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ON A LEMMA OF G. CHOQUET

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0. Introduction. Let $\mathcal{R}$ be an algebra of subsets of a set $\mathcal{E}$, m be a measure on $\mathcal{R}$, m^* be the outer measure induced by m. Then m^* is continuous from below i.e.

$$A_n \subset A_{n+1} (n = 1, 2, \ldots) \Rightarrow m^* \left(\bigcup_{n=1}^{\infty} A_n \right) = \lim_{n \to \infty} m^*(A_n).$$

This fact has been used implicitly and in a more general form in many papers as a lemma. In this note we prove a general form of the lemma and then using it we present straight-forward proofs of some results appearing in literature. Hence the lemma seems to be useful for future applications, too.

1. Theorem. *Let H be a lattice, X, $Y \subset H$, X be a sublattice of H, $a_n \in Y$ $(n = 1, 2, \ldots)$, $a_n \nearrow a$,*) $a \in X \cup Y$. Let $\mu\colon X \cup Y \to \langle -\infty, \infty \rangle$ satisfy the following conditions:*

(i) *μ is non-decreasing.*
(ii) *$\mu(x) + \mu(y) \geqq \mu(x \vee y) + \mu(x \wedge y)$ for every x, $y \in X$.*
(iii) *$\mu | X$ is continuous from below i.e. $x_n \nearrow x$, $x_n \in X$ $(n = 1, 2, \ldots)$, $x \in X$ implies $\mu(x_n) \nearrow \mu(x)$.*
(iv) *$\mu(y) = \inf \{\mu(x);\ y \leqq x \in X\}$ for every $y \in Y$.*
(v) *$\mu(a_1) > -\infty$.*
(vi) *Either $a \in X$ or X is monotonously upper σ-complete (i.e. every non-decreasing bounded sequence has the supremum) and there is $x \in X$ such that $x \geqq a$.*

Then $\mu(a_n) \nearrow \mu(a)$.

Proof. Since $\mu(a_n) \leqq \mu(a)$, we have $\lim_{n \to \infty} \mu(a_n) \leqq \mu(a)$. Hence we can assume that $\lim_{n \to \infty} \mu(a_n) < \infty$. Then to every $\varepsilon > 0$ there are $b_n \in X$, $b_n \geqq a_n$ such that

$$\mu(a_n) + \frac{\varepsilon}{2^n} > \mu(b_n).$$

*) $a_n \nearrow a$ means that $a_n \leqq a_{n+1}$ $(n = 1, 2, \ldots)$ and $a = \sup a_n$.

Put $c_n = \bigvee_{i=1}^{n} b_i$ $(n = 1, 2, \ldots)$. Using (ii) it is easy to prove by induction that

$$(*) \qquad \mu(a_n) + \sum_{i=1}^{n} \frac{\varepsilon}{2^i} > \mu(c_n) \quad (n = 1, 2, \ldots).$$

Now we must distinguish between two cases.

Let $a \in X$. Then we can assume that $b_n \leqq a$ (in the reverse case we could take $b_n \wedge a$). Hence $a_n \leqq b_n \leqq c_n \leqq a$ and therefore $c_n \nearrow a$. Now $(*)$ and (iii)

$$\text{give } \lim_{n\to\infty} \mu(a_n) + \varepsilon \geqq \lim_{n\to\infty} \mu(c_n) = \mu(a).$$

Let the second alternative in (vi) be satisfied. Then we can assume $b_n \leqq x$ $(n = 1, 2, \ldots)$. Put $c = \sup_n c_n = \sup_n b_n$. Then $c \in X$, $c \geqq a$, hence by $(*)$ and (iii)

$$\mu(a) \leqq \mu(c) = \lim_{n\to\infty} \mu(c_n) \leqq \lim_{n\to\infty} \mu(a_n) + \varepsilon.$$

2. Evidently the dual assertion regarding Theorem 1 holds too.

Theorem. *Let H be a lattice, X, $Y \subset H$, X be a sublattice of H, $a_n \in Y$ $(n = 1, 2, \ldots)$, $a_n \searrow a$, $a \in X \cup Y$. Let μ: $X \cup Y \to \langle -\infty, \infty\rangle$ satisfy the following conditions:*

(i) *μ is non-decreasing.*

(ii) *$\mu(x) + \mu(y) \leqq \mu(x \wedge y) + \mu(x \vee y)$ for every x, $y \in X$.*

(iii) *$\mu|X$ is lower continuous, i.e. $x_n \searrow x$, $x_n \in X$ $(n = 1, 2, \ldots)$, $x \in X$ implies $\mu(x_n) \searrow \mu(x)$.*

(iv) *$\mu(y) = \sup\{\mu(x);\ y \geqq x \in X\}$ for every $y \in Y$.*

(v) *$\mu(a_1) < \infty$.*

(vi) *Either $a \in X$ or X is monotonously lower σ-complete (i.e. every non-increasing bounded sequence in X has the infimum) and there is $x \in X$ such that $x \leqq a$.*

Then $\mu(a_n) \searrow \mu(a)$.

