g\cdot\chi_M = f_n\cdot\chi_{T-(M\cup N)} + (f\cdot\chi_{N-M} + g\cdot\chi_M)\cdot\chi_{N\cup M}$. Since $M\cup N\in\mathbf{N}$ and $f'_n\cdot\chi_{T-M} + g\cdot\chi_M\to f\cdot\chi_{T-M} + g\cdot\chi_M$, $f\cdot\chi_{T-M} + g\cdot\chi_M\in\mathcal{B}_1$. Since $M\in\mathbf{N}$ and $g\in R^T$ were arbitrary, $f\in\mathcal{B}$. Thus $\mathcal{B}_1\supset\mathcal{B}\supset\mathcal{B}_{\mathbf{N}}(\mathcal{F})$, hence $\mathcal{B}_1=\mathcal{B}_{\mathbf{N}}(\mathcal{F})$.

Since $\mathcal{S}(\mathbf{N})\subset\mathcal{N}(\mathbf{N})$, $\mathcal{B}(\mathcal{F}\dot{+}\mathcal{S}(\mathbf{N}))\subset\mathcal{B}(\mathcal{F}\dot{+}\mathcal{N}(\mathbf{N}))$. But $\mathcal{N}(\mathbf{N})\subset\mathcal{B}(\mathcal{S}(\mathbf{N}))$ by Theorem 6, hence $\mathcal{B}(\mathcal{F}\dot{+}\mathcal{S}(\mathbf{N}))\supset\mathcal{F}\dot{+}\mathcal{B}(\mathcal{S}(\mathbf{N}))\supset\mathcal{F}\dot{+}\mathcal{N}(\mathbf{N})$. Thus

$\mathscr{B}(\mathscr{F}\dot{+}\mathscr{S}(\mathbf{N})) = \mathscr{B}(\mathscr{B}(\mathscr{F}\dot{+}\mathscr{S}(\mathbf{N}))) \supset \mathscr{B}(\mathscr{F}\dot{+}\mathscr{N}(\mathbf{N}))$, hence $\mathscr{B}(\mathscr{F}\dot{+}\mathscr{N}(\mathbf{N})) = \mathscr{B}(\mathscr{F}\dot{+}\mathscr{S}(\mathbf{N}))$.

Obviously $\mathscr{B}_1^* = \mathscr{B}^*(\mathscr{F}+\mathscr{N}(\mathbf{N})) \subset \mathscr{B}_{\mathbf{N}}^*(\mathscr{F})$. Define $\mathscr{B}^* = \{f: f \in \mathscr{B}_1^*,\ f\cdot\chi_{T-N} + g\cdot\chi_N \in \mathscr{B}_1^*$ for each $N \in \mathbf{N}$ and each $g \in R^{*T}\}$.

Let further $f \in \mathscr{F}$, let $N \in \mathbf{N}$ and let $g \in R^{*T}$. Then $h_n = f - f\cdot\chi_N + (-n) \vee (n \wedge g) \in \mathscr{F}\dot{+}\mathscr{N}(\mathbf{N})$ for each $n = 1, 2, \ldots$, hence $f\cdot\chi_{T-N} + g\cdot\chi_N = \lim h_n \in \mathscr{B}_1^*$. Thus $\mathscr{B}_1^* \supset \mathscr{B}^* \supset \mathscr{F}$. Now, proceeding as above, we obtain that $\mathscr{B}_1^* = \mathscr{B}_{\mathbf{N}}^*(\mathscr{F})$. (Since $T-(M\cup N)$, $N-M$ and M are pairwise disjoint sets, the equality (1) holds in spite of being in R^*).

Using the proved equalities $\mathscr{B}_{\mathbf{N}}^*(\mathscr{F}) = \mathscr{B}^*(\mathscr{F}\dot{+}\mathscr{N}(\mathbf{N})) = \mathscr{B}^*(\mathscr{B}(\mathscr{F}\dot{+}\mathscr{N}(\mathbf{N}))) = \mathscr{B}^*(\mathscr{B}_{\mathbf{N}}(\mathscr{F})) = \mathscr{B}^*(\mathscr{B}(\mathscr{F}\dot{+}\mathscr{S}(\mathbf{N}))) = \mathscr{B}^*(\mathscr{F}\dot{+}\mathscr{S}(\mathbf{N}))$.

2) Clearly $\mathscr{B}_2^* = \mathscr{B}^*([\mathscr{F}+\mathscr{N}^*(\mathbf{N})]+\mathscr{N}^*(\mathbf{N})) \subset \mathscr{B}_{\mathbf{N}}^*(\mathscr{F})$. Define $\mathscr{B}^*$ as above with $\mathscr{B}_2^*$ instead of $\mathscr{B}_1^*$. Then it easily follows that $\mathscr{B}_2^* \supset \mathscr{B}^* \supset [\mathscr{F}\dot{+}\mathscr{N}^*(\mathbf{N})$. Now, in the same way as above in 1), we obtain the desired equality.

Let $\mathbf{S} \subset 2^T$ be a σ-ring. According to the well-known definition, see [7, section 5—1], a function $f: T \to R'$ is called $\mathbf{S}$ — measurable if $\{t: t \in T,\ f(t) \neq 0\} \cap f^{-1}(\mathbf{B}_0') \subset \mathbf{S}$. It is easy to see that $\mathbf{B}_0'$ may be replaced by any class $\mathbf{E} \subset 2^{R'}$ such that $\sigma(\mathbf{E}) = \mathbf{B}_0'$. Particularly, we may take $\mathbf{E} = \{\{x: x \in R',\ x \geqq c\},\ c \in R'\}$.

Important information about $\mathbf{S}$ — measurable functions are contained in the following well-known theorem, see [7, section 5-1].

Theorem 8. *Let $\mathbf{S} \subset 2^T$ be a σ-ring and let $f: T \to R'$. Then the following conditions are equivalent:*
1) *f is $\mathbf{S}$ — measurable,*
2) *$f^+, f^- \in \mathscr{S}(\mathbf{S})^{o+}$,*
3) *there are $f_n \in \mathscr{S}(\mathbf{S})$, $n = 1, 2, \ldots$ such that $f_n \to f$ and $|f| \nearrow |f|$, and*
4) *$f \in \mathscr{B}'(\mathscr{S}(\mathbf{S}))$.*

Corollary 1. *Let $\mathbf{S} \subset 2^T$ be a σ-ring. Then $\mathscr{B}(\mathscr{S}(\mathbf{S})) = \mathscr{B}^*(\mathscr{S}(\mathbf{S})) \cap R^T$.*

Corollary 2. *Let $\mathbf{S} \subset 2^T$ be a σ-ring. Then $\mathbf{B}^*(\mathscr{S}(\mathbf{S})) = \mathbf{B}(\mathscr{S}(\mathbf{S})) = \mathbf{S}$.*

Proof. Obviously $\mathbf{S} \subset \mathbf{B}(\mathscr{S}(\mathbf{S})) \subset \mathbf{B}^*(\mathscr{S}(\mathbf{S}))$. Let $E \in \mathbf{B}^*(\mathscr{S}(\mathbf{S}))$. Then $\chi_E \in \mathscr{S}(\mathbf{S})^{o+}$ by the theorem. Thus there are $f_n \in \mathscr{S}(\mathbf{S})^+$, $n = 1, 2, \ldots$, such that $f_n \nearrow \chi_E$. Each f_n is of the form $f_n = \sum_{i=1}^{r_n} a_{n,i} \cdot \chi_{A_{n,i}}$ with $0 < a_{n,i} \leqq 1$, $A_{n,i} \in \mathbf{S}$, $A_{n,i} \subset E$, and $A_{n,i} \cap A_{n,j} = \emptyset$ for $i \neq j$, $i, j = 1, \ldots, r_n$. Put $E_n = \bigcup_{i=1}^{r_n} A_{n,i}$, $n = 1, 2, \ldots$. Then $E_n \in \mathbf{S}$, $E_n \subset E$, $E_n \nearrow$, and since $f_n \leqq \chi_{E_n}$, $E_n \nearrow E$. Thus $E \in \mathbf{S}$.

Corollary 3. *Let $\mathbf{S}, \mathbf{S}_1 \subset 2^T$ be σ-rings. Then the following conditions are equivalent:*
1) $\mathscr{B}^*(\mathscr{S}(\mathbf{S})) = \mathscr{B}^*(\mathscr{S}(\mathbf{S}_1))$,

2) $\mathscr{B}(\mathscr{S}(\mathbf{S}))=\mathscr{B}(\mathscr{S}(\mathbf{S}_1))$, *and*
3) $\mathbf{S}=\mathbf{S}_1$.

