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ON THE ASYMPTOTIC BEHAVIOUR OF A SOLUTION OF A DIFFERENTIAL EQUATION IN A HILBERT SPACE

IGOR BOCK

1. Introduction

We shall be dealing with the initial value problem

$$A_0(t)\frac{d^m u}{dt^m}+\ldots+A_{m-1}(t)\frac{du}{dt}+A_m(t)u=f(t) \tag{1.1}$$

$$\left.\frac{d^r u}{dt^r}\right|_{t=0}=u_r, \qquad r=0, 1, \ldots, m-1 \tag{1.2}$$

with the abstract functions $u:(R^+\to X)$, $f:(R^+\to X^*)$, the operator functions $A_r(.):(R^+\to L(X, X^*))$ and the elements $u_r\in X$, where $R^+=[0, \infty)$, X is a Hilbert space, X^* is a dual space to X and $L(X, X^*)$ is a space of all linear bounded operators mapping X into X^*.

We shall analyse the behaviour of a solution of (1.1), (1.2) for $t\to\infty$. Due to the results obtained in this paper the solution behaves in the same way as the deflection of a viscoelastic plate made of aging material. These results generalize the results of paper [1], where the problem (1.1), (1.2) with the stationary operator functions $A_r(t)\equiv A_r$ was considered.

First we shall introduce some results from the theory of differential equations in a Banach space proved in [3].

Let X be a complex Banach space, $R^+=[0, \infty)$. We denote by $C(R^+, X)$ the space of all continuous functions mapping R^+ into X and by $C^{(m)}(R^+, X)$ the space of all m-times continuously differentiable functions mapping R^+ into X.

Consider the initial value problem for the differential equation in the space X

$$\frac{du}{dt}=A(t)u+f(t), \qquad t\in R^+ \tag{1.3}$$

$$u(0)=u_0 \tag{1.4}$$

Theorem 1.1 ([3], III. 1.2). *Let* $f \in C(R^+, X)$, $A(\cdot) \in C(R^+, L(X, X))$, $u_0 \in X$. *Then there exists a unique solution* $u \in C^{(1)}(R^+, X)$ *of the problem* (1.3), (1.4).

A solution u of the problem (1.3), (1.4) can be expressed with the help of a solution $U \in C^{(1)}(R^+, L(X, X))$ of the homogeneous operator differential equation in the space $L(X, X)$

$$\frac{dU}{dt} = A(t)U \tag{1.5}$$

$$U(0) = I \quad \text{(the identical operator)} \tag{1.6}$$

There exists for each $t \in R^+$ the inverse operator $U^{-1}(t)$. The operator function $V(.)$ is a solution of the problem

$$\frac{dV}{dt} = -VA(t) \tag{1.7}$$

$$V(0) = I \tag{1.8}$$

A solution u of (1.3), (1.4) can be expressed in the form

$$u(t) = U(t)u_0 + \int_0^t U(t, \tau)f(\tau)\, d\tau, \tag{1.9}$$

where

$$U(t, \tau) = U(t)U^{-1}(\tau). \tag{1.10}$$

The following theorem plays an important role in our further considerations of the asymptotic behaviour of a solution of the problem (1.1), (1.2).

Theorem 1.2 ([3], III. 6.3). *Let* $A(.) \in C(R^+, L(X, X))$, $A_\infty \in L(X, X)$, $\lim_{t \to \infty} \|A(t) - A_\infty\| = 0$. $\operatorname{Re} \lambda < -v_0 < 0$ *for all* $\lambda \in \sigma(A_\infty)$, *where* $\sigma(A_\infty)$ *is the spectrum of the operator* A_∞, $\|\cdot\|$ *is the norm in the space* $L(X, X)$.

Then there exist such constants $v > 0$, N *depending only on* $A(t)$ *that*

$$\|U(t, \tau)\| \leq N e^{-v(t-\tau)}, \qquad \forall t \geq \tau, \forall \tau \in R^+ \tag{1.11}$$

2.The existence and the uniqueness of a solution

Let X be a complex Hilbert space with the scalar product $(.,.)$ and the norm $\|\cdot\|$ and X^* with the norm $\|\cdot\|_*$ the antidual space of all linear bounded functionals over X.

