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ON THE MAXIMAL DEDEKIND COMPLETION OF A LATTICE ORDERED GROUP

ŠTEFAN ČERNÁK

Let $G$ be a partially ordered group. The group operation will be written additively. We denote by $M_1(G)$ the set of all Dedekind cuts of the partially ordered set $(G, \leq)$. The set $G$ can be considered as a subset of $M_1(G)$ under the canonical embedding. The set $M_1(G)$ is partially ordered under the set-inclusion. It is possible to define the operation $+$ on $M_1(G)$ such that $M_1(G)$ turns out to be a partially ordered semigroup having the property that $G$ is a subgroup of the semigroup $M_1(G)$. Denote by $M(G)$ the set of all elements of $M_1(G)$ possessing inverses in $M_1(G)$. Then $M(G)$ is the greatest subgroup of the semigroup $M_1(G)$ (cf. Fuchs [6]). If $G$ is an Abelian group, then $M_1(G)$ is a commutative semigroup and so $M(G)$ is an Abelian group.

C. J. Everett [5] has proved the following theorem:

(A) Let $G$ be a commutative lattice ordered group. Then $M(G)$ is a lattice ordered group.

In this note it will be shown that the assertion (A) holds true for all lattice ordered groups (without supposing the commutativity).

Let $G$ be an $l$-group and suppose that $G$ can be expressed as a mixed product $\Omega A_i (i \in I)$ of linearly ordered groups $A_i$. We denote by $K$ the set of all maximal elements of $I$. It will be proved that $M(G)$ is (up to isomorphisms) the mixed product $\Omega B_i (i \in I)$, where $B_i = M(A_i)$ if $i \in K$ and $B_i = A_i$ if $i \in I - K$. A similar result has been proved by J. Jakubík [8] for the maximal Dedekind completion of an Abelian $l$-group which is the direct product of $l$-groups. Analogous results concerning the Cantor extension are obtained in [3] and [4].

1. The maximal Dedekind completion $M(G)$ of a lattice ordered group $G$

In this paragraph there will be constructed the maximal Dedekind completion $M(G)$ of an arbitrary lattice ordered group $G$. 

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Let $G$ be a lattice ordered group. Let us denote by $X^u(X^l)$ the set of all upper (lower) bounds of a subset $X \subseteq G$ in $G$. Let $G^#$ be the system of all ideals in $G$ of the form $(X^u)$, where $X$ is a nonempty and upper bounded subset of $G$. The system $G^#$ is partially ordered under the set-inclusion. Then $G^#$ is a conditionally complete lattice. The lattice operations in $G^#$ will be denoted by $\land, \lor$. If a system of sets $\{Z_\lambda\}_{\lambda \in \Lambda} \subseteq G^#$ has an upper (lower) bound in $G^#$, then

$$\lor Z_\lambda (\lambda \in \Lambda) = (\lor U_\lambda)^u (\land Z_\lambda (\lambda \in \Lambda)) = \land Z_\lambda (\lambda \in \Lambda).$$

The mapping $q : G \rightarrow G^#$ defined by $q(a) = (\{a\})^u$ is one-to-one and it preserves all intersections and joins existing in $G$. In the next we shall identify $a$ and $q(a)$. Then $G$ is a sublattice of $G^#$ and the following conditions are satisfied:

(i) Every nonempty subset of $G$ bounded from above (below) has the least upper bound (greatest lower bound) in $G^#$. (ii) For each element $z \in G^#$ there exist nonempty subsets $M_1, M_2$ of $G$ such that $M_1$ is bounded from above in $G$, $M_2$ is bounded from below in $G$ and $\sup M_1 = z = \inf M_2$ in the partially ordered set $G^#$.

For an element $z \in G^#$ we denote

$$U(z) = \{h \in G : h \geq z\}, \ L(z) = \{g \in G : g \leq z\}.$$ 

Let $z_1, z_2 \in G^#$. From (ii) it follows that the sets $L(z_1)$ and $L(z_2)$ are nonempty and bounded from above in $G$. Then also the set $Z = \{g_1 + g_2 : g_1 \in L(z_1), g_2 \in L(z_2)\}$ is nonempty and bounded from above in $G$. By (i) there exists $\sup Z$ in $G^#$. Define the operation $+$ in $G^#$ by putting $z_1 + z_2 = \sup Z$. Then $G^#$ is a semigroup (cf. Fuchs [6]). For each $z \in G^#$ we have

$$\text{if } z_1 \leq z_2, \text{ then } z_1 + z \leq z_2 + z, \ z + z_1 \leq z + z_2.$$ 

If $z_1, z_2 \in G$, then the operation $z_1 + z_2$ in $G^#$ coincides with the operation $z_1 + z_2$ in $G$. Thus $G$ is an $l$-subgroup of $G^#$. It should be observed that $G^#$ is not a group in general (cf. [5]).