Corollary 4. *Let* $\mathbf{S}\subset 2^T$ *be a* σ*-ring. Then:*
1) $\mathscr{F}\subset R^T$ *and* $\mathscr{B}(\mathscr{F})=\mathscr{B}(\mathscr{S}(\mathbf{S}))$ *imply* $\mathbf{B}^*(\mathscr{F})=\mathbf{B}(\mathscr{F})=\mathbf{S}$, *and*
2) $\mathscr{F}\subset R^{*T}$ *and* $\mathscr{B}^*(\mathscr{F}) = \mathscr{B}^*(\mathscr{S}(\mathbf{S}))$ *imply* $\mathbf{B}^*(\mathscr{F})=\mathbf{S}$ *and* $\mathscr{B}^*(\mathscr{F})\cap \mathrm{R}^T = \mathscr{B}(\mathscr{S}(\mathbf{S}))$.

If $\mathbf{D}\subset 2^T$ is a δ-ring, then clearly $\mathscr{S}(\sigma(\mathbf{D}))^+ \subset \mathscr{S}(\mathbf{D})^{o+}$. Thus applying Theorem 1—4) we immediately have

Corollary 5. *Let* $\mathbf{D}\subset 2^T$ *be a* δ*-ring and let* $f\colon T\to R'$. *Then the following conditions are equivalent:*
1) f *is* $\sigma(\mathbf{D})$ — *measurable,*
2) $f^+, f^-\in\mathscr{S}(\mathbf{D})^{o+}$,
3) *there are* $f_n\in\mathscr{S}(\mathbf{D})$, $n=1, 2, \ldots$, *such that* $f_n\to f$ *and* $|f|\nearrow|f|$, *and*
4) $f\in\mathscr{B}'(\mathscr{S}(\mathbf{D}))$.

Theorem 9. *Let* $\mathbf{E}, \mathbf{N}\subset 2^T$. *Then* $\mathscr{B}_{\mathbf{N}}(\mathscr{S}(\varrho(\mathbf{E}))) = \mathscr{B}_{\mathbf{N}}(\mathscr{S}(\sigma(\mathbf{E})))$

Proof. Put $\mathbf{B}=\mathbf{B}_{\mathbf{N}}(\mathscr{S}(\varrho(\mathbf{E})))$. Since $\mathscr{S}(\varrho(\mathbf{E}))$ is an R — linear function lattice, $\mathscr{S}(\mathbf{B}) \subset \mathscr{B}_{\mathbf{N}}(\mathscr{S}(\varrho(\mathbf{E})))$, and $\mathbf{B}$ is a σ-ring by Corollary 2-2) of Theorem 2. Since $\mathbf{E}\subset\mathbf{B}$, $\varrho(\mathbf{E}) \subset \sigma(\mathbf{E}) \subset \mathbf{B}$, hence $\mathscr{B}_{\mathbf{N}}(\mathscr{S}(\varrho(\mathbf{E}))) \subset \mathscr{B}_{\mathbf{N}}(\mathscr{S}(\sigma(\mathbf{E}))) \subset \mathscr{B}_{\mathbf{N}}(\mathscr{S}(\mathbf{B})) \subset \mathscr{B}_{\mathbf{N}}(\mathscr{S}(\varrho(\mathbf{E})))$.

Using Corollary 4-1) of Theorem 8 we immediately have the following

Corollary. *Let* $\mathbf{E}\subset 2^T$. *Then* $\mathbf{B}^*(\mathscr{S}(\varrho(\mathbf{E}))) = \mathbf{B}(\mathscr{S}(\varrho(\mathbf{E}))) = \sigma(\mathbf{E})$.

Theorem 10. *Let* $\mathbf{R}\subset 2^T$ *be a ring and let* $\mathbf{N}\subset 2^T$ *be a hereditary ring. Then* $\varrho(\mathbf{R}\dot{\cup}\mathbf{N}) = \mathbf{R}\dot{\triangle}\mathbf{N} = \bigcup_{R\in\mathbf{R}}\bigcup_{N\in\mathbf{N}} \langle R-N, R\cup N\rangle = \varrho_1(\mathbf{R}, \mathbf{N})$, *where* $\langle R-N, R\cup N\rangle = \{A\colon A\in 2^T,\ R-N\subset A\subset R\cup N\}$.

Proof. Since $R-N\subset R\triangle N\subset R\cup N$, $\mathbf{R}\dot{\triangle}\mathbf{N} \subset \varrho_1(\mathbf{R}, \mathbf{N})$. If $R-N\subset A\subset R\cup N$, then $A\triangle R = (A-R) \cup (R-A) \subset N\cup N=N$, hence $A\triangle R\in\mathbf{N}$ ($\mathbf{N}$ is a hereditary class). But $A=R\triangle A\triangle R$, hence $\varrho_1(\mathbf{R}, \mathbf{N}) \subset \mathbf{R}\dot{\triangle}\mathbf{N}$.

Clearly $\varrho(\mathbf{R}\dot{\cup}\mathbf{N}) \supset \mathbf{R}\dot{\triangle}\mathbf{N}$. It remains to show that $\mathbf{R}\dot{\triangle}\mathbf{N}$ is a ring. Let A_1, $A_2\in\mathbf{R}\dot{\triangle}\mathbf{N}$. Then $A_1\cup A_2 = (A_1\triangle A_2) \cup (A_1\cap A_2) = A_1\triangle A_2\triangle(A_1\cap A_2)$, and $A_1-A_2 = A_1\triangle(A_1\cap A_2)$. Thus $\mathbf{R}\dot{\triangle}\mathbf{N}$ will be a ring if and only if it contains $A_1\cap A_2$ for any $A_1, A_2\in\mathbf{R}\dot{\triangle}\mathbf{N}$. Let $A_1 = R_1\triangle N_1$ and $A_2 - R_2\triangle N_2$, where R_1, $R_2\in\mathbf{R}$ and $N_1, N_2\in\mathbf{N}$. Then $A_1\cap A_2 = (R_1\triangle N_1) \cap (R_2\triangle N_2) - [R_1\cap(R_2\triangle N_2)] \triangle (N_1\cap(R_2\triangle N_2)] = (R_1\cap R_2) \triangle (R_1\cap N_2) \triangle [N_1\cap(R_2\triangle N_2)]$.
Since $\mathbf{R}$ is a ring and $\mathbf{N}$ is a hereditary ring, $A_1\cap A_2\in\mathbf{R}\dot{\triangle}\mathbf{N}$, and the theorem is proved.

Corollary 1. *Let* $\mathbf{E}\subset 2^T$ *and let* $\mathbf{N}\subset 2^T$ *be a hereditary ring. Then* $\varrho(\mathbf{E}\dot{\cup}\mathbf{N}) = \varrho(\mathbf{E})\dot{\triangle}\mathbf{N} = \varrho_1(\varrho(\mathbf{E}), \mathbf{N}))$.

In the process of completion of a measure space the following special case is used:

Corollary 2. *Let* $\mathbf{E}\subset 2^T$ *be a ring, let* $\mathbf{N}\subset 2^T$ *be a hereditary ring and let there exist for each* $N\in\mathbf{N}$ *an* $M\in\mathbf{R}\cap\mathbf{N}$ *such that* $N\subset M$. *Then* $\mathbf{R}\mathbin{\dot\triangle}\mathbf{N} = \mathbf{R}\dot{+}\mathbf{N}$, *where* $\dot{+}$ *means disjoint unions.*

Proof. If $A\in\mathbf{R}$, $N\in\mathbf{N}$ and $N\subset M\in\mathbf{R}\cap\mathbf{N}$, then $(A-N)\cup(N-A) = [A-(A\cap M)]\cup[A\cap(M-N)]\cup(N-A)\in\mathbf{R}\dot{+}\mathbf{N}$.

If $\mathbf{R}$, $\mathbf{N}\subset 2^T$ are rings, and if $f\in\mathscr{S}(\mathbf{R})$ and $g\in\mathscr{S}(\mathbf{N})$, then clearly $f+g\in\mathscr{S}(\varrho(\mathbf{R}\cup\mathbf{N}))$. Using this observation, Theorem 7-1), Theorem 9 and its Corollary and Theorem 10, we immediately have the next

Theorem 11. *Let* $\mathbf{R}\subset 2^T$ *be a ring and let* $\mathbf{N}\subset 2^T$ *be a hereditary ring. Then* $\mathscr{B}'_{\mathbf{N}}(\mathscr{S}(\mathbf{R})) = \mathscr{B}'(\mathscr{S}(\mathbf{R}\mathbin{\dot\triangle}\mathbf{N})) = \mathscr{B}'(\mathscr{S}(\sigma(\mathbf{R}\mathbin{\dot\triangle}\mathbf{N})))$, *and* $\mathbf{B}^*_{\mathbf{N}}(\mathscr{S}(\mathbf{R})) = \mathbf{B}_{\mathbf{N}}(\mathscr{S}(\mathbf{R})) = \sigma(\mathbf{R}\mathbin{\dot\triangle}\mathbf{N})$.