We formulate a theorem of the existence and the uniqueness of a solution of the problem (1.1), (1.2)

Theorem 2.1. *Let* $f\in C(R^+, X^*)$, $A_i(.)\in C(R^+, L(X, X^*))$, $i=0, 1, \ldots, m$; $u_r\in X$, $r=0, 1, \ldots, m-1$. *If there exists such a real positive and continuous on* R^+ *function* $\alpha(t)$ *that*

$$\alpha(t)\|x\|^2\leqslant|\langle A_0(t)x, x\rangle, \qquad \forall x\in X, t\in R^+, \tag{2.1}$$

then there exits a unique solution $u\in C^{(m)}(R^+, X)$ *of the problem* (1.1), (1.2).

Proof. Due to (2.1) the operators $A(t)$ and $A(t)^*$ (the adjoint operator to $A(t)$) satisfy the inequalities

$$\begin{aligned}\alpha(t)\|x\|&\leqslant \|A(t)x\|_*\\ \alpha(t)\|x\|&\leqslant \|A(t)^*x\|_*, \qquad \forall x\in X, t\in R^+\end{aligned} \tag{2.2}$$

Using (2.2) and the theorem on the solvalibility of the operator equations ([6], VII. 5) we obtain that there exists the inverse operator $A_0^{-1}(t)\in L(X^*, X)$ satisfying

$$\|A_0^{-1}(t)\|_{L(X^*, X)}\leqslant\alpha(t)^{-1}, \qquad \forall t\in R^+, \tag{2.3}$$

where the function $\alpha(t)^{-1}$ is continuous on R^+. Using the relation

$$A_0^{-1}(t)-A_0^{-1}(t_0)=A_0^{-1}(t_0)\,(A_0(t_0)-A_0(t))A_0^{-1}(t)$$

we can verify easily that the operator-function $A_0^{-1}(.)$ is continuous in each point $t_0\in R^+$ and hence

$$A_0^{-1}(.)\in C(R^+, L(X^*, X)). \tag{2.4}$$

Consider the initial value problem in the Hilbert product space $\chi=[X]^m$

$$\frac{d\boldsymbol{u}}{dt}=\mathscr{A}(t)\boldsymbol{u}+\boldsymbol{F}(t)$$

$$\boldsymbol{u}(0)=\boldsymbol{u}_0 \tag{2.6}$$

with the operator function $\mathscr{A}(.):(R^+\to L(\chi, \chi))$, the function $\boldsymbol{F}(.):(R^+\to\chi)$ and the element $\boldsymbol{u}_0\in\chi$ defined by

$$\mathscr{A}(t)=\begin{pmatrix} 0, & I, & 0, & \ldots, & 0\\ 0, & 0, & I, & \ldots, & 0\\ \cdots & \cdots & \cdots & \cdots & \cdots\\ 0, & 0, & 0, & \ldots, & I\\ -A_0^{-1}(t)\,A_m(t), & & \ldots, & & -A_0^{-1}(t)\,A_1(t)\end{pmatrix} \tag{2.7}$$

$$\boldsymbol{F}(t)=(0, 0, \ldots, A_0^{-1}(t)f(t))^T, \tag{2.8}$$

$$\boldsymbol{u}_0=(u_0, u_1, \ldots, u_{m-1})^T \tag{2.9}$$

Using (2.4) we obtain $\mathscr{A}(.)\in C(R^+, L(\chi, \chi,))$, $\boldsymbol{F}(.)\in C(R^+, \chi)$. There exists, due to

Theorem 1.1, a unique solution $\boldsymbol{u} \in C^1(R^+, \chi)$ of (2.5), (2.6) which has the form

$$(2.10) \qquad \boldsymbol{u}(t) = (u(t), u'(t), \ldots, u^{(m-1)}(t))^T.$$

The function $u \in C^{(m)}(R^+, X)$ is then a unique solution of the problem (1.1), (1.2).

3. On the asymptotic behaviour of a solution

Using the result of Theorem 1.2 we shall investigate the asymptotic behaviour of a solution of the problem (1.1), (1.2).

