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ON EXTENSION PROPERTIES FOR OBSERVABLES

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Results of Mączyński [2] for the case of observables of a separable logic are generalized. Theorems giving necessary and sufficient conditions in order that some families of observables may admit a common extension are similar to the corresponding theorems on extension to homomorphisms of Sikorski for Boolean algebras.

Mączyński [2] studied some properties of spectral σ-measures of a separable Hilbert space. Due to the one-to-one correspondence between spectral σ-measures and self-adjoint operators on H, we may interpret assertion about spectral σ-measures as assertion about self-adjoint operators. This principle is used in the theory of logics — algebraic axiomatic models of quantum mechanics.

Let L be a poset with the first and the last elements 0 and 1, respectively, and an orthocomplementation $\perp: a \mapsto a^\perp$, such that (i) $(a^\perp)^\perp = a$ for all $a \in L$; (ii) if $a < b$, then $b^\perp < a^\perp$; (iii) $a \vee a^\perp = 1$ for all $a \in L$. Two elements a, $b \in L$ are orthogonal, and we write $a \perp b$, if $a < b^\perp$. We further assume that (i) if $a < b$, then there is $c \in L$, $c \perp a$ such that $b = a \vee c$; (ii) if $\{a_i\} \subset L$ is a sequence of mutually orthogonal elements, then $\bigvee_i a_i$ exists in L. A poset L satisfying the above axioms will be called a logic [6].

A logic L is separable if every subset of mutually orthogonal elements of L contains at most countably many non-zero elements.

Let $\mathcal{B}$ be a Boolean σ-algebra. We say that a map x from $\mathcal{B}$ into L is

(a) $\mathcal{B}$-observable if

(i) $x(1) = 1$;

(ii) $x(E) \perp x(F)$ if $E \cap F = 0, E, F \in \mathcal{B}$; (1)

(iii) $x(E \cup F) = x(E) \vee x(F)$ if $E \cap F = 0$;

(b) $\mathcal{B}$ - σ-observable if (i), (ii) of (1) are satisfied and (iii) is in the σ-form, i.e.

(iii)′ $x\left(\bigcup_{i=1}^{\infty}\right) E_i = \bigvee_{i=1}^{\infty} x(E_i),\ E_i \cap E_j = 0,\ i \neq j.$

(Abbrevation observable, σ-observable, respectively, if $\mathcal{B}$ is specified.)

Let $\mathcal{B} = \mathcal{B}(X)$ be the Borel σ-algebra of a topological space X satisfying the second countability axiom and x a σ-observable. We denote by $\sigma(x)$ the smallest closed set $E \subset X$ such that $x(E)=1$. x has a purely pointwise spectrum if $\sigma(x)=\{\lambda_1, \lambda_2, \ldots\}$.

Very important case we have if $\mathscr{B} = \mathscr{B}(R_1)$ is the Borel σ-algebra of the real line R_1 and $L = L(H)$ is the logic that consists of all closed subspaces of a separable Hilbert space H. Then, by Mączyński [2], σ-observables (observables) are said to be σ-spectral measures (spectral measures).

Let $\mathscr{B}_0 \subset \mathscr{B}$ be a Boolean sub-σ-algebra. We say that a map x: $\mathscr{B}_0 \to L$ is (i) partial $\mathscr{B}$-observable if (1) is satisfied for $E, F \in \mathscr{B}_0$; (ii) partial $\mathscr{B}$-σ-observable if (1) and (iii)′ are satisfied for E, F, $E_i \in \mathscr{B}_0$.

Let now $\mathscr{B}_t$, $t \in T$, be a family of Boolean sub-σ-algebras of $\mathscr{B}$. Then every $\mathscr{B}$-σ-observable ($\mathscr{B}$-observable) x induces a family of partial $\mathscr{B}$-σ-observables (partial $\mathscr{B}$-observables) $x_t = x|\mathscr{B}_t$, $t \in T$. Then we say that x is a common extension of x_t, $t \in T$. We shall study the following problem: given a family of partial $\mathscr{B}$-σ-observables x_t, $t \in T$, what are necessary and sufficient conditions for this family to admit a common extension to a σ-observable x.