Using Corollary 2-2) of Theorem 2 we immediately have the following

Corollary. *Let* $\mathscr{F}\subset R^T$ *be an* R *— linear function lattice and let* $\mathbf{N}\subset 2^T$ *be a hereditary ring. Then* $\mathscr{B}'_{\mathbf{N}}(\mathscr{S}(\mathbf{B}_{\mathbf{N}}(\mathscr{F}))) = \mathscr{B}'(\mathscr{S}(\mathbf{B}_{\mathbf{N}}(\mathscr{F})))$.

Theorem 12. *Let* $\mathbf{E}\subset 2^T$ *and let* $\mathbf{N}\subset 2^T$ *be a hereditary σ-ring. Then* $\delta(\mathbf{E}\cup\mathbf{N}) = \delta(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}$ *and* $\sigma(\mathbf{E}\cup\mathbf{N}) = \sigma(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}$.

Proof. $\delta(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}$ and $\sigma(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}$ are rings by Theorem 10. Let $A_n\in\delta(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}$, $n=1,2,\ldots$. Then by Theorem 10 there are $R_n\in\delta(\mathbf{E})$ and $N_n\in\mathbf{N}$, $n=1,2,\ldots$, such that $R_n-N_n\subset A_n\subset R_n\cup N_n$ for each n. But then $\bigcap_{n=1}^{\infty}R_n - \bigcup_{n=1}^{\infty}N_n = \bigcap_{n=1}^{\infty}(R_n-N_n) \subset \bigcap_{n=1}^{\infty}A_n \subset \bigcap_{n=1}^{\infty}R_n \cup \bigcup_{n=1}^{\infty}N_n$, hence $\delta(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}$ is a δ-ring by Theorem 10. If $A_n\in\sigma(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}$, then the inclusions $\bigcup_{n=1}^{\infty}R_n - \bigcup_{n=1}^{\infty}N_n \subset \bigcup_{n=1}^{\infty}(R_n-N_n) \subset \bigcup_{n=1}^{\infty}A_n \subset \bigcup_{n=1}^{\infty}R_n \cup \bigcup_{n=1}^{\infty}N_n$ and Theorem 10 imply that $\sigma(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}$ is a σ-ring.

Since $\mathbf{D}\cap\sigma(\mathbf{D}) = \mathbf{D}$ for a δ-ring $\mathbf{D}\subset 2^T$, similarly as Corollary 2 of Theorem 10, we have the following

Corollary. *Let* $\mathbf{D}\subset 2^T$ *be a δ-ring, let* $\mathbf{N}\subset 2^I$ *be a hereditary σ-ring and let there for each* $N\in\mathbf{N}$ *exist* $M\in\sigma(\mathbf{D})\cap\mathbf{N}$ *such that* $N\subset M$. *Then* $\delta(\mathbf{D}\cup\mathbf{N}) = \mathbf{D}\dot{+}\mathbf{N}$.

From Theorems 11 and 12 we immediately have

Theorem 13. *Let* $\mathbf{E}\subset 2^T$ *and let* $\mathbf{N}\subset 2^T$ *be a hereditary σ-ring. Then* $\mathscr{B}'_{\mathbf{N}}(\mathscr{S}(\varrho(\mathbf{E}))) = \mathscr{B}'(\mathscr{S}(\sigma(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}))$, *and* $\mathbf{B}^*_{\mathbf{N}}(\mathscr{S}(\varrho(\mathbf{E}))) = \mathbf{B}_{\mathbf{N}}(\mathscr{S}(\varrho(\mathbf{E}))) = \sigma(\mathbf{E})\mathbin{\dot\triangle}\mathbf{N}$.

Theorem 14. *Let* $\mathbf{S}\subset 2^T$ *be a σ-ring, let* $\mathbf{N}\subset 2^T$ *be a hereditary σ-ring and let f be* *an R' — valued $\mathbf{S}\dot{\triangle}\mathbf{N}$ — measurable function. Then there is an R' — valued $\mathbf{S}$ — measurable function g such that $f(t)=g(t)$ a.e. $\mathbf{N}$.*

Proof. Since $f=f^+-f^-$, $f^+\cdot f^-=0$, and f^+ and f^- are $\mathbf{S}\dot{\triangle}\mathbf{N}$ — measurable, it is enough to prove the assertion of the theorem for f^+. According to Theorem 8 $f^+\in\mathscr{S}(\mathbf{S}\triangle\mathbf{N})^{\sigma+}$. Hence there is a sequence $f_n\in\mathscr{S}(\mathbf{S}\dot{\triangle}\mathbf{N})$, $n=1, 2, \ldots$, such that $f_n\nearrow f^+$. Each f_n is of the form $f_n=\sum_{i=1}^{r_n}a_{n,i}\cdot\chi_{E_{n,i}\triangle N_{n,i}}$, where $E_{n,i}\in\mathbf{S}$, $N_{n,i}\in\mathbf{N}$ and $a_{n,i}\in R^+$, $i=1, 2, \ldots, r_n$. Put $u_n=\sum_{i=1}^{r_n}a_{n,i}\cdot\chi_{E_{n,i}}$ and $v_n=\bigvee_{i=1}^{n}u_i$, $n=1, 2, \ldots$, and let $g^+=\lim v_n$ if f^+ is R^* — valued, and $g^+=\lim v_n-(+\infty)\cdot\chi_{\{t:t\in T,\, v(t)=+\infty\}}$ if f^+ is R — valued. Then clearly g^+ is an R' — valued $\mathbf{S}$ — measurable function, and $g^+(t)=f^+(t)$ a.e. $\mathbf{N}$.

Corollary 1. *Let $\mathbf{S}\subset 2^T$ be a σ-ring and let $\mathbf{N}\subset 2^T$ be a hereditary σ-ring. Then*

$$\mathscr{B}'_{\mathbf{N}}(\mathscr{S}(\mathbf{S}\dot{\triangle}\mathbf{N}))=\mathscr{B}'_{\mathbf{N}}(\mathscr{S}(\mathbf{S}))=\mathscr{B}'(\mathscr{S}(\mathbf{S}))\dot{+}\mathscr{N}'(\mathbf{N}).$$

By Theorem 8 $\{t:t\in T, f(t)\neq 0\}\in\mathbf{S}$ for each $f\in\mathscr{B}'(\mathscr{S}(\mathbf{S}))$, hence we immediately have

Corollary 2. *Let $\mathbf{S}\subset 2^T$ be a σ-ring, let $\mathbf{N}\subset 2^T$ be a hereditary σ-ring and suppose that for each $N\in\mathbf{N}$ there exists an $M\in\mathbf{S}\cap\mathbf{N}$ such that $N\subset M$. Then each $f\in\mathscr{B}'_{\mathbf{N}}(\mathscr{S}(\mathbf{S}\dot{\triangle}\mathbf{N}))$ can be written in the form $f=g+h$, where $g\in\mathscr{B}'(\mathscr{S}(\mathbf{S}))$, $h\in\mathscr{N}'(\mathbf{N})$ and $g\cdot h=0$.*

Theorem 15. *Let T be a locally compact Hausdorff topological space and let $\mathbf{N}\subset 2^T$. Then $\mathscr{B}'_{\mathbf{N}}(C_{00}(T))=\mathscr{B}'_{\mathbf{N}}(C_0(T))=\mathscr{B}'_{\mathbf{N}}(\mathscr{S}(\sigma(\mathbf{C}_0)))$.*

Proof. By Theorem B in § 51 in [3] each $f\in C_0(T)$ is $\sigma(\mathbf{C}_0)$ — measurable, hence $\mathscr{B}_{\mathbf{N}}(C_{00}(T))\subset\mathscr{B}_{\mathbf{N}}(C_0(T))\subset\mathscr{B}_{\mathbf{N}}(\mathscr{B}(\mathscr{S}(\sigma(\mathbf{C}_0))))\subset\mathscr{B}_{\mathbf{N}}(\mathscr{S}(\sigma(\mathbf{C}_0)))$. On the other hand, $\chi_C\in\mathscr{B}(C_{00}(T))$ for each $C\in\mathbf{C}_0$ by Theorem A in § 55 in [3]. Thus $\mathscr{S}(\sigma(\mathbf{C}_0))\subset\mathscr{B}_{\mathbf{N}}(C_{00}(T))$ by Corollary 2-1) of Theorem 2, hence $\mathscr{B}_{\mathbf{N}}(\mathscr{S}(\sigma(\mathbf{C}_0)))\subset\mathscr{B}_{\mathbf{N}}(C_{00}(T))$. Finally $\mathscr{B}^*_{\mathbf{N}}(C_{00}(T))=\mathscr{B}^*(\mathscr{B}_{\mathbf{N}}(C_{00}(T)))=\mathscr{B}^*(\mathscr{B}_{\mathbf{N}}(\mathscr{S}(\sigma(\mathbf{C}_0))))=\mathscr{B}^*_{\mathbf{N}}(\mathscr{S}(\sigma(\mathbf{C}_0)))$ by Theorem 7-1).