Theorem 3.1. *Assume that the assumptions of Theorem* 2.1 *are fulfilled. Assume, moreover, that there exist such a constant* $\alpha_0 > 0$ *and the operators* $A_i \in L(X, X^*)$, $i = 0, 1, \ldots, m$, *that*

$$(3.1) \qquad \alpha_0 \|x\|^2 \leq |\langle A_0(t)x, x\rangle|, \qquad \forall x \in X, t \in R^+,$$

$$(3.2) \qquad \lim_{t \to \infty} \|A_i(t) - A_{i,\infty}\| = 0, \qquad i = 0, 1, \ldots, m$$

and the polynomial operator

$$(3.3) \qquad D(\lambda) = \lambda^m A_{0,\infty} + \ldots + \lambda A_{m-1,\infty} + A_{m,\infty}; \qquad \lambda \in C$$

possesses the inverse operator $D(\lambda)^{-1} \in L(X^*, X)$ *for all* $\lambda \in C$ *with* $\operatorname{Re} \lambda \geq 0$.
Then the estimate

$$(3.4) \qquad \sum_{i=0}^{m-1} \|u^{(i)}(t)\| \leq Me^{-vt} \Big(\sum_{i=0}^{m-1} \|u_i\| + \int_0^t e^{v\tau} \|f(\tau)\|_* \, d\tau \Big)$$

of a solution $u \in C^{(m)}(R^+, X)$ *of* (1.1), (1.2) *holds with the constants* $M, v > 0$ *depending only on* $A_i(t)$, $i = 0, 1, \ldots, m$.
If there exists such a functional $f_\infty \in X^*$, *that*

$$(3.5) \qquad \lim_{t \to \infty} \|f(t) - f_\infty\|_* = 0,$$

then

$$(3.6) \qquad \lim_{t \to \infty} \|(\|u(t) - A_{m,\infty}^{-1} f_\infty\|_* + \sum_{i=0}^{m-1} \|u^{(i)}(t)\|) = 0$$

Proof. Consider the problem (1.1), (1.2) as the problem (2.5), (2.6) in the space $\chi = [X]^m$. Using (3.1), (3.2) we can see that there exists the inverse operator $A_{0,\infty}^{-1} \in L(X^*, X)$ satisfying the relation

$$(3.7) \qquad \|A_{0,\infty}^{-1}\| \leq \alpha_0^{-1}$$

Using the relations (3.1), (3.7) we obtain

$$(3.8) \qquad \|A_0^{-1}(t) - A_{0,\infty}^{-1}\| = \|A_{0,\infty}^{-1}(A_{0,\infty} - A_0(t))A_0^{-1}(t)\| \leq$$
$$\leq \alpha_0^{-2}\|A_{0,\infty} - A_0(t)\|, \qquad \forall t \in R^+$$

and combining with (3.2) we arrive at

$$(3.9) \qquad \lim_{t\to\infty} \|A_0^{-1}(t) - A_{0,\infty}^{-1}\| = 0$$

Let us define the operator $\mathscr{A}_\infty \in L\mathscr{X}, \mathscr{X})$ by

$$(3.10) \qquad \mathscr{A}_\infty = \begin{pmatrix} 0, & I, & 0, & \dots, & 0 \\ 0, & 0, & I, & \dots, & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0, & 0, & 0, & \dots, & I \\ -A_{0,\infty}^{-1}A_{m,\infty}, & -A_{0,\infty}^{-1}A_{m-1,\infty}, & \dots, & & -A_{0,\infty}^{-1}A_{1,\infty} \end{pmatrix}$$

Combining (2.7), (3.2), (3.8), (3.9) we obtain

$$(3.11) \qquad \lim_{t\to\infty} \|\mathscr{A}(t) - \mathscr{A}_\infty\|_{L(X,X)} = 0.$$

We apply now the results of Theorem 1.2. We must therefore find such a number $v_0 > 0$ that

$$(3.12) \qquad \operatorname{Re}\lambda < -v_0, \qquad \forall\lambda \in \sigma(\mathscr{A}_\infty).$$

It can be verified easily that $\lambda \in \sigma(\mathscr{A}_\infty)$ if and only if $0 \in \sigma(D(\lambda))$, which means that there does not exist the inverse operator $D(\lambda)^{-1}$. Using the assumption (3.3) we obtain that

$$(3.13) \qquad \operatorname{Re}\lambda < 0, \qquad \forall\lambda \in \sigma(\mathscr{A}_\infty)$$

The set $\sigma(\mathscr{A}_\infty)$ is closed in the complex plane ([6], VIII. 2). Then there must exist such $v_0 > 0$ that (3.12) holds. Otherwise there exists such a sequence $\lambda_n \in \sigma(\mathscr{A}_\infty)$ that $\lim_{t\to\infty} \lambda_n = \lambda_0$, $\operatorname{Re}\lambda_0 = 0$, $\lambda_0 \in \sigma(\mathscr{A}_\infty)$, which is in contradiction to (3.13).