Let $M(G)$ be the set of all elements of $G^#$ that have an inverse in $G^#$. Then $M(G)$ is a group; $M(G)$ is a maximal subgroup of the semigroup $G^#$. With respect to (1) $M(G)$ is a partially ordered group. In the following will be shown that $M(G)$ is an $l$-group.

Let $X_1, X_2$ be subsets of $G$ such that $z_1 = \sup X_1, z_2 = \sup X_2$. In a similar manner as above we get that the set $Z' = \{g_1' + g_2' : g_1' \in X_1, g_2' \in X_2\}$ is nonvoid and bounded from above in $G$. Hence by (i) there exists $z' = \sup Z'$ in $G^#$. We intend to show that $\sup Z = \sup Z'$, i. e. that the following statement is true:

**1.1.** $z_1 + z_2 = z'$.

**Proof.** The relations $X_1 \subseteq L(z_1), X_2 \subseteq L(z_2)$ imply $Z' \subseteq Z$ and so $z' \leq z_1 + z_2$. It remains to prove that $z_1 + z_2 \leq z'$, i. e., $U(z') \subseteq U(z_1 + z_2)$. If $u \in U(z')$, then $u \in G$, $u \geq z' \geq g_1' + g_2'$ for every $g_1' \in X_1, g_2' \in X_2$. Hence $-g_1' + u \geq g_2'$ and thus $-g_1' + u \geq$
z_2 \geq g_2 \text{ for each } g_2 \in L(z_2). \text{ From } u - g_2 \geq g_1 \text{ we get } u - g_2 \geq z_1 \geq g_1, u \geq g_1 + g_2 \text{ for each } g_1 \in L(z_1), g_2 \in L(z_2). \text{ Therefore } u \geq z_1 + z_2. \text{ Then } u \in U(z_1 + z_2).

Jakubík [7] introduced the notion of a generalized completion \( D_1(G) \) of an \( l \)-group \( G \). For the operation + on \( D_1(G) \) a relation analogous to 1.1. is valid ([7], Lemma 2.1). Observe that if \( G \) is an Abelian \( l \)-group, then \( D_1(G) \) is an \( l \)-subgroup of \( M(G) \) (see [8]).

For \( H \subseteq G \) denote \( -H = \{-g \in G : g \in H\} \). If \( z \in G^* \), then by (ii) there exist nonvoid subsets \( X, Y \) of \( G \) with the property

\[
(2) \quad z = \sup X = \inf Y.
\]

1.2. Let \( z \in G^* \) and let \( X, Y \) be as in (2). If \( \wedge(y - x; x \in X, y \in Y) = 0 \) in \( G \), then \( z \) has a rightinverse in \( G^* \).

Proof. From (2) it follows that \( -Y \) is a nonvoid and bounded from above in \( G \). According to (i) there is \( z' \in G^* \), \( z' = \sup (-Y) \). We shall show that \( z' \) is a rightinverse to \( z \).

By 1.1. we obtain \( z + z' = \sup \{x + y; x \in X, y \in -Y\} = \sup \{x - y; x \in X, y \in Y\} \) in \( G^* \). Since \( 0 = \inf \{y - x\} = -\sup \{x - y\} \) in \( G \), we conclude that \( \sup \{x - y\} = 0 \) in \( G^* \). Hence \( z + z' = 0 \).

Remark 1. In an analytical way we obtain that \( z' = \inf (-Y) \) is a left-inverse to \( z \) whenever \( \wedge(-x + y; x \in X, y \in Y) = 0 \) holds in \( G \).

1.3. Let \( z \in G^* \) and let (2) be fulfilled. Then \( z \in M(G) \) if and only if the following conditions are satisfied in \( G \):

(a) \( \wedge(y - x; x \in X, y \in Y) = 0 \),
(b) \( \wedge(-x + y; x \in X, y \in Y) = 0 \).