Applying Theorem 13 we immediately have the next

Corollary. *Let T be a locally compact Hausdorff topological space and let $\mathbf{N}\subset 2^T$ be a hereditary σ-ring. Then $\mathscr{B}'_{\mathbf{N}}(C_{00}(T))=\mathscr{B}'_{\mathbf{N}}(C_0(T))=\mathscr{B}'(\mathscr{S}(\sigma(\mathbf{C}_0)\dot{\triangle}\mathbf{N}))$, and $\mathbf{B}^*_{\mathbf{N}}(C_{00}(T))=\mathbf{B}^*_{\mathbf{N}}(C_0(T))=\mathbf{B}_{\mathbf{N}}(C_{00}(T))=\mathbf{B}_{\mathbf{N}}(C_0(T))=\sigma(\mathbf{C}_0)\dot{\triangle}\mathbf{N}$.*

If $f\in R'^T$ and if $T^0\subset T$, then $f/_{T^0}$ will denote the restriction of f to T^0. If T is a separable locally compact metric space, then $\mathbf{C}=\mathbf{C}_0$ and $\sigma(\mathbf{F})=\sigma(\mathbf{C})$, see Theorem E in § 50 in [3]. Further, recall the definition of a Borel measurable function, see § 51 in [3].

Theorem 16. *Let $n\geqq 1$ and let $T^0=R'^n-(0, \ldots, 0)$ be equipped with the relative* *topology. Define* $\Phi_0' = \{f: f: R'^n \to R, f(0, \ldots, 0) = 0, \text{ and } f/_{T^0} \in C_{00}(T^0)\}$, $\mathscr{B}_{00}' = \{f: f: R'^n \to R, f(0, \ldots, 0) = 0, \text{ and } f/_{T^0}$ *is Borel measurable on* $T^0\}$, *and* $\mathscr{B}_0' = \{f: f: R'^n \to R, f(0, \ldots, 0) = 0, \text{ and } f$ *is Borel measurable on* $R'^n\}$.

Then $\mathscr{B}(\Phi_0') = \mathscr{B}_{00}' = \mathscr{B}_0'$.

Proof. Using Theorem 15 we easily obtain that $\mathscr{B}(\Phi_0') = \mathscr{B}_{00}'$. Since T^0 is equipped with the relative topology, $\mathbf{F}^0 = \mathbf{F} \cap T^0$. But then $\sigma(\mathbf{C}^0) = \sigma(\mathbf{F}^0) = \sigma(\mathbf{F}) \cap T^0 = \sigma(\mathbf{C}) \cap T^0$ by Theorem E in § 5 in [3]. Now using the definition of a Borel measurable function it is easy to see that $\mathscr{B}_{00}' = \mathscr{B}_0'$.

Theorem 17. *Let us have the notations of Theorem 16, let* $\mathbf{N} \subset 2^T$, *let* $\mathscr{F} \subset R'^T$ *and let* $\varphi(f_1, \ldots, f_n) \in \mathscr{B}_{\mathbf{N}}''(\mathscr{F})$ *for each* $f_1, \ldots, f_n \in \mathscr{F}$ *and each* $\varphi \in \Phi_0$. *Then* $\varphi(f_1, \ldots, f_n) \in \mathscr{B}_{\mathbf{N}}''(\mathscr{F})$ *for each* $f_1, \ldots, f_n \in \mathscr{B}_{\mathbf{N}}''(\mathscr{F})$ *and each* $\varphi \in \mathscr{B}_0$. *If moreover* $\mathscr{F}$ *is an* R' *— linear function lattice, then* $\mathscr{B}_{\mathbf{N}}^*(\mathscr{F})$ *is an* R^* *— linear function lattice.*

Proof. The first part of the theorem follows from Theorems 4 and 16. For the second part of the theorem we have to realize that the functions $(x, y) \to x \vee y$, $x \wedge y$, $ax + by$: $x, y \in R'$, $a, b \in R$, belongs to $\mathscr{B}_0'$ when $n = 2$.

Corollary. *Let* $\mathscr{F} = \mathscr{S}(\mathbf{R})$ *or* $C_{00}(T)$, *let* $\mathbf{N} \subset 2^T$ *and let us have the notations of Theorem 16. Then* $\varphi(f_1, \ldots, f_n) \in \mathscr{B}_{\mathbf{N}}''(\mathscr{F})$ *for each* $f_1, \ldots, f_n \in \mathscr{B}_{\mathbf{N}}''(\mathscr{F})$ *and each* $\varphi \in \mathscr{B}_0''$. *Particularly* $\mathscr{B}_{\mathbf{N}}^*(\mathscr{F})$ *is an* R^* *— linear function lattice.*

Proof. For given $\mathscr{F}'$ s it is easy to see that $\varphi(f_1, \ldots, f_n) \in \mathscr{F}$ for each $f_1, \ldots, f_n \in \mathscr{F}$ and each $\varphi \in \Phi_0''$.

§ 2. Main results on ($\mathscr{F}$, $\mathbf{N}$) — measurable functions and sets

Theorem 18. *Let* $\mathbf{N} \subset 2^T$ *and let* $\mathscr{F} \subset R^T$. *Then the following conditions are equivalent:*

1) $af + bg$, $f \cdot g \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ *for each* $f, g \in \mathscr{F}$ *and each* $a, b \in R$,
2) $af + bg$, $f \vee g$, $f \wedge g$, $f^{+n} \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ *for each* $f, g \in \mathscr{F}$, $a, b \in R$, *and each* $n = 2, 3, \ldots$,
3) $af + bg$, $f \vee g$, $f \wedge g$, $1 \wedge f^+ \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ *for each* $f, g \in \mathscr{F}$ *and* $a, b \in R$,
4) $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ *is a* σ*-ring and each* $f \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ *is* $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ *— measurable,*
5) $\mathscr{B}_{\mathbf{N}}(\mathscr{F}) = \mathscr{B}(\mathscr{S}(\sigma(\mathbf{B}_{\mathbf{N}}(\mathscr{F}))))$, *and*
6) $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ *is a* σ*-ring and each* $f \in \mathscr{F}$ *is* $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ *— measurable.*

Proof. 1)$\Rightarrow$2). We show that if φ: $R \times R \to R$ is a continuous function with $\varphi(0, 0) = 0$, then $\varphi(f, g) \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ for each $f, g \in \mathscr{F}$. This will prove 2), since $(x, y) \to x \vee y$, $x \wedge y$, $(x \vee 0)^n$ are such functions.

Let $S = R \times R - (0, 0)$ be equipped with the relative topology and consider the restrictions to S of the following functions on $R \times R$:

$$\varphi_1(x, y) = (x^2 + y^2) \cdot e^{-(x^2+y^2)} = r^2 \cdot e^{-r^2}, \quad \text{where } r^2 = x^2 + y^2,$$
$$\varphi_2(x, y) = (x^2 + y^2)^2 \cdot e^{-(x^2+y^2)} = r^4 \cdot e^{-r^2},$$

$$\varphi_3(x, y) = x \cdot e^{-(x^2+y^2)} = x \cdot e^{-r^2}, \quad \text{and}$$
$$\varphi_4(x, y) = y \cdot e^{-(x^2+y^2)} = y \cdot e^{-r^2}.$$

Then S is a separable locally compact metric space and $\varphi_{i/S} \in C_0(S)$, $i = 1, 2, 3, 4$. Clearly $\varphi_1 \wedge \varphi_2 > 0$ on S. We now show that the functions φ_i, $i = 1, 2, 3, 4$ separate the points of S.

The function φ_1 depends only on r, and $\lim_{r\to 0^+} \varphi_1(r) = \lim_{r\to+\infty} \varphi_1(r) = 0$. If r, $r_1 \in (0, 1\rangle$ and $r < r_1$, then $\varphi_1(r) < \varphi_1(r_1)$. If r, $r_1 \in \langle 1, +\infty)$ and $r < r_1$, then $\varphi_1(r) > \varphi_1(r_1)$. Thus the graph of φ_1 is a "crater" with the $\operatorname{maximum}_{r\in(0,+\infty)} \varphi_1(r) = \varphi_1(1) = e^{-1}$. Further, for each $a \in (0, e^{-1})$ there are exactly two points r_1, $r_2 \in (0, +\infty)$, $r_1 \in (0, 1)$ and $r_2 \in (1, +\infty)$ with $\varphi_1(r_1) = \varphi_1(r_2) = a$.

