Mathematica Slovaca

Ján Šipoš

Integration in partially ordered linear spaces

Mathematica Slovaca, Vol. 31 (1981), No. 1, 39--51

Persistent URL: http://dml.cz/dmlcz/136259

Math. Slovaca 31, 1981, No. 1, 39—51

INTEGRATION IN PARTIALLY ORDERED LINEAR SPACES

JÁN ŠIPOŠ

The purpose of this paper is to develop the integration theory for functions $f: \Omega \rightarrow X$ on an arbitrary abstract measure space $(\Omega, \mathcal{S}, \mu)$ with values in a partially ordered linear space X.

When an integration method is to be built up, it is essential to assume that the range space X of a vector valued function has some kind of completeness property. For example, X may be taken a Banach space. Since we will have no metric or uniform structure on X, we will assume some kind of completeness depending on order.

When we are introducing a weak or Gelfand—Pettis type integral, we necessarily need the notion of a dual space. It is also natural to assume that the dual of X separates points of X. In general X is not a topological space, thus our dual space will also be based on order, and we shall assume that it separates points of X.

If one has a topological linear space, it is known that there are cases in which the original space cannot be embedded into its second dual. However, if, for example, the original space happens to be a Banach space, then it can be embedded into its second dual. We shall have a condition with respect to a partial ordering which will imply that this good property of the second dual will be preserved.

Applications of these concepts to the construction of expected value and vector valued martingales will appear in subsequent papers.

Similar problems were studied in [2], [5], [6], [9], [10] and [11].

1. Definitions and preliminary results

A *partially ordered linear space* is a set X endowed with a structure of a partially ordered space and a structure of a real linear space satisfying the following compatibility conditions:

(*i*) If x, y and z are in X and $x \leqq y$, then $x + z \leqq y + z$.

(*ii*) If x and y are in X and c is a non-negative real number, then $x \leqq y$ implies $c \cdot x \leqq c \cdot y$. Θ will denote the neutral element of X. By X_+ we denote the set of all non-negative elements from X.

By $X^<$ we denote the set of all order continuous linear functionals on X, which can be represented as a difference of two monotone linear functionals. (Recall that a linear functional $x^<: X \to R$ — reals is *order continuous* iff $x_n \searrow \Theta$ implies $x^<(x_n) \to 0$.)

We shall say that $X^<$ is the *order dual* of X. As we pointed out, we want to build up the weak integral at first, and so for our considerations it is natural to assume that $X^<$ separates points of X, i.e. for $x \in X$ with $x \neq \Theta$ there exists an $x^< \in X^<$ with $x^<(x) \neq 0$. If this is the case, we shall say that X is *separative*. It is easy to see that $X^<$ separates points of X if and only if the set of all monotone elements from $X^<$ (denoted by $X_+^<$) separates the points of X. The ordering on $X^<$ is the following: $x \leqq y$ iff $y^< - x^< \in X_+^<$.

We say that the partially ordered linear space X is *upward filtering* iff to any x and y in X there exists z in X with $x \leqq z$ and $y \leqq z$.

1. Lemma. (Proposition 3.3.2 [7]) *x is upward filtering iff X_+ spans X (this means that every element of X can be written as a difference of two elements of X_+).*

Let now X be a separative upward filtering linear space. Let x be in X; then we define a map ξ_x: $X^< \to R$ as follows

$$\xi_x(x^<) = x^<(x).$$

Clearly ξ_x is a linear functional on $X^<$ which is a difference of two linear monotone functionals on $X^<$. (Let $x = y - z$ with $y, z \geqq \Theta$; then $\xi_x = \xi_y - \xi_z$, where ξ_y and ξ_z are monotone linear functionals on $X^<$.)

Let now $x_n^< \nearrow x^<$. Let $x \in X_+$; then $x_n^<(x) \nearrow x^<(x)$ and so $\xi_x(x_n) \nearrow \xi_x(x)$ Since every element from X is a difference of two non-negative elements we get that ξ_x or order continuous for every x in X and so that ξ_x is in $X^{<<}$ (the dual of $X^<$). The map $x \mapsto \xi_x$ embeds the space X into its second dual. We note that in this case the space $X^<$ is separative.