Two elements $a, b \in L$ are said to be compatible, and we write $a \leftrightarrow b$ if there are three mutually orthogonal elements a_1, b_1, $c \in L$ such that $a = a_1 \vee c$, $b = b_1 \vee c$. If $A \subset L$ and $a \leftrightarrow b$ for every $a, b \in A$, then A is said to be compatible. If, moreover, for any $a, b \in A$, $a \leftrightarrow b$ the elements a_1, b_1, c belong to the minimal sublogic $L_0(A)$ of L generated by A, then A is said to be strongly compatible. Guz [1] and Neubrunn [3] showed that the strong compatibility of $A \subset L$ is a necessary and sufficient condition in order that $L_0(A)$ should be a Boolean sub-σ-algebra of L.

Let $\mathscr{G}_t$, $t \in T$, be a family of subsets of $\mathscr{B}$. We shall say that a family of maps f_t: $\mathscr{G}_t \to L$, $t \in T$, is strongly compatible if the union of the ranges of all f_t is strongly compatible. It is easy to see that if x is a $\mathscr{B}$-observable, then x is strongly compatible. The strong compatibility of a family of partial $\mathscr{B}$-σ-observables is a necessary condition in order that it may admit a common extension.

A family of strongly compatible maps f_t: $\mathscr{G}_t \to L$, $t \in T$, is atomic if the Boolean sub-σ-algebra generated by the union of the ranges of all f_t is atomic. By the Guz—Neubrunn theorem this sub-σ-algebra exists. If, for example, $\mathscr{B}$ is the Borel σ-algebra of a topological space X satisfying the second countability axiom, then any σ-observable with a purely pointwise spectrum is atomic.

For $a \in L$ we denote $(+1)a \equiv a$, $(-1)a \equiv a^{\perp}$, analogously for the elements of $\mathscr{B}$.

In the following we shall use two notions of Sikorski [5]:

We say that a Boolean σ-algebra $\mathscr{A}'$ has the strong σ-extension property if, for every Boolean σ-algebra $\mathscr{A}$, every map f (from a set $\mathscr{G}$ σ-generating $\mathscr{A}$, into $\mathscr{A}'$) satisfying the following

$$\text{if} \quad \bigcap_{i=1}^{\infty} \varepsilon(i) E_i = 0, \quad \text{then} \quad \bigcap_{i=1}^{\infty} \varepsilon(i) f(E_i) = 0, \tag{2}$$

for every sequence $\{E_i\}_{i=1}^{\infty} \subset \mathscr{G}$, and for every function $\varepsilon(i) = \pm 1$, can be extended to a σ-homomorphism h from $\mathscr{A}$ into $\mathscr{A}'$.

We say that a Boolean σ-algebra $\mathscr{A}'$ has the weak σ-extension property if, for every Boolean σ-algebra $\mathscr{A}$ and for every subalgebra $\mathscr{A}_0$ σ-generating $\mathscr{A}$, every homomorphism f from $\mathscr{A}_0$ into $\mathscr{A}$ satisfying the following:

$$\text{if} \quad \bigcap_{i=1}^{\infty} E_i = 0, \quad \text{then} \quad \bigcap_{i=1}^{\infty} f(E_i) = 0, \tag{3}$$

for every sequence $\{E_i\}_{i=1}^{\infty}$ of elements of $\mathscr{A}_0$, can be extended to a σ-homomorphism h from $\mathscr{A}$ into $\mathscr{A}'$.

The strong σ-extension property always implies the weak σ-extension property.

Theorem 1. *Let L be a separable logic and $\mathscr{G}$ a subset σ-generating $\mathscr{B}$. Then every strongly compatible atomic map f: $\mathscr{G} \to L$ satisfying* (2) *has an extension to a σ-observable.*

Proof. The Guz—Neubrunn theorem implies that there is a Boolean sub-σ-algebra $\mathscr{A}$ of L generated by the range of f. By assumption, $\mathscr{A}$ is atomic. Due to the separability of L, $\mathscr{A}$ is a complete Boolean algebra [7]. Since $\mathscr{A}$ is atomic and complete, $\mathscr{A}$ is isomorphic to the field of all subsets of the set of all atoms [5]. Hence, by theorem 34.1 of Sikorski [5], $\mathscr{A}$ has the strong σ-extension property. This means that our map f can be extended to a $\mathscr{B}$-σ-observable.

Q.E.D.

