Štefan Znám

Decomposition of complete graphs into factors of diameter two

Mathematica Slovaca, Vol. 30 (1980), No. 4, 373--378

Persistent URL: http://dml.cz/dmlcz/136251

Math. Slovaca 30, 1980, No. 4, 373—378

DECOMPOSITION OF COMPLETE GRAPHS INTO FACTORS OF DIAMETER TWO

ŠTEFAN ZNÁM

We say that the system $F_1, \ldots, F_m$ of factors of a graph G presents an edge decomposition of G if every edge of G belongs to exactly one of the factors F_i. Let $f(k)$ be the smallest such natural that the complete graph $K_{f(k)}$ of $f(k)$ vertices can be decomposed into k factors of diameter two. The problem of consideration of the number $f(k)$ has been introduced in [4], where also $f(2)=5$, $f(3)=12$ or 13 are proved. In [3] it is shown that $f(k)$ is finite for any $k \geqq 2$ and that $f(k) \geqq 4k—1$ holds for $k \geqq 3$. *N. Sauer showed in* [5] *that* $f(k) \leqq 7k$ for $k \geqq 2$. In [2] J. Bosák showed that for every $k \geqq 2$ we have

$$6k-52 \leqq f(k) \leqq 6k.$$

Finally B. Bollobás in [1] proved that for $k \geqq 6$ we have

$$f(k) \geqq 6k-9.$$

In our article we prove that for $k \geqq 664$ the inequality $f(k) \geqq 6k-7$ holds. It is very probable that using very similar methods as here (however considering more precisely) the inequality $f(k) \geqq 6k-6$ can be proved.

By the neighbourhood of a set S of vertices in a graph we mean the set of all vertices not belonging to S but adjacent to some vertices of S in this graph.

Now suppose that for a $k \geqq 664$ the complete graph K_{6k-8} is decomposed into k factors of diameter two. Then there exists at least one factor F containing at most

$$\left[\frac{(6k-8)(6k-9)}{2k}\right] = 18k-51$$

edges. We shall state some properties of the factor F.

Lemma 1. *The neighbourhood of any two vertices x, y in F is of cardinality at most $5k-8$.*

Proof. Suppose, the cardinality of the neighbourhood of two vertices x, y in F is at least $5k-7$. Then in the remaining $k-1$ factors there exist at most

$$(6k-10)-(5k-7) = k-3$$

paths of the length 2 between x and y. This is a contradiction with the fact that all factors are of diameter two.

Corollary F does not contain any vertex of degree <3.

Lemma 2. *The maximal degree of vertex in F is at most $3k-6$.*

Proof. A vertex v is of degree at least 3 in all the remaining $k-1$ factors, hence we have

$$3(k-1)+\deg_F v \leqq 6k-9$$

and our assertion follows.

Lemma 3. *Let v be a vertex of degree 3 in F and let it be adjacent to vertices x, y, z. Suppose*

$$\deg_F x \leqq \deg_F y \leqq \deg_F z .$$

Then:

a) *x is of degree at least $2k-3$;*

b) *y is of degree at least $\frac{1}{2}(3k-3)$;*

c) *all three are of degree at least $k-2$.*

Proof. F is of diameter two, hence every vertex of F belongs to the neighbourhood of $\{x, y, z\}$; therefore it has to contain v and $6k-12$ other vertices and so x must be of degree at least $\frac{1}{3}(6k-12)+1$ and the assertion a) follows. Owing to Lemma 2 the vertex x is of degree at most $3k-6$, therefore the neighbourhood of the set $\{z, y\}$ in F contains v and at least $(6k-12) - (3k-7) = 3k-5$ vertices. Both y and z are adjacent to v, hence the sum of degrees of y and z is at least $3k-3$ and the assertion b) follows.

The neighbourhood of the set $\{x, y\}$ is (according to Lemma 1) of cardinality at most $5k-8$ and v belongs to this neighbourhood. Hence there exist at least $(6k-12) - (5k-9) = k-3$ vertices connected with v by a path of length 2 containing z. Therefore z is of degree at least $k-2$. The proof is complete.

Lemma 4. *Let v be a vertex of degree 4 in F adjacent to the vertices x, y, z and t. Suppose*

$$\deg x \geqq \deg y \geqq \deg z \geqq \deg t .$$

Then:

a) $\deg y \geqq k-1$;

b) $\deg z \geqq \frac{1}{2}(k-4)$.

Proof. Owing to Lemma 2 there exist at least $(6k-13) - (3k-7) = 3k-6$

vertices connected with v by a path of length 2 containing one of the vertices y, z, t. Hence the sum of degrees of these three vertices is at least $3k-5$ and a) follows.