Proof. If \( z \in G^* \) and if both conditions (a) and (b) are fulfilled, then 1.2 and Remark 1 imply that \( z' = \sup (-Y) \) is an inverse to \( z \), hence \( z \in M(G) \). Conversely, let \( z \in M(G) \). We shall show that (a) holds true. The assumption implies that \( 0 \leq y - x \) for each \( x \in X, y \in Y \). Let \( g \in G \), \( 0 < g \leq y - x \) for every \( x \in X, y \in Y \). Hence \( g + x \leq y \). From (2) it follows \( g + x \leq z \) and by (1) we get \( x \leq -g + z \). Then \( z \leq -g + z \) because of (2). From the hypothesis \( z \in M(G) \) we conclude that there exists an inverse to \( z \) in \( G^* \). Hence by (1) we have \( 0 \leq -g \), a contradiction. The proof of (b) is analogous.

The question of the independence of the conditions (a) and (b) remains open.

Everett [5] proved the assertion 1.3 under the assumption that (i) \( G \) is commutative and (ii) \( X = L(z), Y = U(z) \).

1.4. If \( z \in M(G) \), then \( z \wedge 0 \in M(G) \) (the operation \( \wedge \) being considered with respect to \( G^* \)).

Proof. Suppose that \( z \in M(G) \) and let \( X, Y \) be as in (2). Since \( G^* \) is a lattice, \( z \wedge 0 \in G^* \). First we prove that \( \wedge(y \wedge 0 - x \wedge 0; x \in X, y \in Y) = 0 \) in \( G \). Using 1.3 and the assumption we get \( \wedge(y - x; x \in X, y \in Y) = 0 \) in \( G \). It is clear that \( 0 \leq y \wedge 0 - x \wedge 0 \). Let there exist \( g \in G \) such that \( 0 < g \leq y \wedge 0 - x \wedge 0 \) for each \( x \in X, y \in Y \),
y \in Y. Applying the result from [1] (p. 296) we get $0 < g \leq y \wedge 0 - x \wedge 0 \leq y - x$ for each $x \in X$, $y \in Y$, a contradiction. Thus $\wedge (y \wedge 0 - x \wedge 0) = 0$ in $G$. Then from the relations $z \wedge 0 = \sup L(z \wedge 0) = \inf U(z \wedge 0)$, $(x \wedge 0)_{x \in X} \subseteq L(z \wedge 0)$, $(y \wedge 0)_{y \in Y} \subseteq U(z \wedge 0)$ it follows $\wedge (y_1 - x_1; x_1 \in L(z \wedge 0), y_1 \in U(z \wedge 0)) = 0$ in $G$. In a similar way it can be proved that $\wedge (-x_1 + y_1; x_1 \in L(z \wedge 0), y_1 \in U(z \wedge 0)) = 0$ in $G$. Then 1.3 completes the proof.

From 1.4 we infer that the partially ordered group $M(G)$ is an $l$-subgroup of $G^*$. Hence $G$ is an $l$-subgroup of $M(G)$. The $l$-group $M(G)$ will be called the maximal Dedekind completion of $G$.

2. The maximal Dedekind completion of the mixed product of linearly ordered groups

Jakubík [8] studied the maximal Dedekind completion of an Abelian $l$-group $G$, which is a direct product of $l$-groups. In this section there will be investigated the maximal Dedekind completion of an $l$-group (without assuming commutativity) that is a mixed product of linearly ordered groups.

The concept of the mixed product of partially ordered groups is a common generalization of the concepts of the complete direct product and the lexicographic product (see Fuchs [6], Conrad [2]). Let us recall the definition of the mixed product.

Let $I \neq \emptyset$ be a partially ordered set and let $A_i$ be a partially ordered group for each $i \in I$. Form the system $H$ of all mappings $f: I \to \bigcup A_i (i \in I)$ such that $f(i) \in A_i$ for each $i \in I$. We denote by $G$ the set of all $f \in H$ such that the set $\sigma(f) = \{i \in I: f(i) \neq 0\}$ fulfills the descending chain condition. If for each $f, g \in G$ and each $i \in I$ we put $(f + g)(i) = f(i) + g(i)$, then $G$ is a group. The set of all minimal elements of the partially ordered set $\sigma(f, g) = \{i \in I: f(i) \neq g(i)\}$ will be denoted by $\min \sigma(f, g)$. Further, we denote $\sigma(f) = \sigma(f, 0)$. We put $f < g$ if and only if $f(i) < g(i)$ for each $i \in \min \sigma(f, g)$; then $G$ is a partially ordered group. It is said to be the mixed product of partially ordered groups $A_i$ and it is denoted by $G = \otimes A_i (i \in I)$.