The function φ_2 behaves similarly as φ_1, but the $\operatorname{maximum}_{r\in(0,+\infty)} \varphi_2(r) = \varphi_2(\sqrt{2})$. If r_1, $r_2 \in (0, +\infty)$, $r_1 \neq r_2$ and if $\varphi_1(r_1) = \varphi_1(r_2)$, then it is easy to compute that $\varphi_2(r_1) \neq \varphi_2(r_2)$. Thus the functions φ_1 and φ_2 separate the points of S with different r. Clearly φ_3 and φ_4 separate the points of S on a given circle $r^2 = x^2 + y^2$.

Denote by Φ the algebra of functions on S generated by the functions $\varphi_{i\,S}$, $i = 1, 2, 3, 4$. Then by the Stone—Weierstrass theorem, see Theorem A in § 38 in [5], $C_0(S)$ is the closure of Φ in the supremum norm.

We extend each $f \in C_0(S)$ to $R \times R$ putting $f(0, 0) = 0$. By $C_0^0(R \times R)$ we denote the space of all thus extended elements of $C_0(S)$, and by Φ^0 the extended members of Φ. It is easy to see that $\mathscr{B}(\Phi^0) = \mathscr{B}(C_0^0(R \times R))$.

According to Theorem 4 it remains to show that $\varphi(f, g) \in \mathscr{B}_N(\mathscr{F})$ for each f, $g \in \mathscr{F}$ and each $\varphi \in \Phi^0$.

Theorem 2 implies that $af + bg$, $f \cdot g \in \mathscr{B}_N(\mathscr{F})$ for each f, $g \in \mathscr{B}_N(\mathscr{F})$ and each a, $b \in R$. By induction $\sum_{i+j=1}^{n} a_{i,j} \cdot f^i \cdot g^j \in \mathscr{B}_N(\mathscr{F})$ for each f, $g \in \mathscr{B}_N(\mathscr{F})$ and each $a_{i,j} \in R$. Using the Taylor development of $e^{-(x^2+y^2)}$ we immediately see that each φ_i, $i = 1, 2, 3, 4$ is a pointwise limit of a sequence of such polynomials. Thus owing to Theorem 4 $\varphi_i(f, g) \in \mathscr{B}_N(\mathscr{F})$ for each f, $g \in \mathscr{B}_N(\mathscr{F})$ and $i = 1, 2, 3, 4$.

Clearly each $\varphi \in \Phi^0$ is of the form $\varphi = \sum_{i+j+k+l=1}^{n} a_{i,j,k,l} \cdot \varphi_1^i \cdot \varphi_2^j \cdot \varphi_3^k \cdot \varphi_4^l$ with $a_{i,j,k,l} \in R$. Hence $\varphi(f, g) \in \mathscr{B}_N(\mathscr{F})$ for each f, $g \in \mathscr{B}_N(\mathscr{F})$ and each $\varphi \in \Phi^0$ again by Theorem 4. Particularly $\varphi(f, g) \in \mathscr{B}_N(\mathscr{F})$ for each f, $g \in \mathscr{F}$ and each $\varphi \in \Phi^0$, what we wanted to show.

2)$\Rightarrow$3). We have to prove that $1 \wedge f^+ \in \mathscr{B}_N(\mathscr{F})$ for each $f \in \mathscr{F}$. Let $S = (0, +\infty)$ be equipped with the relative topology and consider on S the functions $\psi_1(x) = x \cdot e^{-x}$ and $\psi_2(x) = x^2 \cdot e^{-x}$. Then S is a separable locally compact metric space, $\psi_1 \wedge \psi_2 > 0$ on S and the functions ψ_1 and ψ_2 separate the points of S. Denote by Ψ the algebra of functions on S generated by ψ_1 and ψ_2. Then similarly as in 1)$\Rightarrow$2).

above we obtain that $\psi(f)\in\mathscr{B}_{\mathbf{N}}(\mathscr{F})$ for each $f\in\mathscr{B}_{\mathbf{N}}(\mathscr{F})$ and each continuous function $\psi: R\to R$ with $\psi(x)=0$ for $x\in(-\infty, 0\rangle$. Particularly $\psi(x) = 1\wedge(x\vee 0)$ is such a function, hence $1\wedge f^+\in\mathscr{B}_{\mathbf{N}}(\mathscr{F})$ for each $f\in\mathscr{F}$.

3)$\Rightarrow$4). $\mathscr{B}_{\mathbf{N}}(\mathscr{F})$ is an R — linear function lattice by Theorem 2, hence $\mathbf{B}_{\mathbf{N}}(\mathscr{F}) = \mathbf{B}(\mathscr{B}_{\mathbf{N}}(\mathscr{F}))$ is a σ-ring according to Corollary 2-2) of Theorem 2. Since the function $x\to 1\wedge(x\vee 0)$, $x\in R$, is continuous, $1\wedge\mathscr{B}_{\mathbf{N}}(\mathscr{F})^+ \subset \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ by Theorem 2. Since $\mathscr{B}_{\mathbf{N}}(\mathscr{F})$ is an R — linear function lattice, $f\in\mathscr{B}_{\mathbf{N}}(\mathscr{F}) \Leftrightarrow -f\in\mathscr{B}_{\mathbf{N}}(\mathscr{F})$. Thus by Theorem 8 it is enough to prove that each f^+, $f\in\mathscr{B}_{\mathbf{N}}(\mathscr{F})$, is $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ — measurable. Let $f\in\mathscr{B}_{\mathbf{N}}(\mathscr{F})$. Then f^+ will be $\mathscr{B}_{\mathbf{N}}(\mathscr{F})$ — measurable if and only if $\{t: t\in T, f^+(t)\geqq c\} \in \mathbf{B}_{\mathbf{N}}(\mathscr{F})$ for each $c\in(0, +\infty)$. Let $c\in(0, +\infty)$ and take a sequence $c_k\in(0, +\infty)-\{c\}$, $k=1, 2, \ldots$, so that $c_k\nearrow c$. Then $\{t: t\in T, f^+(t)\geqq c\} = \bigcap_{k=1}^{\infty} C_k$, where $C_k=\{t: t\in T, f^+(t)>c_k\}$. Obviously $1\wedge n(f^+-c_k\wedge f^+)\in\mathscr{B}_{\mathbf{N}}(\mathscr{F})^+$ for each n, $k=1, 2, \ldots$, and $1\wedge n(f^+-c_k\wedge f^+)\nearrow\chi_{C_k}$; hence $C_K\in\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ for each $k=1, 2, \ldots$. Since $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ is a σ-ring, $\bigcap_{k=1}^{\infty} C_k\in\mathbf{B}_{\mathbf{N}}(\mathscr{F})$, what we wanted to show.

4)$\Rightarrow$5). is the implication 1).$\Rightarrow$4). of Theorem 8.

5)$\Rightarrow$6). $\mathbf{B}_{\mathbf{N}}(\mathscr{F})=\mathbf{B}(\mathscr{B}_{\mathbf{N}}(\mathscr{F})) = \mathbf{B}(\mathscr{S}(\varrho(\mathbf{B}_{\mathbf{N}}(\mathscr{F})))) = \sigma(\mathbf{B}_{\mathbf{N}}(\mathscr{F}))$ by the Corollary of Theorem 9. The implication 4).$\Rightarrow$1). of Theorem 8 implies that each $f\in\mathscr{F}$ is $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ — measurable.

6)$\Rightarrow$1)...If $f, g\in\mathscr{S}(\mathbf{B}_{\mathbf{N}}(\mathscr{F}))$ and $a, b\in R$, then clearly $af+bg, f\cdot g\in\mathscr{S}(\mathbf{B}_{\mathbf{N}}(\mathscr{F}))$. Applying Theorem 2 we obtain that $af+bg, f\cdot g \in \mathscr{B}(\mathscr{S}\mathbf{B}_{\mathbf{N}}(\mathscr{F}))) \subset \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ for each $f, g\in\mathscr{B}(\mathscr{S}(\mathbf{B}_{\mathbf{N}}(\mathscr{F})))$ and each $a, b\in R$. Since each $f\in\mathscr{F}$ is $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ — measurable, $\mathscr{F}\subset\mathscr{B}(\mathscr{S}(\mathbf{B}_{\mathbf{N}}(\mathscr{F})))$ by implication 1).$\Rightarrow$4). of Theorem 8, hence 1). follows. The theorem is proved.

