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ON A PROBLEM CONCERNING A FUNCTIONAL DIFFERENTIAL EQUATION

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Let R_n be the n-dimensional Euclidean space with the norm $|\cdot|$. Let h be a positive number. Denote by $\mathscr{C}$ the Banach space of continuous functions φ: $[-h, 0]\to R_n$ with the max-norm $\|\cdot\|$ and let $\mathscr{C}_0$ be the subspace of those $\varphi\in\mathscr{C}$, for which $\varphi(0)=0$. Let $T_0<T_1\leqq\infty$ be given numbers. Let $\mathscr{B}$ be the Banach space of bounded continuous functions $[T_0, T_1)\to R_n$ with the sup-norm $\|\cdot\|$. Finally, if x is a function $[T_0-h, T_1)\to R_n$, let x_t, for $t\in[T_0, T_1)$, be the function defined for $s\in[-h, 0]$ by $x_t(s)=x(t+s)$.

Now consider the functional-differential equation

$$x'(t)=f(t, x_t) \tag{1}$$

where f is a continuous function f: $[T_0, T_1)\times\mathscr{C}\to R_n$. Assume that

$$\int_I |f(t, 0)|\, \mathrm{d}t=K<\infty \tag{2}$$

and that there is an integrable function $\beta(t)$ on $I=[T_0, T_1)$ such that

$$|f(t, \varphi)-f(t, \psi)|\leqq\beta(t)\|\varphi-\psi\| \tag{3}$$

for every φ, $\psi\in\mathscr{C}$ and $t\in I$, and

$$\int_I \beta(t)\, \mathrm{d}t=\lambda<1. \tag{4}$$

If $\varphi\in\mathscr{C}$, let $x(t, \varphi)$ denote the unique solution of (1) for $t\in I$, with φ as the initial function (i.e. $x(t, \varphi)=\varphi(t)$ for $t\in[T_0-h, T_0]$). The main aim of this note is to prove the following theorem:

Theorem. *Assume that the conditions (2)—(4) are satisfied. Let $X_1\in R_n$, and $\varphi\in\mathscr{C}_0$ be given. Then there exists such $X_0\in R_n$ that*

$$\lim_{t\to T_1-} x(t, \varphi+X_0)=X_1.$$

Remark. This theorem improves a result of M. Švec [2], where a similar theorem is proved with $\lambda<1/2$. However, the constant 1 in (4) is the best possible as shown by the following example: Let $n=1$, $T_0=0$, $T_1=2$, and let $f(t, \varphi)=a(t)\varphi(t-1)$, where $a(t)\leqq 0$ and such that $a(t)=0$ for $t\in[0, 1]$, and $\int_0^2 a(t)\,dt=-1$. Clearly $\lambda=1$, but for each solution x we have $\lim x(t)=0$ for $t\to 2$.

To prove the theorem the following lemmas will be useful:

Lemma 1. *Assume that the conditions (2)—(4) are satisfied. Then for each* $\varphi\in\mathscr{C}$, $x(t, \varphi)\in\mathscr{B}$.

Proof. Using (2) and (3) we obtain

$$|x(t, \varphi)|\leqq M+\int_{T_0}^t \beta(\xi)\|x_\xi\|\,d\xi,$$

where $M=\|\varphi\|+K$. Let $u(t)=\max|x(\xi, \varphi)|$, for $\xi\in[T_0-h, t]$. Then

$$u(t)\leqq M+\int_{T_0}^t \beta(\xi)u(\xi)\,d\xi,$$

and using the Gronwall lemma and (3) we obtain

$$u(t)\leqq M\cdot\exp\int_{T_0}^t \beta(\xi)\,d\xi<\text{const}.$$

Lemma 2. *Assume that the conditions (2)—(4) are satisfied. Then for each* $\varphi\in\mathscr{C}$ *there exists* $\lim x(t, \varphi)$ *when* $t\to T_1-$.

Proof. For s, $t\in[T_0, T_1)$ we have

$$|x(s, \varphi)-x(t, \varphi)|\leqq\left|\int_s^t \beta(\xi)\|x_\xi\|\,d\xi\right|+\left|\int_s^t |f(\xi, 0)|\,d\xi\right|.$$

Since $\|x_\xi\|$ is bounded, the right-hand side of the inequality vanishes when s, $t\to T_1-$.

Lemma 3. *Assume that the conditions (2)—(4) are satisfied. Let* $\varphi\in\mathscr{C}_0$ *and let* $Z_k\in R_n$ *for* $k=1, 2, \dots$ *be a sequence with* $\lim_{k\to\infty}|Z_k|=\infty$. *Denote* $m_k=\inf|x(t, \varphi+Z_k)|$ *for* $t\in[T_0, T_1]$. *Then* $\lim_{k\to\infty} m_k=\infty$.