Now, what is the type of completeness we have mentioned above? We say that a partially ordered linear space is *monotone σ-complete* iff every monotone increasing bounded sequence $\{x_n\}$ has a limit in X, i.e. the $\lim_n x_n = \bigvee_n x_n$ exists in X.

In his book [1] G. Choquet said the following:

"It is a general philosophical principle among mathematicians that in order to obtain interesting or deep theorems on topological vector spaces, one should assume that the space is locally convex." (pp 333)

By reading these lines the author wondered whether a separative linear space has not to be a locally convex Hausdorff topological space with respect to some topology. Clearly it is so, with respect to the $X^<$ topology (the coarsest topology on X in which every $x^< \in X^<$ is continuous). This follows from the fact that if Y is a linear space and Γ is a family of linear functionals on X, which separates points of X, then with the Γ topology X becomes a locally convex Hausdorff topological space.

We say that a sequence $\{x_n\}$ of elements of a partially ordered linear space X *converges in order to* x (in symbol $x_n \to x$ or $x_n \overset{o}{\to} x$) iff there exist sequences $\{u_n\}$ and $\{v_n\}$ in X with $u_n \leqq x_n \leqq v_n$ and $u_n \nearrow x \searrow v_n$.

We say that a sequence $\{x_n\}$ *converges weakly* to x (in symbol $x_n \overset{w}{\to} x$) iff it converges in the $X^<$ topology. Clearly the order convergence implies the weak convergence. The consequence of this fact is the following:

2. Theorem. *If X is a separative linear space, then there exists a Hausdorff topology (namely the $X^<$ topology) on X such that X is a locally convex linear topological space and every continuous linear functional on X is order continuous.*

If X is a partially ordered linear space, we may say something about the order convergence in X.

3. Lemma. *Let X be a partially ordered linear space. Let $x_n \nearrow x (x_n \searrow x)$ and $y_n \searrow y (y_n \searrow y)$; then*

(i) $x_n + y_n \nearrow x + y \ (x_n + y_n \searrow x + y)$

(ii) $c \cdot x_n \nearrow c \cdot x, \ c \cdot x_n \searrow c \cdot x \ (c \cdot x_n \searrow c \cdot x, \ c \cdot x_n \nearrow c \cdot x)$

according as c is non-negative resp. negative real.

An easy consequence of the last lemma and the definition of the order convergence is the following:

4.Theorem. *Let X be a partially ordered linear space. Then X is a convergence group with respect to the order convergence (i.e. the map $(x, y) \mapsto x - y$ is order dontinous). If X is a monotone σ-complete, separative, upward filtering space, then X is a linear convergence space with respect to the order convergence (i.e. the map $(c, x) \mapsto c \cdot x$ is also order continuous).*

Since we shall deal with spaces of functions on a set with a partially ordered range space X, the following notes about the convergence in such spaces seem to be useful.

Let f, g be functions on Ω with values in X. We say that $f \leqq g$ if $f(\omega) \leqq g(\omega)$ for all ω in Ω. If $\mathcal{F} \subset X^\Omega$, then $f_n \nearrow f$ (in $\mathcal{F}$) means that $f_n \leqq f_{n+1}$ for $n = 1, 2, ..$ and $f \in \mathcal{F}$ is a least upper bound of the family of functions $\{f_n\}$ in $\mathcal{F}$.

If $f_n(\omega) \nearrow f(\omega)$ in X for every ω in Ω, then we say that f_n *converges pointwise to* f on Ω, and we shall write $f_n \overset{p}{\nearrow} f$.

It is clear that $f_n \overset{p}{\nearrow} f$ does not imply $f_n \nearrow f$ in general. (Let $f_n \in C$ $(\langle 0, 1 \rangle)$,

$f_n(x) = 1 - x^n$. Then $f_n \nearrow f \equiv 1$ but $0 = f_n(1) \nearrow f(1) = 1$, and so f_n does not converge to f pointwise. It is also obvious that $f_n \overset{p}{\nearrow} f$ does not imply generally $f_n \nearrow f$ in $\mathscr{F}$, since f need not be even the element of $\mathscr{F}$. However, we are able to prove the following:

