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František Šik

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TOPOLOGY ON REGULATORS OF LATTICE ORDERED GROUPS I. TOPOLOGY INDUCED BY AN l-GROUP

FRANTIŠEK ŠIK

Let an l-group G be a subdirect product of linearly ordered groups $\{G_x : x \in \mathfrak{R}\}$. Then G is a set of functions $f: \mathfrak{R} \to \bigcup\{G_x : x \in \mathfrak{R}\}$. Jakubík [8] defined a topology on the set $\mathfrak{R}$ in such a manner that, on this more general level, he simulated the definition of the weak topology well known from the theory of real valued functions. We shall recall it briefly. Every set of real valued functions C on a given set X defines a topology on X such that every $f \in C$ is a continuous function on the topological space X endowed with this topology. Among these topologies there exists the smallest one, the so-called weak topology induced by the set C on X. If X is a topological space and C the set of all continuous real valued functions on X, then the weak topology induced by C on X has, as a base for closed sets, the system $\{Z(f) : f \in C\}$, where $Z(f) = \{x \in X : f(x) = 0\}$ is the zero set of f [7] 3.5.

In [8] it is shown that the system $\{Z(f) : f \in G\}$, where analogously to the above definition $Z(f) = \{x \in \mathfrak{R} : f(x) = 0\}$, is a base for closed sets of a topology on $\mathfrak{R}$ (topology induced by the l-group G on $\mathfrak{R}$). If we define a mapping $\bigcup : x \in \mathfrak{R} \to \bigcup x$ by the rule $\bigcup x = \{f \in G : f(x) = 0\}$, then the pair $(\mathfrak{R}, \bigcup)$ is the so-called realizer of G (which means that $\bigcup x$ is a prime ideal of G for every $x \in \mathfrak{R}$ and $\bigcap\{\bigcup x : x \in \mathfrak{R}\} = \{0\}$). In the new notation we have $Z(f) = \{x \in \mathfrak{R} : f \in \bigcup x\}$. This topology is used in [12] and [13] and in certain equivalent form also in [5]. We shall show that the main results concerning this topology remain preserved when we omit the requirement of normality for the elements of the set $\{\bigcup x : x \in \mathfrak{R}\}$ in the notion of the realizer, in other words, when $\bigcup$ is a mapping of $\mathfrak{R}$ into the set $\mathcal{P}(G)$ of all prime subgroups of the l-group G with the property $\bigcap\{\bigcup x : x \in \mathfrak{R}\} = \{0\}$. The pair $(\mathfrak{R}, \bigcup)$ shall be called a regulator of the l-group G (cf. sec. 1). We shall investigate the topology on $\mathfrak{R}$ defined as before giving the base for closed sets $\{Z(f) : f \in G\}$, where $Z(f) = \{x \in \mathfrak{R} : f \in \bigcup x\}$. Thus the domain of applicability of the induced topology will be extended from the class of representable l-groups to the class of all l-groups.

If $(\mathfrak{R}, \bigcup)$ is a regulator of G and $\bigcup$ is one-to-one then we may identify the set $\mathfrak{R}$ with the set $\{\bigcup x : x \in \mathfrak{R}\}$, which is a set of prime subgroups of G. Then the topology induced by G on this regulator $\mathfrak{R}$ is the topology inherited on $\mathfrak{R}$ by the hull-kernell topology of the space $\mathscr{P}(G)$ of all prime subgroups of G (see, e.g., [1] and [9]) and $\mathfrak{R}$ is a dense set of the topological space $\mathscr{P}(G)$. This represents another approach to the problem, which will be examined in another paper.

Raising the generalized theory we proceed roughly as in [13]. It will be shown that the generalized theorems often differ from the special ones unessentially. But it is desirable to summarize these generalized results; moreover, a number of present results has no pendant in [13].