Owing to Lemma 1 there exist at least $(6k-13)$ − $(5k-9)$ = $k-4$ vertices connected with v by a path of the length 2 containing z or t and b) follows.

Lemma 5. *Let v be a vertex of degree* 5 *in F. Then v* is adjacent to a vertex of degree at least $k-2$ *and to three vertices of degree at least* $\frac{1}{3}(k-4)$.

The proof is very similar to that of Lemma 4.

Theorem. $f(k) \geqq 6k-7$ for $k \geqq 664$.

Proof. We shall show that K_{6k-8} cannot be decomposed into k factors of diameter 2. Namely we prove the impossibility of the existence of a factor F having the properties stated in Lemmas 1—5 with at most $18k-51$ edges.

Suppose F is such a factor of K_{6k-8}. Denote by A the set of all vertices of degree 3,4 or 5 in F, $|A|=a$; by B the set of all vertices of degree 6, 7, ..., $\left[\frac{1}{3}(k-5)\right]$, $|B|=b$; by C the set of all vertices of degree at least $\frac{1}{3}(k-4)$, $|C|=c$.

If $c \geqq 55$, then the sum of degrees in F is at least

$$3(6k-8)+\frac{55}{3}(k-13)=\frac{109}{3}k-\frac{787}{3},$$

which is a contradiction with the fact that the number of edges is at most $18k-51$. Hence we have $c \leqq 54$.

Now there exist at least $3a$ edges between the sets A and C (see Lemmas 3—5). To every vertex of A choose three edges starting from it to the set C and denote this set of edges by U. Every vertex from B contributes to the sum s of all degrees in F by at least 6 (hence the set B by at least $6b$), the contribution of edges of U is $6a$ and further we have some other edges incident with the vertices of degree 4 or 5 but not considered above.

First suppose there exist at least 325 vertices of degree 4 or 5 in F. Then we have

$$s \geqq 6a+6b+325>6(a+b+c)=6(6k-8)=36k-48.$$

However, this is a contradiction, because the factor F has at most $18k-51$ edges.

Now we shall consider the more complicated case if in F there exist less than 325 vertices of degree 4 or 5. Denote by D the set of vertices of degree 6, 7, ..., $k-3$ in F, $|D|=d$ and by E the set of vertices of degree at least $k-2$, $|E|=e$.

Obviously $e \leqq c \leqq 54$. Suppose $e \geqq 19$. Then the sum of degrees in F is at least $e(k-2)$ + $3(6k-8-54)$ = $(18k+ek)$ − $2e$ − 186, which is for $k \geqq 664$ a contradiction considering the fact that F contains at most $18k-51$ edges. Hence $e \leqq 18$.

However, we shall show that $e \leqq 12$.

We shall distinguish two cases, again.

If $d \geqq 2k$, then the sum of degrees in A and D is at least $3(6k-26) + 3d \geqq 24k-78$. Therefore the sum of degrees in E is at most $(36k-102) - (24k-78) = 12k-24$. Hence $e \leqq 12$.

If $d \leqq 2k$, then we prove first $e \leqq 14$. In this case the number t of vertices of degree 3 in F is at least

$$6k-8-d-18-324=6k-d-350.$$

The sum of degrees in E is at most $36k - 102 - 3(6k-26) - 3d = 18k - 3d - 24 = w$, hence there exist at most 11 vertices of degree at least $\frac{1}{2}(3k-3)=m$ in F.

On the other hand, according to Lemma 3 at least $2t$ edges from vertices of degree 3 go into vertices of degree at least m. Hence the sum of degrees of vertices of E having degree smaller than m is at most

$$w-2t \leqq (18k-3d-24)-2(6k-d-350)=6k-d-676.$$

Hence there exist in E at most 5 vertices of this kind.

Now if the number of vertices of degree at least m in E is $\leqq 8$, we get $e \leqq 14$. However, if the number of vertices of degree at least m is $n=9$, 10, 11, then the sum of degrees of vertices with smaller degree in E is at most

$$w-nm=(18k-3d-24)-\frac{n}{2}(3k-3) = \left(18-\frac{3}{2}n\right)k-3d-24+\frac{3}{2}n\,;$$

hence there exist at most $18-\frac{3}{2}n$ vertices of this kind. Therefore, the number of vertices in E is at most $18-\frac{n}{2}$, which is less than 14.

Hence, in all cases we have $e \leqq 14$. Further we can consider starting from this new information and show that $e \leqq 12$.

Because $e \leqq 14$, the sum of degrees in E is at most

$$36k-102-3(6k-8-14)-3d=18k-36-3d.$$

Under these conditions there exist at least $6k-8-324-2k-14 \geqq 3k-5$ vertices of degree 3 in F, thus due to Lemmas 2 and 3 in E at least two vertices of degree at least $2k-3=r$ exist. We shall deal with two cases.