If $I$ is a trivially ordered set, then the mixed product is the direct product of partially ordered groups $A_i$. If $I = \{1, 2\}$, then for the direct product we shall use the symbol $G = A_1 \times A_2$.

**2.1. If $G$ is a linearly ordered group, $z_1, z_2 \in G^*$, $z_1 < z_2$, then there exists $g \in G$ with the property $z_1 < g \leq z_2$.**

**Proof.** Let $G$ be a linearly ordered group. From the relation $z_1 = \sup L(z_1)$, $z_2 = \sup L(z_2)$, $z_1 < z_2$ it follows that $L(z_1)$ is a proper subset of $L(z_2)$. Hence there exists $g \in L(z_2)$, $g \notin L(z_1)$. Linearity of $G$ implies $z_1 < g \leq z_2$.

If $G$ is an $l$-group, the assertion 2.1 need not hold.

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Example. Let $C$, $Q$ and $R$ be additive groups of all integers, rational and real numbers (with the natural order), respectively. If $G = C \times Q$, then in view of [8] (Theorem 2.7) and [5] (Theorem 7) we obtain $M(G) = M(C) \times M(Q) = C^* \times Q^* = C \times R$. It suffices to set $z_1 = (0, \sqrt{2})$, $z_2 = (1, \sqrt{2})$.

Let $I$ be a partially ordered set and let $A_i$ be a linearly ordered group for each $i \in I$, $A_i \neq \{0\}$. Suppose that $G$ is an $l$-group such that

$$G = \Omega A_i \quad (i \in I).$$

2.2. Let $i_0 \in I$, $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq G$, $\wedge x_\lambda = 0$. Then there exists $\lambda \in \Lambda$ with the property $x_\lambda (i) = 0$ for each $i \in I$, $i < i_0$.

Proof. Assume that for each $\lambda \in \Lambda$ there exists $i \in I$, $i < i_0$ such that $x_\lambda (i) \neq 0$. Then $i_0 \in \min \sigma(x_\lambda)$. There are $a \in A_{i_0}$, $a > 0$ and $g \in G$ such that $g(i_0) = a$, $g(j) = 0$ for each $j \in I$, $j \neq i_0$. Therefore $0 < g \leq x_\lambda$ for each $\lambda \in \Lambda$, contrary to $\wedge x_\lambda = 0$.

From 2.2 it follows that the set $\Lambda(i_0) = \{\lambda \in \Lambda : x_\lambda(i) = 0 \text{ for each } i \in I, i < i_0\}$ is nonempty.

Denote by $K$ the set of all maximal elements of $I$.

2.3. Let $i_0 \in K$, $\{x_\lambda\}_{\lambda \in \Lambda} \subseteq G$, $\wedge x_\lambda (\lambda \in \Lambda) = 0$. Then $\wedge x_\lambda (i_0) (\lambda \in \Lambda(i_0)) = 0$.

Proof. The assumption implies that $x_\lambda \geq 0$ for each $\lambda \in \Lambda$. We have either $x_\lambda (i_0) = 0$ or $i_0 \in \min \sigma(x_\lambda)$ for each $\lambda \in \Lambda(i_0)$. Hence $0 \leq x_\lambda (i_0) = 0$ for each $\lambda \in \Lambda(i_0)$. If there exists $\lambda \in \Lambda(i_0)$ such that $x_\lambda (i_0) = 0$ the statement is evident. Let there exist $a \in A_{i_0}$ such that $0 < a \leq x_\lambda (i_0)$ for each $\lambda \in \Lambda(i_0)$. If $g$ is as in 2.2, in the same way as in 2.2 we arrive at a contradiction with $\wedge x_\lambda (\lambda \in \Lambda) = 0$.

2.4. Let $j \in I - K$, $z \in M(G)$. Then for each $i \in I, i \leq j$ there exists $a_i \in A_i$ with the following properties:

(a) There exist elements $g \in L(z)$, $h \in U(z)$ such that $g(i) = h(i) = a_i$ for each $i \in I$, $i \leq j$.

(b) If $g_1 \in L(z)$, $h_1 \in U(z)$, $g_1(i) = h_1(i)$ for each $i \in I$, $i \leq j$, then $g_1(i) = a_i$ for each $i \in I$, $i \leq j$.