We can now use Theorem 1.2. Combining (1.9), (1.11), (2.5), (2.6) we obtain

$$(3.14) \qquad \|u(t)\| \leq Me^{-vt}(\|u_0\| + \int_0^t e^{v\tau}\|F(\tau)\|\,d\tau), \qquad \forall t \in R^+$$

Using (2.8), (2.9), (2.10), (3.1) we obtain the estimate (3.4) with the constants M, $v > 0$ depending only on $A_i(t)$, $i = 0, 1, \dots, m$.

It remains to verify the second part of the theorem. Let f_∞ be such a functional from X^* that (3.5) holds. We express a solution u of the problem (1.1), (1.2) in the form

$$u(t)=v(t)+A_{m,\infty}^{-1}f_\infty. \tag{3.15}$$

The operator $A_{m,\infty}^{-1}\in L(X^*,X)$ exists, because $D(0)=A_{m,\infty}$. A function $v\in C^{(m)}(R^+,X)$ is a solution of the initial value problem

$$\sum_{i=0}^{m}A_i(t)\frac{\mathrm{d}^{m-i}v}{\mathrm{d}t^{m-i}}=g(t)$$
(3.16)
$$\left.\frac{\mathrm{d}^i v}{\mathrm{d}t^i}\right|_{t=0}=v_i,\qquad i=0,1,\ldots,m-1$$

with $v_i\in X$ and

$$g(t)=f(t)-A_m(t)A_{m,\infty}^{-1}f_\infty. \tag{3.17}$$

Due to the first part of the theorem a function v satisfies

$$\sum_{i=0}^{m-1}\|v^{(i)}(t)\|\le Me^{-\nu t}\Big(\sum_{i=0}^{m-1}\|v_i\|+\int_0^t e^{\nu\tau}\|g(\tau)\|_*\,\mathrm{d}\tau\Big). \tag{3.18}$$

The relations (3.2), (3.5), (3.17) imply

$$\lim_{t\to\infty}\|g(t)\|_*=0. \tag{3.19}$$

If

$$\lim_{t\to\infty}\sum_{i=0}^{m-1}\|v^{(i)}(t)\|=0, \tag{3.20}$$

then the conclusion of the theorem follows from (3.15). Considering (3.18) we see that it suffices to verify

$$\lim_{t\to\infty}e^{-\nu t}\int_0^t e^{\nu\tau}\|g(\tau)\|_*\,\mathrm{d}\tau=0. \tag{3.21}$$

If $\int_0^\infty e^{\nu\tau}\|g(\tau)\|_*\,\mathrm{d}\tau<\infty$, then (3.21) follows immediatly. If $\lim_{t\to\infty}\int_0^t e^{\nu\tau}\|g(\tau)\|_*\,\mathrm{d}\tau=\infty$, then (3.21) follows from (3.19) after using the L'Hospital rule and the proof is complete.

There arise difficulties with verifying the assumption about the operator $D(\lambda)$ by applying Theorem 3.1. The following corollaries show that under some conditions the polynomial operator $D(\lambda)$ defined in (3.3) satisfies the assumption of Theorem 3.3. We shall be dealing with the problem of the first and the second order.

Corollary 3.1. $(m=1)$. *Assume that the operators* $A_{i,\infty} \in L(X, X^*)$, $i=0, 1$ *satisfy the assumptions*

$$\langle A_{0,\infty}x, y\rangle = \langle A_{0\ \infty}y, x\rangle, \qquad \forall x, y \in X, \tag{3.22}$$

$$0 \leqslant \langle A_{0,\infty}x, x\rangle, \qquad \forall x \in X, \tag{3.23}$$

$$\alpha_1\|x\|^2 \leqslant \operatorname{Re}\langle A_{1,\infty}x, x\rangle, \qquad \alpha_1 > 0, \forall x \in X. \tag{3.24}$$

Then the operator

$$D(\lambda) = \lambda A_{0,\infty} + A_{1,\infty} \tag{3.25}$$

possesses the inverse operator $D(\lambda)^{-1}$ *for all* $\lambda \in C$ *with* $\operatorname{Re}\lambda \geqslant 0$.