A short review of results. Let G and $\mathfrak{R}$ be nonempty sets and $\bigcup: \mathfrak{R} \to \exp G$ a mapping. The mappings Ψ and Z defined by

$$\Psi(A) = \bigcap\{\bigcup x : x \in A\} \ (A \subseteq \mathfrak{R}) \quad \text{and} \quad Z(P) = \{x \in \mathfrak{R} : \bigcup x \supseteq P\} \ (P \subseteq G)$$

are dually isotone mappings between the sets $\exp \mathfrak{R}$ and $\exp G$ ordered by inclusion. In particular if G is an l-group and $\bigcup$ a mapping of $\mathfrak{R}$ into $\mathscr{P}(G)$ the set of all prime subgroups of G, then the pair $(\mathfrak{R}, \bigcup)$ is called a regulator of G if $\bigcap\{\bigcup x : x \in \mathfrak{R}\} = \{0\}$. If the regulator $(\mathfrak{R}, \bigcup)$ is standard (i.e. if $\bigcup x \neq G$ for every $x \in \mathfrak{R}$), we can define a topology on $\mathfrak{R}$ the so-called topology induced by G on $\mathfrak{R}$. A basis of closed sets for this topology is $\mathfrak{F} = \{Z(f) : f \in G\}$. The corresponding topological space is denoted by $(\mathfrak{R}, G)$. Moreover, there are defined topologies on the system of all ultraantifilters $\mathfrak{U}(\Xi)$ of the lattice Ξ, where Ξ means $\Gamma(G)$ the lattice of all polars of G or $\Pi'(G)$ the lattice of all dual principal polars of G. One establishes relationship between these topologies and the topologies induced on $\mathfrak{R}$ by different regulators $(\mathfrak{R}, \bigcup)$ of G. In section 2 it is proved that the mapping Ψ maps dually isomorphically the lattice $\mathfrak{M}(\mathfrak{R}, G)$ of all regular closed sets of the space $(\mathfrak{R}, G)$ onto the lattice of polars $\Gamma(G)$ of G, and the lattice $\mathfrak{N}(\mathfrak{R}, G)$ of closed sets of $(\mathfrak{R}, G)$ onto $\Omega(\mathfrak{R}, G)$ the lattice of all $\mathfrak{R}$-subgroups, $\Omega(\mathfrak{R}, G) = \{\Psi(A) : A \subseteq \mathfrak{R}\}$. The restriction of Z onto the corresponding sets is the inverse mapping of Ψ. The extremal disconnectedness of the space $(\mathfrak{R}, G)$ is necessary and sufficient for the lattice $\mathfrak{M}(\mathfrak{R}, G)$ to be a sublattice of the lattice $\mathfrak{N}(\mathfrak{R}, G)$ (2.23). Not every topological space can be represented as $(\mathfrak{R}, G)$ for a suitable l-group G, e.g. any T_1-space which is not T_2-space (1.3). In contrast to this every Hausdorff completely regular space has such a representation (G is the lattice of all continuous real valued functions on the given space).

1. Relations between topologies induced by an l-group

1.1 Definition. Let G be an l-group, $\mathfrak{R} \neq \emptyset$ and $\bigcup : x \to \bigcup x$ a mapping of $\mathfrak{R}$ into the set $\mathscr{P}(G)$ of all prime subgroups of the l-group G. The pair $(\mathfrak{R}, G)$ is called a *regulator* of the l-group G if $\bigcap\{\bigcup x : x \in \mathfrak{R}\} = \{0\}$. A regulator is said to be *standard* if $\bigcup x \neq G$ for every $x \in \mathfrak{R}$ [13] II, 3. The mapping $\bigcup$ defines a decomposition on $\mathfrak{R}$, $\bar{\mathfrak{R}}$, and a one-to-one mapping $\bar{\bigcup}$ of $\bar{\mathfrak{R}}$ onto $\{\bigcup x : x \in \mathfrak{R}\}$. Evidently, the pair $(\bar{\mathfrak{R}}, \bar{\bigcup})$ is a regulator of G. It is called a *simplification* of

$(\mathfrak{R}, \bigcup)$. If we denote by π the projection of $\mathfrak{R}$ onto $\bar{\mathfrak{R}}$, then the mapping $\bar{\bigcup}$ is defined by the rule $\bar{\bigcup}\bar{x} = \bigcup x$, where $\bar{x} \in \bar{\mathfrak{R}}$ and x is an arbitrary element of $\pi^{-1}(\bar{x})$. A regulator $(\mathfrak{R}, \bigcup)$ for which $\bigcup x \parallel \bigcup y$ holds whenever $x, y \in \mathfrak{R}$, $x \neq y$ is said to be *reduced.* A reduced regulator is evidently standard. A regulator $(\mathfrak{R}, \bigcup)$ is called *completely regular* if there holds: $x \in \mathfrak{R}$, $f \in G$, $f \in \bigcup x \Rightarrow$ there exists $g \in G$ with $f\delta g$ and $g \bar{\in} \bigcup x$ (where $f\delta g$ denotes $|f| \wedge |g| = 0$). A completely regular regulator is standard, too. A regulator $(\mathfrak{R}, \bigcup)$ is called *Hausdorff* if for every $x, y \in \mathfrak{R}$, $x \neq y$, there exist elements $f, g \in G$ such that $f\delta g$ and $f \bar{\in} \bigcup x$, $g \bar{\in} \bigcup y$. The last two concepts are due to P. Ribenboim [10] and were introduced for the concept of a realization as defined below (see 1.8). A regulator $(\mathfrak{R}, \bigcup)$ is called a *realizer* if $\bigcup x (x \in \mathfrak{R})$ is a prime ideal of G. In every l-group $\neq \{0\}$ there exists a (reduced completely regular) regulator while the existence of a realizer characterizes representable l-groups.

