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ON TRANSFINITE CONVERGENCE AND GENERALIZED CONTINUITY

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Given a set of functions, which are continuous is some generalized sense, it may or may not be closed with respect to the transfinite convergence. Various criteria are given for such sets to be closed (see the papers [7], [8], [9], [10], [12]). This paper gives some new results in this direction.

Throughout the paper the transfinite sequence means a transfinite sequence of functions of the type Ω, where Ω is the first uncountable ordinal number. In general the function will be defined on a set X assuming the values in a first countable topological space Y. In particular cases X will be supposed to be a topological space of a suitable type. The notion of transfinite convergence will mean the pointwise transfinite convergence. It will be a variant of the classical definition of Sierpiński (see also [4], [6], [12] for some of its modifications).

Definition 1. *A transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of functions defined on a set X and assuming values in a topological space Y is said to be convergent to the function f: $X \to Y$, if for any $x \in X$ there holds* $\lim f_\xi(x) f(x)$. *(The last means that given a neighbourhood V of $f(x)$, there exists an ordinal number $\xi_0 < \Omega$ such that $f(x) \in V$ whenever $\xi \geqq \xi_0$.)*

Two counterexamples

Here we shall prove two theorems which are in fact two counterexamples concerning the closedness of two types of functions.

It is known (see [8]) that the limit of a transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of quasicontinuous functions defined on a separable metric space and assuming values in a metric space is quasicontinuous. The same is true if the quasicontinuity is substituted by cliquishness or somewhat continuity ([8], [10]). It was also proved ([10]) that in the case of quasicontinuity and cliquishness and also in some other cases the assumption of separability of X may be substituted by the local separability. This is not the case when somewhat continuity is consedered. Recall first the definition of somewhat continuity.

Definition 2. (see [2]). *A mapping $f\colon X \to Y$, where X, Y are topological spaces, is said to be somewhat continuous if for any open set $G \subset Y$ such that $f^{-1}(G)$ is nonempty, the set* $\operatorname{int} f^{-1}(G)$ *is nonempty. (Here* $\operatorname{int} A$ *means interior of the set A).*

Theorem 1. *There exists a locally separable metric space X and a transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ $f_\xi\colon X \to R$ of real functions coverging to f such that each f_ξ is somewhat continuous but f is not somewhat continuous.*

Proof. For $\xi < \Omega$ let (X_ξ, ϱ_ξ) be a metric space such that X_ξ contains exactly the points (t, ξ) where $0 \leqq t \leqq 1$ and for $x_1 = (t_1, \xi)$, $x_2 = (t_2, \xi)$, $\varrho_\xi(x_1, x_2) = |t_1 - t_2|$. Put $X = \bigcup_{\xi<\Omega} X_\xi$ and for x, $y \in X$ let

$$\varrho(x, y) = \begin{cases} 1 \text{ if } x \in X_\eta,\ Y\ y_\lambda,\ \eta \neq \lambda \\ \varrho_\xi(x, y) \text{ if } x, y \in X_\xi,\ \xi < \Omega \end{cases}$$

The space X is locally separable. Define now a function $\varphi\colon X \to R$ as follows: If $x \in X_0$, where $x = (t, 0)$, then

$$\varphi(x) = \begin{cases} t \text{ if } t \text{ is rational} \\ 1 \text{ if } t \text{ is irrational.} \end{cases}$$

If $x \in X_\xi$, $\xi \neq 0$ and $x = (t, \xi)$, then put $\varphi(x) = t$. The transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of the functions $f_\xi\colon X \to R$ will be defined as:

$$f_\xi(x) = \begin{cases} \varphi(x) \text{ if } x \in X_0, \text{ or } x \in X_\eta, \text{ where } \eta \geqq \xi \\ 0 \text{ if } x \in X_\eta, \text{ where } \eta < \xi. \end{cases}$$