Proof. Using (3.22) we obtain

$$\operatorname{Re}\langle D(\lambda)x, x\rangle = \operatorname{Re}\lambda\langle A_{0,\infty}x, x\rangle + \operatorname{Re}\langle A_{1,\infty}x, x\rangle. \tag{3.26}$$

Considering (3.23), (3.24) we arrive at

$$\operatorname{Re}\langle D(\lambda)x, x\rangle \geqslant \alpha_1\|x\|^2, \qquad \lambda \in C, \operatorname{Re}\lambda \geqslant 0, \forall x \in X. \tag{3.27}$$

The last inequality implies the existence of the inverse operator $D(\lambda)^{-1}$ for all $\lambda \in C$ with $\operatorname{Re}\lambda \geqslant 0$ and the proof is complete.

Corollary 3.2. $(m=2)$. *Assume that the operators* $A_{i,\infty} \in L(X, X^*)$, $i=0, 1, 2$ *satisfy the next assumptions*

$$\langle A_{j,\infty}x, y\rangle = \langle A_{j,\infty}y, x\rangle, \qquad j=0, 2, \forall x, y \in X, \tag{3.28}$$

$$0 \leqslant \langle A_{0,\infty}x, x\rangle, \qquad \forall x \in X, \tag{3.29}$$

$$\alpha_1\|x\|^2 \leqslant \operatorname{Re}\langle A_{1,\infty}x, x\rangle, \qquad \alpha_1 > 0, \forall x \in X, \tag{3.30}$$

$$\alpha_2\|x\|^2 \leqslant \langle A_{2,\infty}x, x\rangle, \qquad \alpha_2 > 0, \forall x \in X. \tag{3.31}$$

Then the operator

$$D(\lambda) = \lambda^2 A_{0,\infty} + \lambda A_{1,\infty} + A_{2,\infty} \tag{3.32}$$

possesses the inverse operator $D(\lambda)^{-1}$ *for all* $\lambda \in C$ *with* $\operatorname{Re}\lambda \geqslant 0$.

Proof. Assume first that $\lambda = 0$. Then $D(\lambda) = D(0) = A_{2,\infty}$. There exists due to (3.31) the inverse operator $A_{2,\infty}^{-1} = D(0)^{-1}$.

Let $\lambda \neq 0$, $\operatorname{Re}\lambda \geqslant 0$. Consider the operator $T(\lambda) = \lambda^{-1}D(\lambda)$. $T(\lambda)$ can be expressed in the form

$$T(\lambda) = \lambda A_{0,\infty} + A_{1,\infty} + \frac{\bar{\lambda}}{|\lambda|^2}A_{2,\infty}, \qquad \lambda \neq 0$$

With the help of (3.28) we obtain

(3.34) $$\operatorname{Re} \langle T(\lambda)x, x\rangle = \operatorname{Re} \lambda \langle A_{0,\infty}x, x\rangle + \operatorname{Re} \langle A_{1,\infty}x, x\rangle + \frac{\operatorname{Re} \lambda}{|\lambda|^2} \langle A_{2,\infty}x, x\rangle, \qquad \lambda \neq 0, x \in X$$

Using (3.29), (3.30) we obtain the inequality

(3.35) $$\operatorname{Re} \langle T(\lambda)x, x\rangle \geqslant \alpha_1 \|x\|^2, \qquad \forall x \in X, \lambda \in C, \operatorname{Re} \lambda \geqslant 0, \lambda \neq 0,$$

which implies the existence of the operator $T(\lambda)^{-1}$. Then, however, there exists the inverse operator $D(\lambda)^{-1} = \lambda^{-1}T(\lambda)^{-1}$ for all $\lambda \neq 0$ with $\operatorname{Re} \lambda \geqslant 0$ and the proof is complete.

Remark 3.1. The previous results can be applied to the case of the real Hilbert space X, too. We can extend the space X onto the complex Hilbert space $\hat{X} = \{\hat{x} = \{x_1, x_2\} \in X \times X\}$ with the scalar product $[\hat{x}, \hat{y}] = (x_1, y_1) + (x_2, y_2) + i((x_2, y_1) - (x_1, y_2))$. The operator $A \in L(X, X^*)$ can be extended onto the operator $\hat{A} \in L(\hat{X}, \hat{X}^*)$ by $\langle \hat{A}\hat{x}, \hat{y}\rangle = \langle Ax_1, y_1\rangle + \langle Ax_2, y_2\rangle + i(\langle Ax_1, y_2\rangle - \langle Ax_2, y_1\rangle)$.