Instead of $(\mathfrak{R}, \bigcup)$ we often write $\mathfrak{R}$ only supposing tacitly that the mapping $\bigcup$ is fixed. In [13] the symbol $x(\in \mathfrak{R})$ is identified with the associated subgroup $\bigcup x$ and by a regulator we understand there an indexed system of prime subgroups of G whose intersection is $\{0\}$.

Given $f \in G$ we define

$$Z(f) = \{x \in \mathfrak{R} : f \in \bigcup x\}.$$

1.2 Theorem. *Let* $(\mathfrak{R}, \bigcup)$ *be a standard regulator of an* l*-group* $G(\neq \{0\})$. *Then the set*

$$\mathfrak{F} = \{Z(f) : f \in G\}$$

is a basis of closed sets for a topology on the set $\mathfrak{R}$.

Proof. ([13] I 1.5) By a topology we mean a topology in the sense of Bourbaki. It suffices to prove (1) $\mathfrak{R} \in \mathfrak{F}$; (2) $A \cup B \in \mathfrak{F}$ for every $A, B \in \mathfrak{F}$; (3) $\bigcap \{Z(f) : f \in G\} = \emptyset$. There holds (1) $Z(0) = \mathfrak{R}$; (2) $Z(f) \cup Z(g) = Z(|f| \wedge |g|)$ for every $f, g \in G$ ([13] III 6.3); (3) $x \in \bigcap \{Z(f) : f \in G\} \Rightarrow f \in \bigcup x$ for every $f \in G \Rightarrow G = \bigcup x$, a contradiction.

This topology is called a topology *induced* by the l-group G on the set $\mathfrak{R}$. The corresponding topological space is denoted by $(\mathfrak{R}, G)$. If we take a regulator for a topological space, its topology will always be the induced one.

1.3 Definition. Let G be an l-group and $\Gamma(G)$ the Boolean algebra of all polars of G. By the symbol K' we mean the complement of $K \in \Gamma(G)$ in $\Gamma(G)$. There holds $K' = \{g \in G : f\delta g$ for every $f \in K\}$, where $f\delta g$ denotes $|f| \wedge |g| = 0$. By the symbol $\Pi'(G)$ or $\Pi(G)$ we mean the set $\{f' : f \in G\}$ of all dual principal polars f' of G or the set $\{f'' : f \in G\}$ of all principal polars f'' of G, respectively. Here $f' = \{f\}' = \{g \in G : f\delta g\}$, $f'' = (f')'$. Thus $K' = \bigcap \{f' : f \in K\}$. We call the polars K and K' *complementary* (in $\Gamma(G)$).

1.4 Theorem. *Let* $(\mathfrak{R}, \bigcup)$ *be a standard regulator of an* l*-group* G*. The following conditions are equivalent.*

1. $(\mathfrak{R}, G)$ *is a* T_1*-space (i.e. singletons are closed sets).*
2. $(\mathfrak{R}, G)$ *is a* T_2*-space (i.e. distinct points are separated; a Hausdorff space).*
3. *The regulator* $(\mathfrak{R}, \bigcup)$ *is reduced.*
4. *The regulator* $(\mathfrak{R}, \bigcup)$ *is Hausdorff.*

(Cf. [13] IV 8.1)

Proof. $1 \Rightarrow 3$. Given $x, y \in \mathfrak{R}$, $x \neq y$, there exist by supposition $f, g \in G$ with $x \in (\mathfrak{R} \setminus Z(f)) \cap Z(g)$, $y \in Z(f) \cap (\mathfrak{R} \setminus Z(g))$. It follows that $f \in \bigcup y \setminus \bigcup x$, $g \in \bigcup x \setminus \bigcup y$, thus $\bigcup x$ and $\bigcup y$ are incomparable sets, whence 3.