Remark. In virtue of Lemma 2 we take clearly

$$x(T_1, \varphi+Z_k)=\lim x(t, \varphi+Z_k) \quad \text{for} \quad t\to T_1.$$

Proof. Assume on the contrary that $\lim_{k\to\infty} m_k$ is not ∞. Then there is a bounded subsequence of $\{m_k\}$. We may assume without loss of generality that $\{m_k\}$ is this subsequence. Since each $x(t, \varphi+Z_k)$ for fix k is bounded and continuous in $[T_0, T_1]$, there exist s_k, $t_k \in [T_0, T_1]$ such that $m_k = |x(s_k, \varphi+Z_k)|$ and $\|x(t, \varphi+Z_k)\| = |x(t_k, \varphi+Z_k)| \geqq |Z_k|$. Now using (2), (3) and (4) we obtain

$$|x(s_k, \varphi \times Z_k) - x(t_k, \varphi+Z_k)| \leqq \left| \int_{s_k}^{t_k} \beta(\xi)\|x_\xi\| \, d\xi \right| + K$$
$$\leqq \lambda(|x(t_k, \varphi+Z_k)| + \|\varphi\|) + K,$$

hence

$$(|x(s_k, \varphi+Z_k) - x(t_k, \varphi+Z_k)| - K)(|x(t_k, \varphi+Z_k)| +$$
$$+ \|\varphi\|)^{-1} \leqq \lambda < 1.$$

But the left-hand side of the inequality tends to 1 when $k \to \infty$ and this is a contradiction.

Now the theorem follows easily from the following general topological principle.

Proposition. *Let $\mathcal{F}$ be a continuous mapping from R_n into the Banach space $\mathcal{B}$ of continuous functions $[T_0, T_1] \to R_n$. Denote by $\mathcal{F}_t(X)$ the value of $\mathcal{F}(X) \in \mathcal{B}$ at the point $t \in [T_0, T_1]$. Assume that*

$$v(X) = \inf_t |\mathcal{F}_t(X)| \to \infty \tag{5}$$

uniformly for $|X| \to \infty$ and that $\mathcal{F}_{T_0}(X) = X$ for $X \in R_n$. Then for each $t \in [T_0, T_1]$, $\mathcal{F}_t(R_n) = R_n$.

Proof. Clearly it suffices to show that for each $X_1 \in R_n$ there is some $X_0 \in R_n$ such that $\mathcal{F}_{T_1}(X_0) = X_1$. For $r \geqq 0$ denote by $S_{n-1}(r)$ the $(n-1)$-dimensional sphere $\{X \in R_n ; |X| = r\}$. By (5) there is some $r_0 > |X_1|$ such that $X_1 \notin \mathcal{F}_t(S_{n-1}(r_0))$ for each $t \in [T_0, T_1]$. Hence $S_{n-1}(r_0) = \mathcal{F}_{T_0}(S_{n-1}(r_0))$ separates the points X_1 and ∞ of the extended space $R_n^* = R_n \cup \{\infty\}$, which is topologically equivalent to the sphere $S_n(1)$ (i.e. $S_{n-1}(r_0)$ separates "the north and the south poles" of the "sphere" R_n^*). Now $\mathcal{F}_{T_1}(S_{n-1}(r_0))$ is obtained by a continuous deformation of $\mathcal{F}_{T_0}(S_{n-1}(r_0))$ $= S_{n-1}(r_0)$ without passing through ∞ (since $\{\mathcal{F}_t(X); t \in [T_0, T_1]$ and $X \in S_{n-1}(r_0)\}$ is compact and hence bounded in R_n) and X_1. Consequently $\mathcal{F}_{T_1}(S_{n-1}(r_0))$ separates the "poles" X_1 and ∞ (see Theorem 2 in [1], p. 350). Now diam $\mathcal{F}_{T_1}(S_{n-1}(r_0)) \to 0$ whenever $r_0 \to 0$, hence by the theorem of balayage ([1], Theorem 4 on p. 350) there is some $r \geqq 0$ such that $X_1 \in \mathcal{F}_{T_1}(S_{n-1}(r))$, and hence there is some $X_0 \in S_{n-1}(r)$ with $\mathcal{F}_{T_1}(X_0) = X_1$, q.e.d.

Proof of the theorem. Fix some $\varphi \in C_0$. Define a mapping $\mathcal{F}: R_n \to B$ in the following way. For $t < T_1$ let $\mathcal{F}_t(X) = x(t, \varphi+X)$, and let $\mathcal{F}_{T_1}(X) = \lim \mathcal{F}_t(X)$ for $t \to T_1$. Clearly $\mathcal{F}$ is continuous (see [2]) and in view of Lemma 3, (5) is satisfied. To finish the proof it suffices to apply the proposition.

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ОБ ОДНОЙ ПРОБЛЕМЕ КАСАЮЩЕЙСЯ ФУНКЦИОНАЛЬНОГО ДИФФЕРЕНЦИАЛЬНОГО УРАВНЕНИЯ

Кристина Смиталова

Резюме

В этой статье доказывается следующая теорема: Если функция в уравнении $x'(t)=f(t, x_t)$ удовлетворяет условиям (2), (3), (4), тогда для фиксированых $T_0 \in R_1, (T_1, X_1) \in R^{n+1}$ и непрерывной для $t \leqq T_0$ начальной функции $\varphi(\varphi(T_0)=0)$ существует $X_0 \in R^n$ для которого предел

$$\lim_{t\to T_1-} x(t, X_0+\varphi)=X_1.$$