Corollary. *Let* $\mathbf{N}\subset 2^T$ *be a hereditary* σ*-ring, let* $\mathscr{F}\subset R^T$ *be an* R *— linear function lattice and let* $1\wedge\mathscr{F}^+\subset\mathscr{B}(\mathscr{F})$. *Then* $\mathscr{B}_{\mathbf{N}}(\mathscr{F}) = \mathscr{B}(\mathscr{S}(\mathbf{B}(\mathscr{F})\triangle\mathbf{N}))$ *and* $\mathbf{B}_{\mathbf{N}}(\mathscr{F}) = \mathbf{B}(\mathscr{F})\triangle\mathbf{N}$.

Proof. $\mathscr{B}(\mathscr{F}) = \mathscr{B}(\mathscr{S}(\mathbf{B}(\mathscr{F})))$ by the theorem, hence $\mathscr{B}_{\mathbf{N}}(\mathscr{F}) = \mathscr{B}_{\mathbf{N}}(\mathscr{B}(\mathscr{F})) = \mathscr{B}_{\mathbf{N}}(\mathscr{B}(\mathscr{S}(\mathbf{B})\mathscr{F})))) = \mathscr{B}_{\mathbf{N}}(\mathscr{S}(\mathbf{B}(\mathscr{F}))) = \mathscr{B}(\mathscr{S}(\mathbf{B}(\mathscr{F})\triangle\mathbf{N}))$ and $\mathbf{B}_{\mathbf{N}}(\mathscr{F}) = \mathbf{B}(\mathscr{F})\triangle\mathbf{N}$ by Theorem 13.

The following simple example shows that the condition $1\wedge\mathscr{F}^+ \subset \mathscr{B}(\mathscr{F})$ is in general weaker for R — linear function lattices than Stone's condition $1\wedge\mathscr{F}\subset\mathscr{F}$.

Example. Let $T=\langle 0, 1\rangle$ and let $\mathscr{F}=\{f: f\in R^T, f$ is continuous and there exist $c\in(0, 1)$ and $a\in R$ such that $f(t)=a\cdot t$ for $t\in\langle c, 1\rangle\}$. Then clearly $\mathscr{F}$ is an R — linear function lattice, $1\wedge\mathscr{F}\not\subset\mathscr{F}$ (consider the function $t\to 2t$), but $1\wedge\mathscr{F}^+ \subset \mathscr{B}(\mathscr{F})$.

§ 3. Applications to the Daniell integral

We suppose that the reader is familiar with the Daniell integral (for a clear exposition of it see [7]). $(T, \mathscr{F}, I)$ will be a given elementary integral and

$I^o: \mathscr{F}^o \to (-\infty, +\infty\rangle$ will be the corresponding extension of I to over functions, see sections 6-1 and 6-2 in [7].

For each $f \in R^{*T}$ we define its upper integral $\bar{I}(f)$ and its lower integral $\underline{I}(f)$ by the equalities

$$\bar{I}(f) = \inf\{I^o(h), h \in \mathscr{F}^o, h \geqq f\} \quad (\inf\{\emptyset\} = +\infty),$$

and

$$\underline{I}(f) = -\bar{I}(-f).$$

If $f, g \in R^{*T}$ and $\bar{I}(f) + \bar{I}(g)$ ($\underline{I}(f) + \underline{I}(g)$) is not of the form $(+\infty) + (-\infty)$ or $(-\infty) + (+\infty)$, then $\bar{I}(f+g) \leqq \bar{I}(f) + \bar{I}(g)$ ($\underline{I}(f+g) \geqq \underline{I}(f) + \underline{I}(g)$). Further $\underline{I}(f) \leqq \bar{I}(f)$ for each $f \in R^{*T}$.

The class $\mathbf{N}$ of all I — null sets is defined by the equality

$$\mathbf{N} = \{E : E \subset T, \bar{I}(\chi_E) = 0\}.$$

$\mathbf{N}$ is a hereditary σ-ring and $\mathscr{N}(\mathbf{N}) \subset \mathscr{N}^*(\mathbf{N}) \subset \bar{\mathscr{F}}$, since $\mathscr{F}$ is an R — linear function lattice.

Our concept of measurable functions and sets for the Daniell integral is given by the next

Definition 5. *Elements of $\mathscr{B}'_{\mathbf{N}}(\mathscr{F})$ are called R' — valued measurable functions and elements of $\mathbf{B}'_{\mathbf{N}}(\mathscr{F})$ are called R' — measurable sets.*

Corollaries 1 and 2 of Theorem 2 imply that $\mathbf{B}'_{\mathbf{N}}(\mathscr{F})$ is a σ-ring, that $\mathscr{B}_{\mathbf{N}}(\mathscr{F})$ is an R — linear function lattice and that $\mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$ is a lattice of functions. If the conditions of Theorem 18 are valid, then $\mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$ is an R^* — linear function lattice by the Corollary of Theorem 17.

Since $\mathscr{F}$ is an R — linear function lattice and since $\mathscr{N}(\mathbf{N}) \subset \mathscr{N}^*(\mathbf{N}) \subset \bar{\mathscr{F}}$, $\mathscr{B}_{\mathbf{N}}(\mathscr{F}) \subset \mathscr{B}^*_{\mathbf{N}}(\mathscr{F}) \subset \bar{\mathscr{F}}$ owing to Theorem 7.

The class $\mathscr{L}'$ of R' — valued summable or integrable functions is defined by the equality

$$\mathscr{L}' = \{f : f : T \to R', -\infty < \underline{I}(f) = \bar{I}(f) < +\infty\}.$$

If $f \in R^{*T}$ and $\underline{I}(f) = \bar{I}(f)$, then this common value is denoted by $I(f)$.

We easily have: $\mathscr{N}'(\mathbf{N}) = \{f : f \in \mathscr{L}', I(f) = 0\} = \{f : f \in R'^T, \bar{I}(|f|) = 0\}$, $\mathscr{L} = \mathscr{L} \dot{+} \mathscr{N}(\mathbf{N})$, $\mathscr{L}^* = \mathscr{L}^* \dot{+} \mathscr{N}^*(\mathbf{N}) = \mathscr{L} \dot{+} \mathscr{N}^*(\mathbf{N})$, and $\mathscr{L} = \mathscr{L}^* \cap R^T$. The last equality implies that $\{E : \chi_E \in \mathscr{L}\} = \{E : \chi_E \in \mathscr{L}^*\}$. Let $\mathbf{P} = \{E : \chi_E \in \mathscr{L}\}$, and for $E \in \mathbf{P}$ put $\mu(E) = I(\chi_E)$. Elements of $\mathbf{P}$ are called summable or integrable sets. The Lebesgue dominated convergence theorem, see [7, Theorem 6-3IV (c)], and the simple properties of I and $\mathscr{L}$ imply that $\mathbf{P}$ is a δ-ring and that $\mu : \mathbf{P} \to \langle 0, +\infty)$ is a complete countably additive measure. Clearly $\sigma(\mathbf{P}) \subset \mathbf{B}_{\mathbf{N}}(\mathscr{F})$.

It is well known, see [7, Theorems 6-4V and 6-4VI], that for each $f \in \mathscr{L}'$ there are $f_n \in \mathscr{F}$, $n = 1, 2, \ldots$, such that $f_n(t) \to f(t)$ a.e. $\mathbf{N}$. Thus $\mathscr{L}' \subset \mathscr{B}'_{\mathbf{N}}(\mathscr{F})$. Moreover we have

Theorem 19. $\mathscr{B}''_{\mathbf{N}}(\mathscr{F}) = \mathscr{B}''_{\mathbf{N}}(\mathscr{L}') = \mathscr{B}''(\mathscr{L}') = \mathscr{B}''(\mathscr{L})$.

Proof. Suppose $\mathscr{L}' = \mathscr{L}$. Since $\mathscr{L} = \mathscr{L} \dotplus \mathscr{N}(\mathbf{N})$, $\mathscr{B}''_{\mathbf{N}}(\mathscr{L}) = \mathscr{B}''(\mathscr{L})$ by Theorem 7-1). Since $\mathscr{F} \subset \mathscr{L} \subset \mathscr{B}''_{\mathbf{N}}(\mathscr{F})$, $\mathscr{B}''_{\mathbf{N}}(\mathscr{F}) = \mathscr{B}''(\mathscr{L})$. Let now $\mathscr{L}' = \mathscr{L}^*$. Since $\mathscr{L}^* = \mathscr{L} \dotplus \mathscr{N}^*(\mathbf{N}) \subset \mathscr{B}(\mathscr{L}) \dotplus \mathscr{B}^*(\mathscr{L}) \subset \mathscr{B}^*(\mathscr{L})$, $\mathscr{B}^*(\mathscr{L}^*) = \mathscr{B}^*(\mathscr{L})$ by the Corollary of Theorem 3. Since $[\mathscr{L}^* \dotplus \mathscr{N}^*(\mathbf{N})] \dotplus \mathscr{N}^*(\mathbf{N}) = \mathscr{L}^*$, $\mathscr{B}^*_{\mathbf{N}}(\mathscr{L}^*) = \mathscr{B}^*(\mathscr{L}^*)$ by Theorem 7-2). The remaining equality $\mathscr{B}^*_{\mathbf{N}}(\mathscr{F}) = \mathscr{B}^*_{\mathbf{N}}(\mathscr{L}^*)$ follows from the inclusions $\mathscr{F} \subset \mathscr{L}^* \subset \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$.