5. Theorem. *Let X be a monotone σ-complete poset; Let $\mathscr{F} \subset X^{\Omega}$ be closed with respect to the convergence ${}^p\nearrow$, then the convergence $\nearrow$ in $\mathscr{F}$ is equivalent to the convergence $\overset{p}{\nearrow}$.*

Proof. Let $f_n \nearrow f$ in $\mathscr{F}$; then $f_n(\omega) \leqq f_{n+1}(\omega) \leqq f(\omega)$ for every ω in Ω. X is monotone σ-complete, hence there exists an element $g(\omega)$ in X with $f_n(\omega) \nearrow g(\omega) \leqq f(\omega)$. By the definition of g we have $g \leqq f$ and $f_n \leqq g$. Since f is a least upper bound of the functions $\{f_n\}$, we have $f \leqq g$ and so $f = g$. The other implication being similar, the theorem is proved.

2. Examples

We give examples to present objects we are interested in.

6. Example. Let $\mathscr{F}$ be a linear space of real valued functions on an abstract space E. The ordering on $\mathscr{F}$ is pointwise, i.e. $f \leqq g$ means $f(x) \leqq g(x)$ for all x in E. Let $f_n \nearrow f$ in $\mathscr{F}$ imply $f_n \overset{p}{\nearrow} f$. Let $\xi_x(f) = f(x)$. Obviously the map ξ_x is a linear, monotone, order continuous functional on $\mathscr{F}$. Hence $\mathscr{F}_+^<$ separates points of $\mathscr{F}$.

7. Example. Let P_m be the set of all polynomials of the form

$$a_0 + a_1 x + \ldots + a_{m-1} x^{m-1} + a_m x^m$$

(m — fixed) with the pointwise ordering. Let

$$f_n(x) = a_{0,n} + a_{1,n} x + \ldots + a_{m-1,n} x^{m-1} + a_{m,n} x^m$$

and let $f_n \leqq f_{n+1} \leqq f_0$. Then $f_n(x) \leqq f_{n+1}(x) \leqq f_0(x)$ for every real x. Put

$$f(x) = \lim_n f_n(x).$$

Then f is in P_m (See [8] pp 153), and so P_m is monotone σ-complete.

By the above argumentation it is also clear that $f_n \nearrow f$ in P_m implies $f_n \overset{p}{\nearrow} f$ and so (see Example 6) P_m is separative.

If m is even, then P_m is upward filtering; if m is odd, P_m is not upward filtering.

8. Example. Let $\mathscr{F}$ be the set of all Baire functions on $\langle 0, 1 \rangle$ with the pointwise ordering. It is easy to see that $\mathscr{F}$ is a separative, monotone σ-complete linear space. It is a known fact that $\mathscr{F}$ with the order convergence (this is the same

as the pointwise convergence on $\mathscr{F}$) is a linear convergence space. The sequential closure of the set need not be closed (via Baire classification), hence this convergence is not a topological one.

9. Example. Let $\mathfrak{M}$ be the set of all finite signed measures (finitely additive at empty set vanishing set function) defined on all subsets of a space F. The ordering on $\mathfrak{M}$ is pointwise, i.e. $\mu \leqq \nu$ means $\mu(A) \leqq \nu(A)$ for all $A \subset F$. $\mathfrak{M}$ is clearly a separative, monotone σ-complete linear space. $\mathfrak{M}$ is not a lattice.

For the terminology and the proofs in the following examples see [3], [4] and [5].

10. Example. If Y is a locally convex topological space with a lattice ordering given by a closed cone such that every linear continuous functional on Y is order continuous, then Y is separative.

11. Example. If Z is a vector lattice, regularly ordered by a cone such that every order bounded linear functional is order continuous, then Z is separative.

3. The weak integral

The realization that the totality of linear continuous functionals on a Banach space transfers the convergence and related properties from the original space to the space of real numbers has inspired Gelfand with new ideas. Pettis has generalized his suggestions and discusses the properties of the integral and its relation to other integral definitions. We will follow now their ideas. The main difference is in the notion of a dual space. In our next considerations X will be a separative linear space.