Proof. From $z \in M(G)$ and from 1.3 we get $\wedge (h - g ; h \in U(z), g \in L(z)) = 0$. There exists $j' \in I$, $j' > j$. According to 2.2 there is $h \in U(z)$, $g \in L(z)$ such that $(h - g) (i) = 0$ and so $g(i) = h(i)$ for each $i \in I, i \leq j$. For the elements $g$, $h$ with the mentioned property and for each $i \leq j$ denote $a_i = g(i) = h(i)$. Thus (a) is valid. Let $g_1$ and $h_1$ fulfill the assumption of the condition (b). Suppose that there exist $i' \in I$, $i' \leq j$ such that $g_1(i') \neq a_i$. Hence $g(i') = h(i') \neq g(i') = h(i')$. Let $i_0 \in I$, $i_0 \leq i'$, $i_0 \in \min \sigma(g_1, h)$. Then $i_0 \in \min \sigma(g_1, h)$. Since $g_1(i) < h(i) = g(i_0)$, $g(i_0) < h_1(i_0) = g_1(i_0)$, a contradiction.

From (b) it follows that for each $i \in I - K$ the element $a_i$ is uniquely determined by $z \in M(G)$ (it does not depend on $j \in I$).

Let $z \in M(G)$, $i_0 \in I$ and suppose that $i_0$ is not minimal in $I$. Denote
Now let \( i_0 \) be a minimal element of \( I \). We define

\[
L^{i_0}(z) = \{ g \in L(z) : g(i) = a_i \text{ for each } i \in I, \ i < i_0 \}, \quad U^{i_0}(z) = \{ h \in U(z) : h(i) = a_i \text{ for each } i \in I, \ i < i_0 \}.
\]

if \( i_0 \) is not maximal in \( I \) and

\[
L^{i_0}(z) = L(z), \quad U^{i_0}(z) = U(z)
\]

if \( i_0 \) is a maximal element of \( I \). Further, for any \( i_0 \in I \) denote

\[
L^{i_0}(z)(i_0) = \{ u \in A_{i_0} : \text{there exists } g \in L^{i_0}(z), g(i_0) = u \}, \quad U^{i_0}(z)(i_0) = \{ v \in A_{i_0} : \text{there exists } h \in U^{i_0}(z), h(i_0) = v \}.
\]

From 2.4 we infer that \( U^{i_0}(z) \neq 0 \), \( U^{i_0}(z) \neq 0 \) and so \( U^{i_0}(z)(i_0) \neq 0 \). Because of \( u \leq v \) for each \( u \in L^{i_0}(z)(i_0), v \in U^{i_0}(z)(i_0) \), we have that \( l^{i_0}(z)(i_0) \) (\( U^{i_0}(z)(i_0) \)) is a set bounded from above (below). Hence there exist \( c \in A^\#_{i_0} \) and \( d \in A^\#_{i_0} \), \( c = \sup L^{i_0}(z)(i_0) \), \( d = \inf U^{i_0}(z)(i_0) \) in \( A^\#_{i_0} \). Clearly \( c \leq d \). According to 1.3 we obtain \( \wedge (h - g ; g \in L(z), h \in U(z)) = 0 \).

Let \( i_0 \) be a maximal element of \( I \). Using the definition of the sets \( L^{i_0}(z) \) and \( U^{i_0}(z) \) we obtain that the equality \( (h - g)(i) = 0 \) is valid for each \( i \in I, \ i < i_0 \) and for each \( g \in L^{i_0}(z), h \in U^{i_0}(z) \). We conclude from 2.3 that \( \wedge (h(i_0) - g(i_0) ; g \in L^{i_0}(z), h \in U^{i_0}(z)) = 0 \). Similarly we get \( \wedge (-g(i_0) + h(i_0) ; g \in L^{i_0}(z), h \in U^{i_0}(z)) = 0 \). Using 1.3 it is easily verified that \( c \in M(A_{i_0}) \). Analogously it can be proved that \( d \in M(A_{i_0}) \). We intend to show that \( c = d \). If \( c < d \), i.e., \( d - c > 0 \), then by 2.1 there exists \( a \in A_{i_0} \), \( 0 < a \leq d - c = h(i_0) - g(i_0) \) for each \( g \in L^{i_0}(z), h \in U^{i_0}(z) \), a contradiction. Let us denote \( a^*_{i_0} = c = d \). The definition of \( a^*_{i_0} \) implies that \( a^*_{i_0} \in M(A_{i_0}) \).

\[
(3) \quad a^*_{i_0} = \sup L^{i_0}(z)(i_0) = \inf U^{i_0}(z)(i_0).
\]

From (3) we conclude that for each \( i_0 \in K \) the elements \( a^*_{i_0} \) are uniquely determined by \( z \in M(G) \).