Theorem 20. *Let $f \in \mathscr{B}'_{\mathbf{N}}(\mathscr{F})$, let $g \in \mathscr{L}'$ and let $|f(t)| \leqq |g(t)|$ a.e. $\mathbf{N}$. Then $f \in \mathscr{L}'$.*

Proof. Put $\mathscr{B}' = \{f : f \in \mathscr{B}_{\mathbf{N}}'(\mathscr{F}), |f(t)| \leqq |g(t)| \text{ a.e. } \mathbf{N} \Rightarrow f \in \mathscr{L}'\}$. Then $\mathscr{F} \subset \mathscr{B}'$, and if $f_n \in \mathscr{B}'$, $n = 1, 2, \ldots$, and $f_n(t) \to f(t)$ a.e. $\mathbf{N}$, then $f \in \mathscr{B}'$ by the Lebesgue dominated convergence theorem, see [7, Theorem 6-3IV (c)]. In this way $\mathscr{B}' = \mathscr{B}'_{\mathbf{N}}(\mathscr{F})$.

Corollary 1. *$f \in \mathscr{L}'$ if and only if $f \in \mathscr{B}'_{\mathbf{N}}(\mathscr{F})$ and $|f| \in \mathscr{L}'$.*

Corollary 2. $\mathscr{L}' = \{f : f \in \mathscr{B}'_{\mathbf{N}}(\mathscr{F}), \bar{I}(|f|) < +\infty\}$.

Proof. If $f \in \mathscr{B}'_{\mathbf{N}}(\mathscr{F})$ and $\bar{I}(|f|) < +\infty$, then there is an $h \in F^{o+}$ such that $|f| \leqq h$ and $I^o(h) < +\infty$. But then $h \in \mathscr{L}^*$, see [7, Theorem 6-3I (b)], hence $f \in \mathscr{L}'$ by Corollary 1 above.

Corollary 3. $\mathbf{P} = \{E : E \in \mathbf{B}_{\mathbf{N}}(\mathscr{F}), \bar{I}(\chi_E) < +\infty\} = \{E : E \in \mathbf{B}^*_{\mathbf{N}}(\mathscr{F}), \bar{I}(\chi_E) < +\infty\}$.

Theorem 21. *Let $f \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})^+$. Then $\underline{I}(f) = \bar{I}(f)$.*

Proof. If $\bar{I}(f) < +\infty$, then $\underline{I}(f) = \bar{I}(f)$ by the preceding Corollary 2. Suppose that $\bar{I}(f) = +\infty$. Since $f \in \tilde{\mathscr{F}}^+$, there are $h_n \in \mathscr{F}^+$, $n = 1, 2, \ldots$, such that $h_n \nearrow h \geqq f$. According to the preceding Corollary 2 $h_n \wedge f \in \mathscr{L}^+$ for each n. But then $\lim I(h_n \wedge f) = +\infty$, because otherwise $f \in \mathscr{L}^+$ by the monotone convergence theorem, see [7, Theorem 6-3III]. Since $\underline{I}(f) \geqq \underline{I}(h_n \wedge f) = I(h_n \wedge f)$ for each n, $\underline{I}(f) = +\infty$.

Corollary 1. *Let $f \in \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$ and let $I(f^+) - I(f^-)$ be not of the form $(+\infty) - (+\infty)$. Then $\underline{I}(f) = \bar{I}(f) = I(f^+) - I(f^-)$.*

Proof. $I(f^+) - I(f^-) = I(f^+) + \underline{I}(-f^-) \leqq \underline{I}(f) \leqq \bar{I}(f) \leqq I(f^+) + \bar{I}(-f^-) = I(f^+) - I(f^-)$ under the assumptions of the corollary.

Corollary 2. *Let $f \in \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$, let $I(f^+) - I(f^-)$ be not of the form $(+\infty) - (+\infty)$ and let $g \in \mathscr{L}^*$. Then $\underline{I}(f+g) = \bar{I}(f+g) = I(f) + I(g)$.*

Proof. $\underline{I}(f) = \bar{I}(f)$ by the preceding corollary, hence $I(f) + I(g) \leqq \underline{I}(f+g) \leqq \bar{I}(f+g) \leqq I(f) + I(g)$.

Theorem 22. *Let $f_n \in \mathscr{B}'_{\mathbf{N}}(\mathscr{F})$, $n = 1, 2, \ldots$, let $g \in \mathscr{L}'$ and let $g(t) \leqq f_n(t)$ a.e. $\mathbf{N}$ for each n. Then $\underline{I}(f_n) = \bar{I}(f_n)$ for each n, and $I(\liminf_n f_n) \leqq \liminf_n I(f_n)$. If moreover, $f_n \nearrow f \in R'^T$ a.e. $\mathbf{N}$, then $f \in \mathscr{B}'_{\mathbf{N}}(\mathscr{F})$ and $I(f_n) \nearrow I(f)$.*

Proof. Since $0 \leqq f_n(t) - g(t)$ a.e. $\mathbf{N}$, and $f_n - g \in \mathscr{B}'_{\mathbf{N}}(\mathscr{F})$, $\underline{I}(f_n - g + g) = \bar{I}(f_n - g + g)$ by Corollary 2 of Theorem 21. For the same reason $\underline{I}(\liminf_n f_n) = \bar{I}(\liminf_n f_n)$. The inequality $I(\liminf_n f_n) \leqq \liminf_n I(f_n)$ is a direct consequence of Fatou's lemma, see [7, Theorem 6-3IV (b)]. The final assertion now trivially follows from the monotonicity of I.

Corollary 1. *Let $f_n \in \mathscr{B}'_{\mathbf{N}}(\mathscr{F})^+$, $n = 1, 2, \ldots$. Then $I\left(\sum_{n=1}^{\infty} f_n\right) = \sum_{n=1}^{\infty} I(f_n)$.*

Corollary 2. *For $E \in \mathbf{B}'_{\mathbf{N}}(\mathscr{F})$ put $\mu'(E) = I(\chi_E)$. Then $\mu' : \mathbf{B}'_{\mathbf{N}}(\mathscr{F}) \to \langle 0, +\infty\rangle$ is a complete countably additive measure.*

Theorem 23. *Let $f \in R^{*T}$ be $\mathbf{B}'_{\mathbf{N}}(\mathscr{F})$ — measurable and let $\int_T f^+ \mathrm{d}\mu' - \int_T f^- \mathrm{d}\mu'$ be not of the form $(+\infty) - (+\infty)$. Then $\underline{I}(f) = \bar{I}(f) = \int_T f \, \mathrm{d}\mu'$.*

Proof. According to Theorem 21 and its Corollary 1 it is enough to prove the theorem for $f^+ \cdot f^+ \in \mathscr{S}(\mathbf{B}'_{\mathbf{N}}(\mathscr{F}))^{\sigma+}$ by Theorem 8, hence there are $f_n \in \mathscr{S}(\mathbf{B}'_{\mathbf{N}}(\mathscr{F}))$, $n = 1, 2, \ldots$, such that $f_n \nearrow f$. But then $I(f_n) \nearrow I(f)$ by Theorem 22, hence the monotone convergence theorem implies that $\int_T f_n \, \mathrm{d}\mu' \nearrow \int_T f \, \mathrm{d}\mu'$. Since $I(f_n) = \int_T f_n \, \mathrm{d}\mu'$ for each n, the theorem is proved.