Let $(\Omega, \mathscr{S}, \mu)$ be a measure space. A function $f\colon \Omega \to X$ is called *weakly measurable* iff a real function $x^<(f)$ is measurable for every $x^<$ in $X^<$. f is said to be a *simple integrable function* iff it has a form

$$f = \sum_{i=1}^{n} x_i \chi_{A_i}, \quad \text{where if } x_i \neq \Theta, \text{ then } A_i \in \mathscr{S} \text{ and } \mu(A_i) < \infty .$$

Let f be a non-negative weakly measurable function. If $x^<(f)$ is integrable for all $x^<$ in $X^<$, then this expression defines a linear monotone transformation on the space $X^<$ to the space of the Lebesgue integrable real functions $\mathscr{L}_1(\Omega, \mathscr{S}, \mu, R)$. The map $x^< \mapsto \int_E x^<(f)\, \mathrm{d}\mu$ is a linear monotone, order continuous functional on $X^<$ for every E in $\mathscr{S}$, and so it is an element of $X^{<<}$, denoted by $x^{<<}_{E,f}$, hence

$$x^{<<}_{E,f}(x^<) = \int_E x^<(f)\, \mathrm{d}\mu .$$

It is obvious that $x^{<<}_{E,f}$ need not be an element of X. It seems natural that we should like our integral values in the space X with which we started. Thus we define the integral in the following way:

A weakly measurable function f on Ω is weakly integrable iff for every measurable set E in $\mathcal{S}$ there exists an element f_E of X such that

$$x^{<}(f_E)=\int_E x^{<}(f)\,\mathrm{d}\mu$$

for every $x^{<}$ in $X^{<}$. Then we define

$$f_E=w-\int_E f\,\mathrm{d}\mu\,.$$

f_E is the weak integral of f on the set E.

4. The properties of the weak integral

1°. *The integral is well defined.*

This is a consequence of the fact that X is separative.

2°. *A weakly measurable function f is weakly integrable iff for any $E\in\mathcal{S}$ there exists an $f_E\in X$ with*

$$x^{<}(f_E)=\int_E x^{<}(f)\,\mathrm{d}\mu$$

for every $x^{<}$ in $X^{<}_{+}$.

3°. *If f is a simple integrable function with $f=\sum_{i=1}^{n}x_i\chi_{A_i}$, then*

$$f_E=w-\int_E f\,\mathrm{d}\mu=\sum_{i=1}^{n}x_i\mu(A_i\cap E)\,.$$

Proof. Let $x^{<}\in X^{<}$; then

$$x^{<}\Big(\sum_{i=1}^{n}x_i\mu(A_i\cap E)\Big)=\sum_{i=1}^{n}x^{<}(x_i)\mu(A_i\cap E)=$$

$$=\sum_{i=1}^{n}\int_E x^{<}(x_i)\chi_{A_i}\,\mathrm{d}\mu=\int_E\sum_{i=1}^{n}x^{<}(x_i)\chi_{A_i}\,\mathrm{d}\mu=\int_E x^{<}(f)\,\mathrm{d}\mu\,.$$

The assertion is now a consequence of the fact that X is separative.

4°. *The integral on a set E is a linear operator.*

If x, y are in X, $x\overset{w}{\leqq}y$ means $x^{<}(x)\leqq x^{<}(y)$ for every $x^{<}$ in $X^{<}_{+}$. $f\overset{w}{\leqq}g$ means $f(\omega)\overset{w}{\leqq}g(\omega)$ for every ω in Ω.

5°. *The integral on a set E is a weakly monotone operator, i.e. if $f\overset{w}{\leqq}g$, then $f_E\overset{w}{\leqq}g_E$.*

6°. *Let f_n, f, g and h be weakly integrable functions. Let $g \overset{w}{\leqq} f_n \overset{w}{\leqq} h$ and let $f_n \overset{w}{\to} f$; then*

$$w-\int_E f_n \,\mathrm{d}\mu \overset{w}{\to} w-\int_E f \,\mathrm{d}\mu .$$

Proof. Let $x^< \in X_+^<$. By assumption

$$x^<(g) \overset{w}{\leqq} x^<(f_n) \overset{w}{\leqq} x^<(h), \quad x^<(f_n) \to x^<(f)$$

and the real functions $x^<(f_n)$, $x^<(f)$, $x^<(g)$ and $x^<(h)$ are Lebesgue integrable. So by the Lebesgue dominated convergence theorem

$$\int_E x^<(f_n)\,\mathrm{d}\mu \to \int_E x^<(f)\,\mathrm{d}\mu .$$

By 2° we get $f_{n,E} \overset{w}{\to} f_E$.