2.4'. Let \( j \in I - K, \ z \in M(G) \) and let \( X, Y \) be as in (2). Then the following conditions are valid.

(a') There exist elements \( x \in X, y \in Y \) such that \( x(i) = y(i) = a_i \) for each \( i \leq j \).

(b') If \( x_1 \in X, y_1 \in Y, x_1(i) = y_1(i) \) for each \( i \leq j \), then \( x_1(i) = a_i \) for each \( i \leq j \).

The proof of this assertion is analogous to that of 2.4.

If the symbols \( X^{i_0}, Y^{i_0}, X^{i_0}(i_0), Y^{i_0}(i_0) \) have an analogical meaning with \( L^{i_0}(z), U^{i_0}(z), L^{i_0}(z)(i_0), U^{i_0}(z)(i_0) \), in the same way as above we get the following statement.

2.5. \( a^*_{i_0} = \sup X^{i_0}(i_0) = \inf Y^{i_0}(i_0) \) for each \( i_0 \in K \).

2.6. \( a_i \) is the greatest (least) element of the set \( L^i(z)(i) \) (\( U^i(z)(i) \)) for each \( i \in I - K \).

Proof. Let \( i \in I - K \). By 2.4 there exist elements \( g \in L(z), h \in U(z) \), \( g(j) = h(j) = a_j \) for each \( j \in I, j \leq i \). Since \( g \in L^i(z), h \in U^i(z) \), we have \( a_i = g(i) \in 310 \)
Let $X, Y$ be as in (2). Since $X \subseteq L(z), Y \subseteq U(z)$, with respect to 2.6 the following assertion is valid.

**2.7.** $a_i$ is the greatest (least) element of the set $X'(i)$ ($Y'(i)$) for any $i \in I \setminus K$.

**2.8.** There exists an element $a \in G$ such that $a(i) = a_i$ for each $i \in I - K$.

Proof. Let us denote $A = \{i \in I - K : a_i \neq 0\}$. We have to show that each nonempty set $I_1 \subseteq A$ contains a minimal element. If $i_0 \in I_1$ is not minimal in $I_1$, then $I_2 = \{i \in I_1 : i < i_0\} \neq \emptyset$. By 2.4 there exists $g \in L(z)$, $g(i) = a_i$ for each $i < i_0$ and we have $I_2 \subseteq \sigma(g)$. From the fact $g \in G$ it follows that every nonempty subset of $\sigma(g)$ has a minimal element. Consequently, $I_2$ contains a minimal element $i'$. Hence $i'$ is a minimal element of $I_1$, too.

Let us form $B = \Omega B_i$ ($i \in I$), where $B_i = A_i$ for each $i \in I - K$ and $B_i = M(A_i)$ for each $i \in K$. In view of 2.8 there exist elements $z_1, z_2 \in B$ such that $z_1(i) = a_i$, $z_2(i) = 0$, whenever $i \in I - K$ and $z_1(i) = 0$, $z_2(i) = a^*_i$ whenever $i \in K$. Hence $z_1 + z_2 = z' \in B$.

(4) $z'(i) = a_i$ if $i \in I - K$ and $z'(i) = a^*_i$ if $i \in K$.

Let $X, Y$ be as in (2). Let $X_i, Y_i$ be in (2). Since $A_i \subseteq M(A_i)$, we have $X \subseteq B_i, Y \subseteq B_i$.