4. Bending of viscoelastic plates with aging

The previous theory can be applied to the initial boundary value problem, which expresses a bending of a viscoelastic plate made of aging material with a short memory ([5], IV.). We suppose, that the central surface of the plate is the bounded region $\Omega \subset E_2$ with the Lipschitz boundary $\partial\Omega$ (def. [4]). We assume that $\partial\Omega = \bar{\Gamma}_1 \cup \bar{\Gamma}_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$. A plate is clamped on Γ_1 and simply supported on Γ_2. The case $\Gamma_1 = \partial\Omega$, or $\Gamma_2 = \partial\Omega$ is always possible. The bending $u(x_1, x_2, t)$ of the plate is a solution of the initial boundary value problem

(4.1) $$\sum_{r=0}^{m} K_{ijkl}(t) \frac{\mathrm{d}^{m-r}}{\mathrm{d}t^{m-r}} u_{,ijkl} = f(x_1, x_2, t), \qquad (x_1, x_2, t) \in \Omega \times R^+$$

(4.2) $$\left.\frac{\mathrm{d}^r u}{\mathrm{d}t^r}\right|_{t=0} = u_r, \qquad r = 0, 1, \ldots, m-1$$

(4.3) $$u = 0 \text{ on } \partial\Omega \times R^+$$

(4.4) $$\frac{\partial u}{\partial n} = 0 \text{ on } \Gamma_1 \times R^+$$

(4.5) $$M(t)u = \sum_{r=0}^{m} K_{ijkl}^{(r)}(t) \frac{\mathrm{d}^{m-r}}{\mathrm{d}t^{m-r}} u_{,ij} \cos(n, x_k) \cos(n, x_l) = 0$$

$$\text{on } \Gamma_2 \times R^+.$$

We denote by n the exterior normal to $\partial\Omega$. The above problem with constant coefficients is investigated in [2]. We use the notation $u_{,ijkl}=\dfrac{\partial^4 u}{\partial x_i \partial x_j \partial x_k \partial x_l}$, $i, j, k, l \in \{1, 2\}$. Summation over repeated subscripts i, j, k, l is implied. We assume that the coefficients $K^{(r)}_{ijkl}(t)$ are symmetric

$$K^{(r)}_{ijkl}(t)=K^{(r)}_{jikl}(t)=K^{(r)}_{klij}(t), \qquad \forall t \in R^+, \tag{4.6}$$

continuous on R^+ and uniformly positive definite, i.e.

$$K^{(r)}_{ijkl}(t)\varepsilon_{ij}\varepsilon_{kl} \geqq c_r \varepsilon_{ij}\varepsilon_{ij}, \; c_r > 0, \tag{4.7}$$

$$r = 0, 1, \ldots, m, \; \{\varepsilon_{ij}\} \in E_4, \; \varepsilon_{ij} = \varepsilon_{ji}, \; t \in R^+$$

We introduce a weak solution of the problem (4.1)—(4.5). Let $H^2(\Omega)$ be the Sobolev space of all functions from the space $L_2(\Omega)$, whose generalized derivatives up to the 2-nd order belong to $L_2(\Omega)$. The scalar product in $H_2(\Omega)$ is defined by

$$(u, v)_2 = \sum_{|i| \leqq 2} \int_\Omega D^i u D^i v \, d\Omega \tag{4.8}$$

$$\left(D^i u = \frac{\partial^{|i|} u}{\partial x_1^{i_1} \, \partial x_2^{i_2}}, \qquad i=(i_1, i_2), \; |i| = i_1 + i_2\right).$$

We denote by $W(\Omega)$ the space of all functions from $H^2(\Omega)$ which satisfy the essential (or geometrical) boundary conditions (4.3), (4.4) in the sense of traces (def. [4]). It can be verified with the help of the Fridrichs and Poincarré inequalities ([4]), that $W(\Omega)$ is a Hilbert space with the scalar product

$$(u, v) = \sum_{|i|=2} \int_\Omega D^i u \, D^i v \, d\Omega \tag{4.9}$$

and the norm

$$\|u\| = \left(\sum_{|i|=2} \int_\Omega (D^i u)^2 \, d\Omega\right)^{1/2}, \tag{4.10}$$

which is equivalent to the original norm in the space $H^2(\Omega)$. Let us denote by $W(\Omega)^*$ the space dual to $W(\Omega)$. We define now a weak solution of the problem (4.1—(4.5).