7°. *Let f_n and f be weakly integrable functions and let $f_n \nearrow f (f_n \searrow f)$; then*

$$w-\int_E f_n \,\mathrm{d}\mu \overset{w}{\to} w-\int_E f \,\mathrm{d}\mu .$$

Proof. Since $f_n \nearrow f$ implies $f_n \to f$ and $f_1 \overset{w}{\leqq} f_n \overset{w}{\leqq} f$ by 6°, we have

$$f_{n,E} \overset{w}{\to} f_E .$$

Since we shall be interested in the construction of expected values, we give another result of a weak integration.

We say that a set function v: $\Omega \to X$ is *weakly σ-additive* iff the real set function $x^<(v): \Omega \to R$ is σ-additive for every $x^<$ in $X^<$. We say that v is *absolute continuous* with respect to μ (in symbol $v \ll \mu$) iff $v(E) = \Theta$ for every E in $\mathscr{S}$ for which $\mu(E) = \Theta$.

8°. *If f is a weakly integrable function, then v: $E \mapsto f_E$ is a weakly σ-additive set function and is absolute continuous with respect to μ.*

9°. *If Y is a separative linear space and T: $X \to Y$ is a linear, order continuous transformation, then if f: $\Omega \to X$ is a weakly integrable function, Tf is also weakly integrable and*

$$w-\int_E Tf\,\mathrm{d}\mu = T\left(w-\int_E f\,\mathrm{d}\mu\right).$$

15. Lemma. *If $m \leqq n$ then*

$$\mathscr{I}_n \mathscr{I}_m = \mathscr{I}_m .$$

16. Lemma. *$\mathscr{I}_n$ is a continuous operator on $\mathscr{L}_n$ with resprect to the convergence $\overset{p}{\nearrow}$ and $\overset{p}{\searrow}$.*

Proof. Let $f_k \in \mathscr{L}_0$ with $f_k \overset{p}{\searrow} \Theta \quad \chi_\Omega$. Take $x^< \in X_+^<$; then $x^<(f_k) \searrow 0$. Denote by z the limit of the decreasing bounded sequence $\mathscr{I}_0 f_k$. Then

$$x^<(z) \quad \lim_k x^<(\mathscr{I}_0 f_k) = \lim_k \int x^<(f_k)\, d\mu = 0$$

for all $x^<$ in X_+, and so $z = \Theta$. The proof of the continuity of $\mathscr{I}_n$ with respect to the convergence $\overset{p}{\nearrow}$ is similar.

17. Lemma. *Let $f_k \overset{p}{\nearrow} f \leqq g$, f_k, $g \in \mathscr{L}_n$ and $f \in \mathscr{L}_{n+1}$; then*

$$\mathscr{I}_n g \leqq \lim_k \mathscr{I}_n f_k .$$

Proof. Let $n = 0$. Let us assume first that X be a lattice. Since $f_k \wedge g \overset{p}{\nearrow} f \wedge g = g$ (this is true in any vector lattice), by the continuity of $\mathscr{I}_0$ on $\mathscr{L}_0$ we get

$$\mathscr{I}_0 g = \lim_k \mathscr{I}_0(f_k \wedge g) \leqq \lim_k \mathscr{I}_0 f_k .$$

Let now $X_+^<$ determine the order in X. Let $x^< \in X_+^<$; then

$$x^<(\mathscr{I}_0 g) = \int x \quad (g)\, d\mu \leqq \lim_k \int x^<(f_k)\, d\mu \leqq$$

$$\leqq \lim_k x^<(\int f_k)\, d\mu) = x^<(\lim_k \mathscr{I}_0 f_k) .$$

Since $X_+^<$ determines the order in X; we get

$$\mathscr{I}_0 g \leqq \lim_k \mathscr{I}_0 f_k .$$

The proof for $n \geqq 1$ is similar.