Now if $x \in X$ and $x \in X_0$ we have $f_\xi(x) = \varphi(x)$ for every $\xi < \Omega$, hence $\lim_{\xi<\Omega} f_\xi(x) = \varphi(x)$. If $x \notin X_0$, then $x \in X_\eta$, where $\eta > 0$. Thus for $\xi > \eta$ we have $f_\xi(x) = 0$ and so $\lim_{\xi<\Omega} f_\xi(x) = 0$. The sequence $\{f_\xi\}_{\xi<\Omega}$ is convergent and the limit function f is

$$f(x) = \begin{cases} \varphi(x) \text{ if } x \in X_0 \\ 0 \quad \text{ if } x \notin X_0. \end{cases}$$

Evidently f is not a somewhat continuous function. Each of the functions f_ξ, $1 \leqq \xi < \Omega$ is somewhat continuous. In fact, let G be open and such that $f^{-1}(G) \neq \emptyset$. Then G necessarily contains an open interval $(a, b) \subset (0, 1)$. By the definition of f_ξ the set $f_\xi^{-1}(a, b)$ contains the set of all (t, ξ), where $a < t < b$, which is an open set in X. Thus $\operatorname{int} f^{-1}(G) \supset \operatorname{int} f^{-1}(a, b) \neq \emptyset$. The theorem is proved.

There is another known fact that the limit of a transfinite sequence of real continuous functions of a real variable is a continuous function. The last fact was proved by Sierpiński in [11]. It was generalized by T. Šalát in [12] for transfinite sequences of mappings from X to Y, where X, Y are arbitrary metric spaces. The separability, even the local separability of X, was not supposed. The extension of the result of [12] to the case when X is the first countable topological space is immediate. (It will be seen also from Corollary 2 of Theorem 3). But we shall prove by means of the continuum hypothesis that it is not possible to extend the mentioned result to an arbitrary topological space.

Theorem 2. *There exists (under the continuum hypothesis) a topological space X and a transfinite sequence $\{f_\xi\}_{\xi<\Omega}$ of real continuous functions defined on X converging to f such that f is not continuous.*

Proof. Consider the space $X=(-\infty, \infty)$ with the density topology (see [3]). Let $\{f_\xi\}_{\xi<\Omega}$ be a sequence of approximately continuous functions such that the transfinite limit f of $\{f_\xi\}_{\xi<\Omega}$ is not approximately continuous. Such a sequence exists under the assumption of the continuum hypothesis according to a result of J. S. Lipiński [6]. Since in the density topology a function is continuous precisely if it is approximately continuous (see [3]), we have that $\{f_\xi\}_{\xi<\Omega}$ is a sequence of continuous functions on X, the limit of which is not continuous. The theorem is proved.

Weak continuity and transfinite convergence

Weak continuity is defined in the following way (see [5]).

Definition 2. *A mapping $f\colon X \to Y$ (X, Y are topological spaces) is said to be weakly continuous at the point $x \in X$ if for any open neighbourhood V of $f(x)$ there exists a neighbourhood U of x such that $f(U) \subset \bar{V}$ ($\bar{A}$ denotes the closure of the set A). If f is weakly continuous at any $x \in X$, then it is said to be wekly continuous.*

The fact that any continuous function is weakly continuous is evident. If we consider only the function with the values in regular topological spaces, then it can be easily seen that the notions of weak continuity and continuity coincide. In this case the question whether the limit of a transfinite sequences of weakly continuous functions is weakly continuous reduces to the problem of a transfinite limit of continuous functions. But in general this is not the case, and so the question whether the transfinite limit of weakly continuous functions is weakly continuous has its own meaning. If the space where a function takes its values is not regular, then there may exist a weakly continuous function which is not continuous. This may occur even in the case when the space Y is first countable. We shall give such an example. (The example may be found in another context in [1]).