Definition 4.1. *Let* $f \in C(R^+, W(\Omega)^*)$, $u_i \in W(\Omega)$, $i = 0, 1, \ldots, m-1$, $K^{(r)}_{ijkl}(.) \in C(R^+)$, $r = 0, 1, \ldots, m$; $i, j, k, l \in \{1, 2\}$. *A function* $u \in C^{(m)}(R^+, W(\Omega))$, *which is for each* $h \in W(\Omega)$ *a solution of the initial value problem*

$$\sum_{r=0}^{m} \int_\Omega K^{(r)}_{ijkl} \frac{d^{m-r}}{dt^{m-r}} u_{,ij}(t) h_{,kl} \, d\Omega = \langle f(t), h \rangle \tag{4.11}$$

$$\left.\frac{d^r u}{dt^r}\right|_{t=0} = u_r, \qquad r = 0, 1, \ldots, m-1, \tag{4.12}$$

is a weak solution of the problem (4.1)—(4.5).

If we define the operators $A_r(t)$ by

$$\langle A_r(t)u, h\rangle = \int_\Omega K^{(r)}_{ijkl}(t)u_{,ij}h_{,kl}\, \mathrm{d}\Omega, \tag{4.13}$$

$$u, h \in W(\Omega),\ t \in R^+,\ r = 0, 1, \ldots, m,$$

then the operators $A_r(t)$ (extended to $\hat{A}_r(t)$ according to Remark 3.1) satisfy all the assumptions of Theorem 2.1 with $X = W(\Omega)$, $X^* = W(\Omega)^*$ and hence there exists a unique weak solution of the problem (4.1)—(4.5).

If $\lim_{t\to\infty} K^{(r)}_{ijkl}(t) = K^{r,\infty}_{ijkl}$, $r = 0, 1, \ldots, m$; $\lim_{t\to\infty} \|f(t) - f_\infty\|_* = 0$, $f_\infty \in W(\Omega)^*$, then the assumptions of Corollaries 3,1, 3.2 are fulfilled and hence a weak solution u of (4.1)—(4.5) satisfies in the cases $m = 1, 2$ the relation

$$\lim_{t\to\infty} \|u(t) - u_\infty\| = 0, \tag{4.14}$$

where $u_\infty \in W(\Omega)$ is a weak solution of the corresponding elastic problem, i.e.

$$\int_\Omega K^{m,\infty}_{ijkl} u_{,ij}\, h_{,kl}\, \mathrm{d}\Omega = \langle f_\infty, h\rangle, \qquad \forall h \in W(\Omega). \tag{4.15}$$

This result corresponds with the physical experience.

REFERENCES

[1] BOCK, I : On quasiparabolical differential equations of higher order. Math. slov. 3, 26, 1976.

[2] BRILLA, J.: Variational methods in mathematical theory of viscoelasticity. Proceedings of Equadiff III, Brno 1972.

[3] DALECKIJ, J. L., KREJN, M. G.: Ustočivosť rešenij differenciaľnych uravnenij v Banachovom prostranstve, Moskva, "Nauka" 1970.

[4] NEČAS, J.: Les mèthodes directes en théorie des éguations elliptiques. Academia, Praha 1967.

[5! RŽANICYN, Q. R.: Teoria polzučesti, Moskva, "Strojizdat" 1968.

[6] YOSIDA, K.: Functional analysis. Springer Verlag 1965.

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АСИМПТОТИЧЕСКОЕ ПОВЕДЕНИЕ РЕШЕНИЯ ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ В ПРОСТРАНСТВЕ ГИЛЬБЕРТА

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Резюме

В этой работе изучается начальная задача (1.1), (1.2) в пространстве Гильберта X с операторными функциями $A_r(.) \in C(R^+, L(X, X^*))$. Если оператор A_0 коэрцивный для любого $t \in R^+$, то для любой функции $f \in C(R^+, X^*)$ и для любых элементов $u_r \in X$ существует единственное решение задачи (1.1), (1.2). Если выполнены некоторые предположения и если

$$\lim_{t \to \infty} \|A_r(t) - A_{r,\infty}\| = \lim_{t \to \infty} \|f(t) - f_\infty\|_* = 0, \text{ то } \lim_{t \to \infty} \|u(t) - A_{m,\infty}^{-1} f_\infty\|_* = 0.$$

Полученные результаты используются для решения начально краевых задач, решения которых определяют изгибы вязкоупругих плит со свойствами зависящими от времени.