**2.9.** $z' = \sup X = \inf Y$ in $B$.

Proof. We intend to show that $z' = \sup X$ in $B$. Pick out any $x \in X$. If $x = z'$, then in view of (4), 2.7 and (3) $z'$ is the greatest element of $X$ and the assertion follows. Let $x \neq z'$, $i_0 \in \min \sigma(x, z')$. Hence $x(i) = z'(i) = a_i$ whenever $i \in I, i < i_0$. Since $x \in X_0$, we get $x(i_0) \in X^{0}(i_0)$. If $i_0 \in I - K$, we infer from 2.7 that $x(i_0) < a_0 = z'(i_0)$. If $i_0 \in K$, by using (3) and 2.5 we obtain $z'(i_0) = a^*_0 = \sup X^{0}(i_0)$ and thus $x(i_0) < z'(i_0)$. Therefore $x \leq z'$. Let $u \in B$. $u \geq x$ for each $x \in X$ and let $i_0 \in \min \sigma(u, z')$. If $i_0 \in I - K$, by 2.4 there is $x \in X, x(i) = a_i$ for each $i \leq i_0$. Hence $x \in X^{0}$ and $i_0 \in \min \sigma(u, x)$. Then $u(i_0) > x(i_0) = a_0 = z'(i_0)$. If $i_0 \in K$, then either $u(i_0) = x(i_0)$ or $i_0 \in \min \sigma(u, x)$. Thus $u(i_0) \geq x(i_0)$. This inequality is valid for each $x \in X^{0}$. From $a^*_0 = \sup X^{0}(i_0)$ it follows that $u(i_0) > a^*_0 = z'(i_0)$. Thus $u \geq z'$. The proof of the relation $z' = \inf Y$ is analogous.

Denote $A = \{g \in G : g \leq z'\}$.

**2.10.** $L(z) = A$.

Proof. Since $z = \sup L(z)$ in $M(G)$, by 2.9 we get $z' = \sup L(z)$ in $B$. Hence $L(z) \subseteq A$. Let $g \in A$. Because of $z = \inf U(z)$ in $M(G)$, by 2.9 we obtain $z' = \inf U(z)$ in $B$. Thus $g \leq h$ for each $h \in U(z)$. Then $g \leq z$, i.e. $g \in L(z)$.

**2.11.** If $z_1, z_2 \in M(G)$, then $z_1 + z_2 = \sup Z$ in $B$, where $Z = \{g_1 + g_2 : g_1 \in L(z_1), g_2 \in L(z_2)\}$.

Proof. From $z_1 = \sup L(z_1), z_2 = \sup L(z_2)$ in $M(G)$ and from 2.9, we infer that $z'_1 = \sup L(z_1), z'_2 = \sup L(z_2)$ in $B$. Hence $z'_1 \geq g_1, z'_2 \geq g_2$ for every $g_1 \in L(z_1), g_2 \in L(z_2)$. Thus $z'_1 + z'_2 \geq g_1 + g_2$, i.e. $z'_1 + z'_2$ is an upper bound of $Z$ in $B$. Let
Let \( b \in B, b \geq g_1 + g_2 \) for each \( g_1 \in L(z_1), g_2 \in L(z_2) \) and let \( i_0 \in \min \sigma(b, z'_1 + z'_2) \). For \( z_n \) \((n = 1, 2)\) let \( a_n \) and \( a^*_n \) have an analogous meaning as \( a \) and \( a^* \) have for the element \( z \). If \( i_0 \in K \), then by 2.4 and (4) there exists \( g_1 \in L(z_1), g_2 \in L(z_2) \) such that \( g_1(i) = a_{i_0} = z'_1(i_0), g_2(i) = a_{i_0} = z'_2(i_0) \) for each \( i \in i_0 \). We will show that \( b(i_0) > (z'_1 + z'_2)(i_0) = z'_1(i_0) + z'_2(i_0) \). If \( b(i_0) < z'_1(i_0) + z'_2(i_0) = g_1(i_0) + g_2(i_0) \), then because of \( i_0 \in \min \sigma(b, g_1 + g_2) \) we obtain \( b \neq g_1 + g_2 \), which is impossible. Now we prove that \( b(i_0) > z'_1(i_0) + z'_2(i_0) \) for \( i_0 \in K \). Suppose that \( b(i_0) < z'_1(i_0) + z'_2(i_0) \). According to (4) we get \( z'_1(i_0) = a^*_i(i_0) = sup L'^0(z_1)(i_0), z'_2(i_0) = a^*_i(i_0) = sup L'^0(z_2)(i_0) \) in \( M(A_{i_0}) \). The definition of the operation \( + \) in \( M(A_{i_0}) \) and 1.1 imply \( b(i_0) < z'_1(i_0) + z'_2(i_0) = sup \{g_1(i_0) + g_2(i_0) : g_1 \in L'^0(z_1), g_2 \in L'^0(z_2)\} \) in \( M(A_{i_0}) \). From the fact that \( A_{i_0} \) is a linearly ordered set it follows that we can find \( g'_1 \in L'^0(z_1) \subseteq L(z_1), g'_2 \in L'^0(z_2) \subseteq L(z_2) \) with \( b(i_0) < g'_1(i_0) + g'_2(i_0) \). From \( g'_1 \in L'^0(z_1), g'_2 \in L'^0(z_2) \) we conclude that \( g'_1(i) - a_{i_0} = z_1(i_0) \) for each \( i \in I, i < i_0 \). Then \( i_0 \in \min \sigma(b, g'_1 + g'_2) \). Thus \( b \neq g'_1 + g'_2 \), a contradiction.