$3 \Rightarrow 4$. For $x, y \in \mathfrak{R}$, $x \neq y$ there exist $f, g \in G$ such that $f \in \bigcup y \setminus \bigcup x$ and $g \in \bigcup x \setminus \bigcup y$. One can suppose $f \geqslant 0$, $g \geqslant 0$. For the elements $p = f - (f \wedge g)$, $q = g - (f \wedge g)$ there holds $x \bar{\in} Z(p)$, $y \bar{\in} Z(q)$, because $x \in Z(p) \equiv p \in \bigcup x \equiv f - (f \wedge g) \in \bigcup x \equiv f \in \bigcup x + (f \wedge g) = \bigcup x$, a contradiction. Similarly for y. Finally $Z(p) \cup Z(q) = \mathfrak{R}$. In fact, $x \in Z(p \wedge q) \equiv p \wedge q \in \bigcup x \equiv p \in \bigcup x$ or $q \in \bigcup x \equiv x \in Z(p) \cup Z(q)$. Then $Z(p) \cup Z(q) = Z(p \wedge q) = Z(0) = \mathfrak{R}$.

$4 \Rightarrow 2 \Rightarrow 1$ is evident.

1.5 Definitions and known results. An *antifilter* on a $\vee$-semilattice Ξ is a subset $\emptyset \neq x \subseteq \Xi$ with the following properties: 1. x does not contain the greatest element of Ξ (provided it exists); 2. $K \in x$, $L \in \Xi$, $K \geqslant L \Rightarrow L \in x$; 3. $K, L \in x \Rightarrow K \vee L \in x$. A maximal (with respect to the inclusion) antifilter is called an *ultraantifilter*. The set of all ultraantifilters on Ξ is denoted by $\mathfrak{U}(\Xi)$.

Let $\Xi = \Gamma(G)$ or $= \Pi'(G)$ or $= \Pi(G)$, respectively, and $x \in \mathfrak{U}(\Xi)$. Then by $\bigcup x$ we define $\bigcup \{K \colon K \in x\}$. If $\bigcup x \neq G$, we speak of a *standard* ultraantifilter on Ξ, [13] II 4.10. The set of all standard ultraantifilters on $\Gamma(G)$ will be denoted by $\mathfrak{U}_s(\Gamma)$. Every $x \in \mathfrak{U}(\Pi')$ is standard (provided $G \neq \{0\}$), [13] II 4.11; $x \in \mathfrak{U}(\Gamma)$ is standard iff $x \cap \Pi' \neq \emptyset$, [13] II 4.12. Thus non-standard ultraantifilters on $\Gamma(G)$ ($G \neq \{0\}$) exist only in l-groups without any weak unit and these are exactly the ultraantifilters on $\Gamma(G)$ which contain $\Pi(G)$ (since an ultraantifilter on a Boolean algebra Ξ contains either an element $a \in \Xi$ or its complement a', for every $a \in \Xi$). If G has a weak unit, then every ultraantifilter $x \in \mathfrak{U}(\Pi)$ is standard, if not (and $G \neq \{0\}$), then $x = \Pi(G)$ is a unique ultraantifilter on $\Pi(G)$ and $\bigcup x = G$ (for given $x \in \mathfrak{U}(\Pi)$ and $a \in G$ then $a \in \bigcup x \equiv a'' \subseteq \bigcup x \equiv a'' \in x$, [13] II 4.7).

If $\Xi = \Gamma$ or $= \Pi'$ or $= \Pi$, respectively, and $x \in \mathfrak{U}(\Xi)$, then $\bigcup x$ is a prime subgroup of G. If $G \neq \{0\}$, $(\mathfrak{U}_s(\Gamma), \bigcup)$ and $(\mathfrak{U}(\Pi'), \bigcup)$ — briefly written $\mathfrak{R}_\Gamma$ and $\mathfrak{R}_{\Pi'}$, respectively — are standard regulators of G, the latter is reduced and completely regular, [13] II 4.15 and 4.16. $\mathfrak{R}_\Gamma$ or $\mathfrak{R}_{\Pi'}$ is called the Γ*-regulator* or the Π'*-regulator*, respectively.