3. Let B be a boundedly σ-complete sublattice of a given lattice H. Let there exists to every $x \in H$ a $b \in B$ such that $b \geqq x$. Let J_0: $B \to \langle -\infty, \infty\rangle$ satisfy the following conditions:

(i) *J_0 is non-decreasing*

(ii) *$J_0(x) + J_0(y) \geqq J_0(x \vee y) + J_0(x \wedge y)$ for every x, $y \in B$.*

(iii) *If $x_n \nearrow x$, $x_n \in B$, $x \in B$, then $J_0(x_n) \nearrow J_0(x)$.*

Define further for $y \in H$

$$J^*(y) = \inf\{J_0(x);\ y \leqq x \in B\}.$$

Now we can put $X = B$, $Y = H$, $\mu = J^*$. Evidently $\mu|X = J_0$, hence all assumptions of Theorem 1 are satisfied by the second part of (vi). Therefore

$$y_n \nearrow y, \quad y_n \in H, \quad y \in H \Rightarrow J^*(y_n) \nearrow J^*(y).$$

The last implication is the assertion of Lemma 1.4 of [3].

4. Let $\mathcal{R}$ be an algebra of subsets of a set E, m be a measure on $\mathcal{R}$. Denote by H the family of all subsets of E, by B the family of all sets of the form $\bigcup_{n=1}^{\infty} A_n$, where $A_n \in \mathcal{R}$ and define J_0 by the formula

$$J_0\left(\bigcup_{n=1}^{\infty} A_n\right) = \lim_{n\to\infty} m\left(\bigcup_{i=1}^{n} A_i\right).$$

It is not difficult to prove that the definition is correct and that B and J_0 satisfy the assumptions of the assertion presented in 3. Therefore J^* is upper continuous. But J^* is the outer measure induced by m. Hence we have obtained the result stated in the Introduction.

5. In [4] M. Šabo starts with a sublattice A of a given lattice S and a mapping J: $A \to R$ which is non-decreasing, satisfies the valuation identity $J(a)+J(b) = J(a \vee b) + J(a \wedge b)$ and is lower continuous. Moreover to every $x \in S$ there exists an $a \in A$ with $a \geqq x$. In Theorem 2 of [4] a sequence $(a_n)_{n=1}^{\infty}$ of elements of A is given converging to a given element $O \in A$, where $a_n \geqq O$ $(n = 1, 2, \ldots)$ and $J(O) = 0$. The theorem states that $J(a_n) \to 0$.

We show that the mentioned theorem is a corollary of Theorem 2. Put $H = S$, $X = A$, $Y = A^+ = \{x \in S;\ \exists b_n \in A,\ b_n \nearrow x\}$ and $\mu(x) = \lim_{n\to\infty} J(b_n)$ for $x \in Y = X \cup Y$. If $a_n \to 0$, $a_n \geqq 0$, then $\bigvee_{i=n}^{\infty} a_i \searrow O (n \to \infty)$, hence by Theorem 2 $\mu\left(\bigvee_{i\ n}^{\infty} a_i\right) \searrow 0$. Further $O \leqq J(a_n) = \mu(a_n) \leqq \mu\left(\bigvee_{i=n}^{\infty} a_i\right)$, and therefore $J(a_n) \to 0$. (Here the first possibility in (vi) was satisfied, because $O \in A$.)

6. Another consequence of Theorem 2 in [4] is the following theorem (Theorem 4): Let A, J satisfy the assumptions given in 5. Let $A^* = \{x \in S;\ \exists b_n \in A,\ b_n \to x\}$, $J^*(x) = \lim_{n\to\infty} J(b_n)$, $x \in A^*$. Then $a_n \searrow O$, $a_n \in A^*$ $(n = 1, 2, \ldots)$ implies $J^*(a_n) \searrow 0$.

To prove the statement put $X = A^- = \{x \in S;\ \exists b_n \in A,\ b_n \searrow x\}$, $Y = A^*$, $\mu = J^*$ (of course, $X \subset Y$). Lemma 5 in [4] gives (iv), Lemma 6 gives (iii), (ii) is easy to prove. Hence Theorem 2 implies Šabo's theorem 4.

7. Similar considerations have been used by E. Futáš in [1]. He also starts with a sublattice A of a lattice H and J: $A \to R$ satisfying the same assumptions as we have mentioned in 5. Only Futáš's construction is different. He puts $A_\sigma = \{x;$

$\exists b_n \in A,\ b_n \nearrow x\},\ J_1 : A_\sigma \to R,\ J_1(x) = \lim_{n\to\infty} J(b_n)$. A very important Lemma 2.2.18 in [1] states that $x_n \in A_\sigma,\ x \in A,\ x_n \searrow x$ implies $\lim_{n\to\infty} J_1(x_n) = J_1(x)$.

This lemma follows from Theorem 2. It suffices to put $X = A,\ Y = A_\sigma,\ \mu = J_1$.

8. Since Futáš's lemma 2.2.19 is dual to the result mentioned above, it follows immediately from our Theorem 1.

REFERENCES

[1] FUTÁŠ, E.: Extension of continuous functionals. Mat. Čas., 21, 1971, 191—198.

[2] CHOQUET, G.: Theory of capacities. Ann., Inst. Fourier, 5, 1953—54, 131—295.

[3] RIEČAN, B.: On the Carathéodory method of the extension of measures and integrals. Math. slov., 27, 1977, 365—374.

[4] ŠABO, M.: On an extension of finite functional by the transfinite induction. Math. slov., 26, 1976, 193—200.

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Резюме

Статья посвящена абстрактной подстановке того факта, что внешняя мера индуцированная мерой является непрерывной снизу.