1. Suppose there exist exactly two vertices of degree at least r. Any vertex of degree 3 is connected with at least one of them. There exist at least 5 vertices of degree smaller than r but not smaller than m in E. Choose 5 vertices of this kind. Thus the sum of degrees of remaining vertices of E (without those two vertices and the chosen 5 vertices) is at most

$$18k-3d-36-t-5m\leqq$$
$$\leqq 18k-3d-36-(6k-d-350)-\frac{15}{2}k-\frac{15}{2}=$$
$$=4{,}5k-2d-321{,}5.$$

Hence, according to the condition $k\geqq 664$ we get that E contains at most 4 further vertices and in this case we have $e\leqq 11$.

2. If E contains at least 3 vertices of degree at least r and at least 4 further vertices of degree at least m, then the sum of degrees of 7 vertices with the greatest degree in F is at least $3r+4m=12k-15$, hence the sum of the degrees of the remaining vertices in E is at most $18k - 3d - 36 - 12k + 15 = 6k - 21 - 3d < 6(k-2)$. Therefore, there exist at most 5 further vertices in E. Hence in all cases we have $e\leqq 12$.

All the vertices of degree 3 are connected with 3 vertices of E, all the vertices of degree 4 with at least 2 vertices of E and every vertex of degree 5 is adjacent to some vertex in E. If we denote F_1 the factor of F which arises deleting the edges connecting two vertices of E from F, then the sum of degrees of vertices in F_1 is at least

$$(6k-8-e)6=36k-48-6e.$$

For $e\leqq 8$ this gives a contradiction, because for such an e we have $36k-48-6e > 36k-102$.

Suppose $e\geqq 9$. Let v_0 be a vertex of degree 3 in F. Then every vertex of E not adjacent to v_0 is adjacent to at least one vertex of the neighbourhood of v_0. Hence there exist at least $e-3$ edges with both endopoints in E and the sum of degrees in F is at least

$$(36k-48-6e)+2(e-3)=36k-4e-54$$

which is for $e=9$, 10, 11 more than $36k-102$.

Suppose $e=12$. If there exist at least 10 edges with both endpoints in E, then we get a contradiction again. Suppose, there exist exactly 9 such edges. Denote by H the factor of F induced by the set E. Let $V=\{v_1, v_2, v_3\}$ be the neighbourhood of v_0 in F and let

$$\deg_H v_1\geqq\deg_H v_2\geqq\deg_H v_3. \tag{I}$$

Then we have the only possibility:

$$\deg_H v_1+\deg_H v_2+\deg_H v_3=9 \tag{II}$$

and all the vertices not belonging to V are of degree one in H. We supposed $d \leqq 2k$, hence $t > 3k - 6$. Thus due to Lemma 2 there exists a vertex v_4 of degree 3 in F not adjacent to v_1. Let $\{v_5, v_6, v_7\}$ be the neighbourhood of v_4 in F. Then due to (I) and (II) the sum of degrees of vertices v_5, v_6 and v_7 is at most 7 in H, which is a contradiction, because then the diameter of F would be greater than 2. Thus, according to Theorem 1 of [4], the proof of out theorem is complete.

REFERENCES

[1] BOLLOBÁS, B.: Extremal grapf theory. Academic Press, London 1978.

[2] BOSÁK, J.: Disjoint factors of diameter two in complete graphs, J. Combinatorial Theory Ser. B, 16, 1974, 57—63.

[3] BOSÁK, J.—ERDÖS, P.—ROSA, A.: Decomposition of complete graphs into factors with diameter two, Mat. Čas., 21, 1971, 14—28.

[4] BOSÁK, J.—ROSA, A.—ZNÁM, Š.: On decomposition of complete graphs into factors with given diameters. In Theory of Graphs (proc. Colloq. Tihany, 1966) Academic Press, New York and Akadémiai Kiadó, Budapest 1968, 37—56.

[5] SAUER, N.: On the factorization of complete graphs into factors of diameter 2, J. Combinatorial Theory, 9, 1970, 423—426.

Received October 31, 1978

Katedra algebry a teórie čísel
Prírodovedeckej fakulty UK
Mlynská dolina
816 31 Bratislava

РАЗЛОЖЕНИЕ ПОЛНОГО ГРАФА НА ФАКТОРЫ ДИАМЕТРА ДВА

Штефан Знам

Резюме

Доказывается, что полный граф с $6k-8$ вершинами невозможно разложить на k факторы диаметра 2, если $k \geqq 664$.

Пользуясь теми же методами, но рассуждая точнее, вероятно возможно показать: полный граф с $6k-7$ вершинами тоже невозможно разложить на k факторы диаметра 2.