18. Lemma. (*i*) *$\mathscr{I}_n$ is a monotone operator.*

(*ii*) *If f, $g \in \mathscr{L}_n$, then $f + g \in \mathscr{L}_n$ and*

$$\mathscr{I}_n(f+g) = \mathscr{I}_n f + \mathscr{I}_n g .$$

(*iii*) *If $f \in \mathscr{L}_n$ and c is a real number, then $c \cdot f \in \mathscr{L}_{n+1}$ and*

$$\mathscr{I}_{n+1}(c \cdot f) = c \quad \mathscr{I}_n f .$$

Proof. (*i*) follows by Lemma 17. (*ii*) Let $n=1$. Let f_k and g_k be in $\mathscr{L}_0$ such that $f_k \overset{p}{\nearrow} f$, $g_k \overset{p}{\nearrow} g$, $\{\mathscr{I}_0 f_k\}$ and $\{\mathscr{I}_0 g_k\}$ are bounded. By Lemma 3 $f_k + g_k \overset{p}{\nearrow} f+g$. $\mathscr{I}_0(f_k+g_k)$ is bounded, hence $f+g$ is in $\mathscr{L}_1$. Take now $x^< \in X_+^<$; then by Lemma 14 (*ii*)

$$x^<(\mathscr{I}_1(f+g)) = \int x^< \, (f+g) \, d\mu = \int x^<(f) \, d\mu + \int x^<(g) \, d\mu =$$
$$= x^<(\mathscr{I}_1 f) + x^<(\mathscr{I}_1 g) = x^<(\mathscr{I}_1 f + \mathscr{I}_1 g)$$

and so

$$\mathscr{I}_1(f+g) = \mathscr{I}_1 f + \mathscr{I}_1 g \, .$$

The proof for a general n and the proof of (*iii*) are similar.

We can formulate the results of this section as follows:

19. Theorem *$\mathscr{L}$ is a partially ordered linear space and $\mathscr{I}: \mathscr{L} \to X$ is a linear, monotone and continuous operator on $\mathscr{L}$ with respect to the convergence $\overset{p}{\nearrow}(\overset{p}{\searrow})$.*

20. Theorem. *Let $\{f_k\}$ be an increasing sequence of functions from $\mathscr{L}$. Let $\{\mathscr{I} f_k\}$ be bounded and let there exist a function $g \in X^\Omega$ with $f_k \leqq g$. Then there exists a function f in $\mathscr{L}$ such that $f_k \overset{p}{\nearrow} f$ and $\mathscr{I} f_k \nearrow \mathscr{I} f$.*

Proof. $f_k(\omega) \leqq g(\omega)$ for every ω and $f_k(\omega)$ is an increasing sequence. Denote by $f(\omega)$ its limit. This has to exists, since X is monotone σ-complete. Then $f_k \overset{p}{\nearrow} f$. Take an odd ordinal n such that $f_k \in \mathscr{L}_{n-1}$ for $k = 1, 2, \ldots$ Then since the sequence $\{\mathscr{I} f_k\} = \{\mathscr{I}_{n-1} f_k\}$ is bounded, $f \in \mathscr{L}_n \subset \mathscr{L}$ by definition of $\mathscr{L}_n$ and again by the definition of $\mathscr{I}_n$ and $\mathscr{I}$

$$\mathscr{I} f = \mathscr{I}_n f = \lim_k \mathscr{I}_{n-1} f_k = \lim_k \mathscr{I} f_k \, .$$

We define for E in $\mathscr{S}$ and f in $\mathscr{L}$

$$\int_E f \, d\mu = \int \chi_E \cdot f \, d\mu \, .$$

(It is easy to see that $\chi_E \cdot f$ is in $\mathscr{L}$ if $f \in \mathscr{L}$.)

21. Proposition. *Let f be in $\mathscr{L}$; then for every E in $\mathscr{S}$*

$$x^<\left(\int_E f \, d\mu\right) = \int_E x^<(f) \, d\mu$$

and so f is weakly integrable and

$$w-\int_E f\,\mathrm{d}\mu=\int_E f\,\mathrm{d}\mu .$$

Let f, g be in λ. We say that f is equivalent to $g(f\sim g)$ iff $\int_E f\,\mathrm{d}\mu=\int_E g\,\mathrm{d}\mu$ for all E in $\mathscr{S}$.