Our basic result is the following

Theorem 24. *The following conditions are equivalent:*

1) $f \cdot g \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ *for each* $f, g \in \mathscr{F}$,
2) $f^{+n} \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ *for each* $f \in \mathscr{F}$ *and each* $n = 2, 3, \ldots$,
3) $1 \wedge f^+ \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ *for each* $f \in \mathscr{F}$,
4) $1 \wedge f^+ \in \mathscr{L}$ *for each* $f \in \mathscr{L}$,
5) $1 \wedge f^+ \in \mathscr{L}$ *for each* $f \in \mathscr{F}$,
6) *each* $f \in \mathscr{F}$ *is* $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ — *measurable,*
7*) *each* $f \in \mathscr{L}^*$ *is* $\sigma(\mathbf{P})$ — *measurable,*
7) *each* $f \in \mathscr{L}$ *is* $\sigma(\mathbf{P})$ — *measurable,*
8*) $\mathscr{L}^* = \mathscr{L}^*(T, \sigma(\mathbf{P}), \mu)$,
8) $\mathscr{L} = \mathscr{L}(T, \sigma(\mathbf{P}), \mu)$,
9*) $\mathscr{B}^*_{\mathbf{N}}(\mathscr{F}) = \mathscr{B}^*(\mathscr{S}(\mathbf{P}))$,
9) $\mathscr{B}_{\mathbf{N}}(\mathscr{F}) = \mathscr{B}(\mathscr{S}(\mathbf{P}))$,

and each of them implies that $\mathbf{B}^*_{\mathbf{N}}(\mathscr{F}) = \mathbf{B}_{\mathbf{N}}(\mathscr{F}) = \sigma(\mathbf{P})$ and that $I(f) = \int_T f \, \mathrm{d}\mu$ for each $f \in \mathscr{L}^*$.

Proof. 1)$\Rightarrow$2)$\Rightarrow$3) by Theorem 18.

3)$\Rightarrow$4) Since $x \to 1 \wedge (x \vee 0)$ is a continuous function from R to R, $1 \wedge f^+ \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ for each $f \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ by Theorem 2. But $\mathscr{L} \subset \mathscr{B}_{\mathbf{N}}(\mathscr{F})$, hence $1 \wedge f^+ \in \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ for each $f \in \mathscr{L}$. Since $1 \wedge f^+ \leqq f^+$, $1 \wedge f^+ \in \mathscr{L}$ for each $f \in \mathscr{L}$ by Theorem 20.

4)$\Rightarrow$5), since $\mathscr{F} \subset \mathscr{L}$.

5)$\Rightarrow$3), since $\mathscr{L} \subset \mathscr{B}_{\mathbf{N}}(\mathscr{F})$.

3)$\Rightarrow$6) by Theorem 18.

6)$\Rightarrow$7*) By Theorem 8 each $f \in \mathscr{B}_{\mathbf{N}}(\mathscr{F}) \supset \mathscr{L}$ is $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ — measurable. Since $\mathscr{L}^* = \mathscr{L} \dot{+} \mathscr{N}^*(\mathbf{N})$, each $f \in \mathscr{L}^*$ is also $\mathbf{B}_{\mathbf{N}}(\mathscr{F})$ — measurable. Let $f \in \mathscr{L}^*$. According to Theorem 8 take $f_n \in \mathscr{S}(\mathbf{B}_{\mathbf{N}}(\mathscr{F}))^+$, $n = 1, 2, \ldots$, so that $f_n \nearrow f^+$. Since $0 \leqq I(f_n) \leqq I(f^+) < +\infty$, $f_n \in \mathscr{S}(\mathbf{P})$ for each n by Corollary 3 of Theorem 20. Thus f^+ is $\sigma(\mathbf{P})$ — measurable. Similarly f^- is $\sigma(\mathbf{P})$ — measurable, hence f is also.

7*)$\Rightarrow$7) trivially.

7)$\Rightarrow$8*). $\mathscr{L}^*(T, \sigma(\mathbf{P}), \mu) \subset \mathscr{L}^*$ by Theorem 23. Let $f \in \mathscr{L}^*$. Since $\mathscr{L}^* = \mathscr{L} \dot{+} \mathscr{N}^*(\mathbf{N}))$ and since each $u \in \mathscr{N}^*(\mathbf{N})$ is $\sigma(\mathbf{P})$ — measurable, f is $\sigma(\mathbf{P})$ — measurable. Applying Theorem 23 to f^+ and f we have $f \in \mathscr{L}^*(T, \sigma(\mathbf{P}), \mu)$.

8*)$\Rightarrow$8) trivially.

8)$\Rightarrow$9*). Since each $f \in \mathscr{L}$ is $\sigma(\mathbf{P})$ — measurable, $\mathscr{S}(\mathbf{P}) \subset \mathscr{L} \subset \mathscr{B}(\mathscr{S}(\mathbf{P}))$ by Corollary 5 of Theorem 8. Hence $\mathscr{B}^*(\mathscr{S}(\mathbf{P})) = \mathscr{B}^*(\mathscr{L})$. But $\mathscr{B}^*(\mathscr{L}) = \mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$ by Theorem 19.

9*)$\Rightarrow$9) trivially.

9)$\Rightarrow$1). Clearly $f \cdot g \in \mathscr{S}(\mathbf{P})$ for each $f, g \in \mathscr{S}(\mathbf{P})$. Since $(x, y) \to x \cdot y$ is a separately continuous function from R^2 to R, $f \cdot g \in \mathscr{B}(\mathscr{S}(\mathbf{P})) = \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ for each $f, g \in \mathscr{B}(\mathscr{S}(\mathbf{P})) = \mathscr{B}_{\mathbf{N}}(\mathscr{F})$ by Theorem 2.

9) and Corollary 4 of Theorem 8 give the equality $\mathbf{B}^*_{\mathbf{N}}(\mathscr{F}) = \mathbf{B}_{\mathbf{N}}(\mathscr{F}) = \sigma(\mathbf{P})$. 8*) and Theorem 23 give the equality $I(f) = \int_T f \, d\mu$ for each $f \in \mathscr{L}^*$. The theorem is proved.

Addendum

In [8] the class of measurable functions Λ is defined by 9.20: $\Lambda = \{f: T \to R^*$: there are $f_n \in \mathscr{L}$, $n = 1, 2, \ldots$ such that $f_n \to f\}$, and using limit theorems for the integral it is shown that 9.42 (g): $f_n \in \Lambda$, $n = 1, 2, \ldots$ and $f_n(t) \to f(t)$ a.e. $N \Rightarrow f \in \Lambda$, and that 9 D: $f \in \Lambda$ if and only if there are $f_n \in \mathscr{F}$, $n = 1, 2, \ldots$ such that $f_n(t) \to f(t)$ a.e. N.

From 9.D and 9.42 (g) it is evident that our class of R^* — measurable functions $\mathscr{B}_{\mathbf{N}}(\mathscr{F})$ equals the class of measurable functions Λ introduced by J. Lukeš. Hence 9.D and 9.20 give further valuable information about our common class of measurable functions.

For some further conditions equivalent to those given in Theorem 24 see [9].

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ОБ ОДНОЙ КОНЦЕПЦИИ ИЗМЕРИМОСТИ ДЛЯ ИНТЕГРАЛА ДАНИЭЛЯ

Иван Добраков

Резюме

В работе определена и исследована новая концепция измеримости для схемы интегрирования Даниэля. Эта концепция оказывается более естественной и эффективной чем обще принятая концепция Стоуна.

Пусть $\mathscr{F}$ некоторая векторная решетка вещественных функций на непустом множестве T, и пусть $\mathbf{N} \subset 2^T$. Тогда класс $\mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$ так называемых $R^* = \langle -\infty, +\infty \rangle$ — значных $(\mathscr{F}, \mathbf{N})$ — измеримых функций определяется как наименьший класс R^* — значных функций на T, содержащий $\mathscr{F}$, и замкнутый относительно образования точечных пределов $\mathbf{N}$ — почти всюду своих последовательностей. Подмножество $E \subset T$ называется $R^* - (\mathscr{F}, \mathbf{N})$ — измеримым, если его характеристическая функция принадлежит $\mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$. Класс всех $R^* - (\mathscr{F}, \mathbf{N})$ — измеримых подмножеств T обозначим $\mathbf{B}^*_{\mathbf{N}}(\mathscr{F})$. В случае, когда $\mathbf{N}$ пусто и $\mathscr{F}$ векторная решетка $\mathbf{S}$ — ступенчатых функций, где $\mathbf{S}$ сигма кольцо подмножеств T, то $\mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$ совпадает с обычным классом всех $\mathbf{S}$ — измеримых R^* — значных функций на T, и $\mathbf{B}^*_{\mathbf{N}}(\mathscr{F}) = S$. Когда $\mathbf{N}$ пусто $i\ \mathscr{F} = C_0(T)$, где T локально компактное хаусдорфово пространство, то мы получаем обычную измеримость в смысле Бэра.

В основной теореме 18 дается пять эквивалентных условий необходимых и достаточных для того, чтобы класс $\mathbf{B}^*_{\mathbf{N}}(\mathscr{F})$ — ступенчатых функций порождал весь класс $\mathscr{B}^*_{\mathbf{N}}(\mathscr{F})$.

В § 3 эти результаты применяются к интегралу Даниэля, и в теореме 24 установлен ряд новых необходимых и достаточных условий для возможности представления интеграла Даниэля как интеграла по индуцированной мере.