Example. Let $X=\langle 0, 1\rangle$ with its ordinary topology. Let $Y=\langle 0, 1\rangle$ with a topology defined by the following system of neighbourhoods. If $x \neq 0$, then the neighbourhoods of x are the ordinary neighbourhoods. The neighbourhoods of $x=0$ will be the sets $\langle 0, t) - \left\{\frac{1}{2}, \frac{1}{3}, \ldots, \frac{1}{n}, \ldots\right\}$, $0<t<1$. The identical mapping from X onto Y is weakly continuous but not continuous.

Theorem 3. *Let X, Y be topological spaces satisfying the first countability axiom. Let Y be a Hausdorff space. Let $\{f_\xi\}_{\xi<\Omega}$ be a transfinite sequence of weakly continuous functions on X with values in Y, converging to f. Then f is a weakly continuous function.*

To prove the theorem we shall use the following lemma.

Lemma. *If $\{f_\xi\}_{\xi<\Omega}$ is a transfinite sequence of functions defined on X with values in Y, where Y is the first countable Hausdorff topological space and $S \subset X$ a countable set such that $\{f_\xi(x)\}$ converges to $f(x)$ for any $x \in S$, then there exists such a ξ_0 that for $\xi \geqq \xi_0$ we have $f_\xi|S = f|S$.*

Proof. We may suppose with no loss of generality that S is a set of values of a sequence $\{x_n\}_{n=1}^{\infty}$. For n fixed choose a base $\{V_k\}_{k=1}^{\infty}$ of neighbourdhoods of $f(x_n)$. Since Y is Hausdorff we have $\bigcap\limits_{k=1}^{\infty} = f\{(x_n)\}$. It follows from the convergence of $\{f_\xi(x_n)\}_{\xi<\Omega}$ to $f(x_n)$ that to any V_k there exists ξ_k such that if $\xi \geqq \xi_k$, then $f_\xi(x_n) \in V_k$. Choosing an ordinal number η_n such that $\eta_n \geqq \xi_k$ for $k=1, 2, \ldots$ and $\eta_n < \Omega$, we have $f_\xi(x_n) \in V_k$ for every $\xi \geqq \eta_n$ and any $k=1, 2, \ldots$ Hence $f_\xi(x_n) \in \bigcap V_k = f(x_n)$ for $\xi \geqq \xi_n$. This being true for any $n=1, 2, \ldots$ we choose ξ_0 such that $\xi_0 > \eta_n$ for $n=1, 2, \ldots$ and $\xi_0 < \Omega$. Thus for $\xi \geqq \xi_0$ we have $f_\xi(x_n) = f(x_n)$ for $n=1, 2, \ldots$ The lemma is proved.

Proof of theorem 3. Suppose f not to be weakly continuous at a point $x_0 \in X$. Then there exists a neighbourhood V of $f(x_0)$ such that for any neighbourhood U of x_0 there exists $x \in U$ with $f(x) \notin \bar{V}$. Since X satisfies the first countability axiom we may choose a base of neighbourhoods $\{U_n\}_{n=1}^{\infty}$ at x_0 with a point $x_n \in U_n$ and $f(x_n) \notin \bar{V}$. According to the Lemma there exists $\xi_0 < \Omega$ such that $f_\xi(x_n) = f(x_n)$ for $\xi \geqq \xi_0$, $n=0, 1, 2, \ldots$ Hence $f_{\xi 0}(x_n) = f(x_n)$ for $n=0, 1, 2, \ldots$ We obtain from the weak continuity of f_{ξ_0} at x_0 the existence of a neighbourhood U of x_0 with $f_{\xi_0}(U) \subset \bar{V}$. Since $\{U_n\}_{n=1}^{\infty}$ is a base at x_0, there exists n_0 such that $U_{n_0} \subset U$, hence $f(x_{n_0}) \in \bar{V}$. This is a contradiction.

In connection with the weak continuity let us mention the notion of Θ-continuity defined by Fomin in [1].