22. Theorem. *(i)* $\mathscr{L}=\mathscr{L}(\Omega,\mathscr{S},\mu,X)$ *is a monotone* σ*-complete linear space.*

(ii) If $f\not\sim g$, *then there exists a linear order continous monotone functional* ξ *on* $\mathscr{L}$ *such that*

$$\xi(f)\neq\xi(g).$$

(iii) $\mathscr{I}$ *is a linear, monotone, order contidnuous operator on* $\mathscr{L}$.

Proof. *(i)* Let $f_k, g\in\mathscr{L}$ with $f_k\leqq f_{k+1}\leqq g$. Then $f_k(\omega)\leqq f_{k+1}(\omega)\leqq g(\omega)$. Denote $f(\omega)=\lim_k f_k(\omega)$. Then $f_k\overset{p}{\nearrow}f$. By Theorem 20 $f\in\mathscr{L}$. It is obvious that $f_k\nearrow f$ in $\mathscr{L}$.

(ii) Let $f\not\sim g$; then there exists a set $E\in\mathscr{S}$ with

$$y=\int_E f\,\mathrm{d}\mu\neq\int_E g\,\mathrm{d}\mu=z .$$

Let $x^<$ be in $X_+^<$ with $x^<(y-z)\neq 0$. Then the functional ξ defined as

$$\xi(h)=x^<\Big(\int_E h\,\mathrm{d}\mu\Big)$$

has the desirable property.

(iii) Let $f_k\nearrow f$ in $\mathscr{L}$. Then by Theorem 5 $f_k\overset{p}{\nearrow}f$ and so by Theorem 20 $\mathscr{I}f_k\nearrow\mathscr{I}f$.

We define the linear space L in the usual way by putting $L=\mathscr{L}/\sim$. As a consequence of this definition and the last theorem we get:

23. Theorem. $L=L(\Omega,\mathscr{S},\mu,X)$ *is a monotone* σ*-complete, separative linear space and* $\mathscr{I}$ *is a linear, monotone, order continuous operator on* L.

The following example shows that the family $\mathscr{L}$ of integrable functions may be very "small" in some pathological cases.

24. Example. Let X be an infinite dimensional partially ordered linear space with a discrete order. Then $\mathscr{L}=\mathscr{L}_0$.

REFERENCES

[1] CHOQUET, G.: Lectures on Analysis, VI, W. A. Benjamin, INC. 1969.

[2] FREMLIN, D. H.: A direct proof of the Mathes-Wright integral extension theorem. J. London M. S., 11, 1975, 276—284.

[3] LUXEMBURG, W. A., ZAANEN, A. C.: Riesz spaces 1. Amsterdam 1971.
[4] JAMESON, G.: Ordered Linear Spaces, Berlin 1970.
[5] POTOCKÝ, R.: On random variables having values in a vector lattice, Math. Slovaca 27, 1977, 267—276.
[6] RIEČAN, B.: O prodolženii operatorov s značeniami v linejnich poluuporiadočennych prostranstvách, Čas. Pěst. Mat., 93, 1968, 459—471.
[7] SEMADENI, Z.: Banach Spaces of Continuous Functions, Warszawa.
[8] SIKORSKI, R.: Funkcje rzeczywiste, Warszawa, 1958.
[9] VOLAUF, P.: On extension of maps with values in ordered space, Math. Slov. 30, 1980, 351—361.
[10] VONKOMEROVÁ, M.: Extension of operators, Math. Slov. 30, 1980, 351—361.
[11] WRIGHT, J. D. M.: The measure extension problem for vector lattices, Ann. Inst. Fourier (Grenoble) 21 Fasc. 4 (1971) 65—85.

Received May 17, 1978

Katedra matematiky a deskriptívnej geometrie
Stavebnej fakulty SVŠT
Radlinského 11
884 20 Bratislava

ИНТЕГРИРОВАНИЕ В ЧАСТИЧНО УПОРЯДОЧЕННОМ ПРОСТРАНСТВЕ

Ян Шипош

Резюме

В статье излагается теория слабого и сильного интеграла для функции со значениями в некотором частично упорядоченном линейном пространстве.