Theorem 2. *Let L be a separable logic and let x_t, $t \in T$, be an atomic strongly compatible family of partial $\mathscr{B}$-σ-observables. Suppose that the union of the domains of x_t σ-generates $\mathscr{B}$. Then there exists a $\mathscr{B}$-σ-observable x which is a common extension of all x_t, $t \in T$ iff for every countable subset $T' \subset T$ and every indexed system E_t, $t \in T'$, such that $E_t \in D(x_t)$ ($D(x_t)$ is the domain of x_t), there hold*

$$\bigcap_{t \in T'} E_t = 0 \quad \textit{implies} \quad \bigwedge_{t \in T'} x_t(E_t) = 0. \tag{4}$$

Proof. Analogously as in the proof of Theorem 1, we infer that union of all ranges of x_t σ-generates a complete atomic Boolean algebra $\mathscr{A}$ which is isomorphic to a complete field of sets and therefore it has the strong σ-extension property. By application of [5, Theorem 34.3], we conclude the proof of theorem. Q.E.D.

Remark. The next theorem (especially condition (5)) is known in the theory of the Hilbert space. But it is proved by means of the methods of functional analysis [4, Theorem 5.5]. Mączyński [2] showed that this fact can be proved by algebraic reasoning in the logic $L(H)$.

Theorem 3. *Let L be a separable logic and x an atomic partial $\mathscr{B}$-observable such that the domain $D(x)$ of x σ-generates $\mathscr{B}$. Then each of the following conditions is* *necessary and sufficient in order that x should have an extension to $\mathscr{B}$-σ-observable:*

$$x\left(\bigcup_{i=1}^{\infty} E_i\right)=\bigvee_{i=1}^{\infty} x(E_i); \tag{5}$$

$$\bigcup_{i=1}^{\infty} E_i=1 \quad \textit{implies} \quad \bigvee_{i=1}^{\infty} x(E_i)=1; \tag{6}$$

both conditions (5) *and* (6) *being satisfied for any sequence* $\{E_i\}_{i=1}^{\infty}$ *of disjoint elements from* $D(x)$, *for which* $\bigcup_{i=1}^{\infty} E_i \in D(x)$;

$$\bigcup_{i=1}^{\infty} E_i=1 \quad \textit{implies} \quad \bigvee_{i=1}^{\infty} x(E_i)=1, \tag{7}$$

for any $\{E_i\}_{i=1}^{\infty} \subset D(x)$;

$$\bigcap_{i=1}^{\infty} E_i=0 \quad \textit{implies} \quad \bigwedge_{i=1}^{\infty} x(E_i)=0. \tag{8}$$

Proof. In the first place we show that the conditions (5)—(8) are equivalent. The equivalence of (5) and (6) is evident. Now let (6) hold. By the Guz—Neubrunn theorem we have that $\bigvee_{i=1}^{\infty} x(E_i)$ exists in L. If $\bigcup_{i=1}^{\infty} E_i=1$, $E_i \in D(x)$, we define $F_1=E_1$, $F_i=E_i-\bigcup_{k=1}^{i-1} E_k$, $i \geqslant 2$. Then $F_i<E_i$ and $\{F_i\}_{i=1}^{\infty}$ is a disjoint covering of 1 in $D(x)$. Hence, by (6), we have $\bigvee_{i=1}^{\infty} x(F_i)=1$ and consequently $\bigvee_{i=1}^{\infty} x(E_i)=1$; therefore (7) holds. (7) implies (6) evidently. Now, by de Morgan's laws we have that (7) and (8) are equivalent.

Analogously as in the proof of Theorem 1 we have that the range of x generates an atomic complete Boolean algebra $\mathscr{A} \subset L$, which is isomorphic to a complete field of sets. Therefore $\mathscr{A}$ has the weak σ-extension property (3) and consequently x can be extended to a $\mathscr{B}$-σ-observable.

Q.E.D.

Theorem 4. *Let L be a separable logic and x an atomic partial $\mathscr{B}$-observable. Then x can be extended to a $\mathscr{B}$-observable.*

Proof. We show easily that x is a homomorphism from the Boolean subalgebra $D(x)$ into a complete atomic Boolean subalgebra $\mathscr{A} \subset L$. Hence, by the Sikorski theorem on the extension of homomorphisms [5, Theorem 33.1], x can be extended to a $\mathscr{B}$-observable.

Q.E.D.

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О СВОЙСТВАХ РАСШИРЕНИЯ ДЛЯ НАБЛЮДАЕМЫХ

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Резюме

Результаты Манчиньского обобщены для случая наблюдаемых сепарабельной логики. Теоремы дадут необходимые и достаточные условия для того, чтобы некоторые семейства наблюдаемых допускали совместное расширение. Эти теоремы аналогичны теоремам Сикорского о расширении до гомоморфизмов для булевых алгебр.