Define a mapping \( \varphi: M(G) \rightarrow B \) by the rule \( \varphi(z) = z' \). With respect to 2.10 we have \( L(z_1) = \{g \in G : g \sim z'_1\}, L(z_2) = \{g \in G : g \leq z'_2\} \). Then \( z'_1 = z'_2 \) if and only if \( L(z_1) = L(z_2) \). Hence \( \varphi \) is a one-to-one mapping. Since \( z_1, z_2 \) if and only if \( L(z_1) \subseteq L(z_2) \), by 2.9 and 2.10 we obtain \( z_1 \leq z_2 \) if and only if \( z'_1 \sim z'_2 \). Now we show that \( \varphi \) is a mapping \( M(G) \rightarrow B \). Let \( b \in B, B_1 = \{g \in G : g \leq b\} \). Since \( b(i) \in M(A_i) \) for each \( i \in I \), the sets \( \{a \in A_i : a \leq b(i)\}, \{a \in A_i : a \geq b(i)\} \) are nonempty for any \( a \in A \). There are elements \( g, h \in G \) such that \( g(i) = h(i) = b(i) \in A_i \) for \( i \in I - K \) and \( g(i) = u_i \), where \( u_i \in A_i \in A_i \), \( v_i \geq b(i) \) for \( i \in K \). Then \( B_i \neq \emptyset, B^*_i \neq \emptyset \), since \( g \in B_i, h \in B^*_i \). Hence by (i) there is \( z \in G^*, z = sup B_1 \). Now we show that \( z \in M(G) \). Denote \( U_i = \{u \in G : u(i) = b(i) \} \) for each \( i \in I - K \) and \( u(i) \in A_i, \in A_i \). Also \( u(i) \leq b(i) \) for each \( i \in K \), according to 1.3 we obtain \( \wedge(u(i) - v(i)) = 0 \) for each \( i \in I - K \) and \( v(i) \in A_i, v(i) \geq b(i) \) for each \( i \in K \). Therefore \( u(i) - v(i) = 0 \) for \( u \in U_i, v \in V_i \) for each \( i \in I - K \). Since \( b(i) \in M(A_i) \) for each \( i \in K \), we have \( \wedge(u(i) - v(i)) = 0 \) for each \( i \in K \). From \( U_i \subseteq U(z), V_i \subseteq L(z), z = sup L(z) = inf U(z) \) and from 1.3 we conclude \( z \in M(G) \). In view of 2.9 we obtain \( z' = sup B_1 = b = \varphi(z) \). It is easily seen that \( \varphi \) preserves the group operation. In fact, using 2.9 and 2.11 from \( z_1 + z_2 = sup Z \) in \( M(G) \) it follows that \( (z_1 + z_2)' = sup Z = z'_1 + z'_2 \) in \( B \).

We have proved that the following theorem is true.

**Theorem.** Let \( G \) be a lattice ordered group that can be written as a mixed product \( G = \Omega A_i \) \((i \in I)\), where \( A_i \) is linearly ordered for each \( i \in I \). Put \( B_i = M(A_i) \) if \( i \) is maximal in \( I \) and \( B_i = A_i \), otherwise. Then there exists an isomorphism \( \varphi \) of \( M(G) \) onto \( \Omega B_i \) \((i \in I)\) such that \( \varphi(g) = g \) for each \( g \in G \).
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МАКСИМАЛЬНОЕ ДЕДЕКИНДОВО ПОПОЛНЕНИЕ СТРУКТУРНО УПРАЯДОЧЕННОЙ ГРУППЫ

Штефан Чернак

Резюме

Эверетт доказал, что максимальное дедекиндово пополнение коммутативной структурно упорядоченной группы есть структурно упорядоченная группа. В этой статье результат Эверетта обобщается для всех структурно упорядоченных групп. Доказаны некоторы свойства максимального дедекиндового пополнения смешанного произведения линейно упорядоченных групп.