Definition 3. *A function f: $X \to Y$ (X, Y are topological spaces) is said to be Θ-continuous at $x_0 \in X$ if for any neighbourhood V of $f(x_0)$ there exists a neighbourhood U of x_0 such that $f(\bar{U}) \subset \bar{V}$. If f is Θ-continuous at any $x \subset X$, then it is said to be Θ-continuous.*

It is evident from the definition that Θ-continuity implies the weak continuity.

Since $\overline{f^{-1}(V)} \subset f^{-1}(\bar{V})$ for any continuous f (see [14] p. 80), we obtain that the continuity implies Θ-continuity. The function in the above example is an example of a Θ-continuous function with values in a first countable space, which is not continuous. If X is a regular space, the Θ-continuity and the weak continuity coincide. From the last fact and from Theorem 3 the following result follows.

Corollary 1. *Let X, Y be first countable topological spaces. Moreover, let X be regular and Y Hausdorff. Then the limit of any transfinite sequence of Θ-continuous function defined on X and taking values in Y is Θ-continuous.*

Since for the regular space Y the weak continuity and Θ-continuity coincide with the notion of continuity, we obtain another result generalizing the mentioned result of [12].

Corollary 2. *Let X, Y be first countable topological spaces and moreover, let Y be a regular Hausdorff space. Then the limit of any transfinite sequence of continuous functions defined on X, with values in Y is continuous.*

There is a question, whether there exists a weakly continuous function which is not Θ-continuous. We found in the literature notes that such a function exists (see [13]) but we did not find any example of such a function. The following example is due to T. Šalát.

Example. Let $X = \{a, b, c, d\}$ and $\mathcal{T} = \{\emptyset, X, \{b\}, \{c\}, \{b, c\}, \{a, b\}, \{a, b, c\}, \{b, c, d\}\}$. Define $f\colon X \to X$ as follows

x	a	b	c	d
$f(x)$	c	d	b	a

We prove that f is weakly continuous. Since f is continuous at b, c, (b, c are isolated) it is sufficient to verify the weak continuity at a, d.

The point a. Each neighbourhood $V(f(a)) = V(c)$ of the point $f(a)$ contains $\{c\}$, hence $\overline{V(f(a))} \supset \overline{\{c\}} = \{c, d\}$. Take $U(a) = \{a, b\}$ as a neighbourhood of a. Then $f(U(a)) \subset \overline{V(f(a))}$, since $f(U(a)) = f(\{a, b\}) = \{c, d\}$.

The point d. Each neighbourhood $V(f(d)) = V(a)$ of the point $f(d)$ contains $\{a, b\}$, and so $\overline{V(f(d))} \supset \overline{\{a, b\}} = \{a, b, d\}$. We can take $\{b, c, d\}$ as a neighbourhood $U(d)$ of the point d. Then $f(\{b, c, d\}) = \{a, b, d\} \subset \overline{V(f(d))}$.

Now we prove that f is not Θ-continuous at the point a. Choose $V(f(a)) = V(c) = \{c\}$. Then $\overline{V(f(a))} = \overline{\{c\}} = \{c, d\}$. Each neighbourhood of the point a contains $\{a, b\}$, therefore $\overline{U(a)} \supset \overline{\{a, b\}} = \{a, b, d\}$. But $f(d) = a \notin \{c, d\}$, and so $f(\overline{U(a)}) \not\subset \overline{V((f)a)}$.

Added in proofs. While in proofs the author proved Theorem 2 without continuum hypothesis.

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О ТРАНСФИНИТНОЙ СХОДИМОСТИ И ОБОБЩЕННОЙ НЕПРЕРЫВНОСТИ

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Резюме

Результаты, касающиеся вопроса сохранения какого-то типа непрерывности при трансфинитной сходимости последовательностей были изучены в [6–12]. В работе исследуется (кроме других) этот вопрос для слабо непрерывных функций и функций Θ непрерывных, определенных Фоминым в [1]. Показывается, что в случае слабо непрерывных функций, определенных на топологическом пространстве X и принимающих значение в топологическом пространстве Y (X, Y обладает первой аксиомой счетности) слабая непрерывность сохраняется.