The pair $(\mathfrak{U}(\Pi), \bigcup)$ is not generally any regulator, because $\bigcap \{\bigcup x \colon x \in \mathfrak{U}(\Pi)\} = \{0\}$ does not hold in general; the equality holds iff the following condition (a) is fulfilled ($\mathfrak{A}(G)$ denotes the set of all elements of the l-group G which are not weak units of G, i.e. $a \in \mathfrak{A}(G) \Leftrightarrow a'' \neq G$):

(a) $0 \neq a \in \mathfrak{A}^+(G) \Rightarrow$ there exists $b \in \mathfrak{A}^+(G)$ such that $a \vee b \bar{\in} \mathfrak{A}(G)$. The condition (a) can be reformulated as follows:

(b) $0 \neq a \in G \Rightarrow$ there exists $b \in G$ with $\{0\} \neq b' \subseteq a''$.

[12] Lemma 3, [13] II 4.10.

If (a) is fulfilled, we can speak of the Π-*regulator* $(\mathfrak{U}(\Pi), \bigcup)$ (briefly denoted by $\mathfrak{R}_\Pi$).

Put $\mathfrak{U} = \mathfrak{U}(\Xi)$, where Ξ is a $\vee$-semilattice or $\mathfrak{U} = \mathfrak{U}_s(\Gamma(G))$ ($G \neq \{0\}$ an l-group). We define

$\Sigma = \{\mathfrak{U}K: K \in \Xi\}$ or $\Sigma' = \{\mathfrak{U}f' : f \in G\}$, respectively,

where $\mathfrak{U}K = \{x \in \mathfrak{U}: K \in x\}$.

Then Σ or Σ' is a basis of closed sets for a topology on the set $\mathfrak{U}(\Xi)$ or $\mathfrak{U}_s(\Gamma)$, respectively, [5] 1.9, 1.10; [6] 1.5; [12] sec. IV. The corresponding topological space is denoted by $(\mathfrak{U}(\Xi), \Sigma)$ or $(\mathfrak{U}_s(\Gamma), \Sigma')$, respectively.

1.6 Lemma. *Put* $\mathfrak{U} = \mathfrak{U}(\Gamma)$ *or* $= \mathfrak{U}(\Pi')$, *respectively, and pick* $x \in \mathfrak{U}$ *and* $f \in G$. *Then*

$$f' \in x \equiv f \bar{\in} \bigcup x.$$

[12] Lemma 1; [13] II 4.6.

1.7 Theorem. *Let* $G \neq \{0\}$ *be an* l*-group. Then*

1. *The topological spaces* $(\mathfrak{U}_s(\Gamma), \Sigma')$ *and* $(\mathfrak{R}_\Gamma, G)$ *are homeomorphic.*
2. *The topological spaces* $(\mathfrak{U}(\Pi'), \Sigma)$ *and* $(\mathfrak{R}_{\Pi'}, G)$ *are homeomorphic.*
3. *The topological space* $(\mathfrak{U}(\Gamma), \Sigma)$ *is Hausdorff and compact.*

Proof. (Cf. [5] 2.1; [13] II 4.18) Put $\mathfrak{U} = \mathfrak{U}_s(\Gamma)$ or $= \mathfrak{U}(\Pi')$, respectively. The assertions 1 and 2 follow from the fact that, according to Lemma 1.6, the bases of closed sets for the compared topological spaces are identical. Indeed, $\mathfrak{U} \setminus \mathfrak{U}f' = \{x \in \mathfrak{U}: f' \bar{\in} x\} = \{x \in \mathfrak{U}: f \in \bigcup x\} = Z(f)$. The assertion 3 is Teorema 3 [12].

1.8 Let G be a representable l-group and $(\mathfrak{R}, \bigcup)$ a standard realizer of G. Then $G/\bigcup x = G_x$ $(x \in \mathfrak{R})$ is a linearly ordered group. Let φ be the corresponding representation of G and $\varphi G = \bar{G} = (G_x: x \in \mathfrak{R})$ the canonical *realization* of G, which means that the φ-image of an element $f \in G$, $\varphi f = \bar{f} \in \bar{G} = (G_x: x \in \mathfrak{R})$ is a function $\mathfrak{R} \to \bigcup\{G_x: x \in \mathfrak{R}\}$ defined by the formula $\bar{f}(x) = f + \bigcup x \in G/\bigcup x$ $(= G_x)$ for every $x \in \mathfrak{R}$.

The topology *induced* by the realization $\bar{G} = (G_x: x \in \mathfrak{R})$ on $\mathfrak{R}$ is the topology, of which the basis for closed sets is the system $\{Z(\bar{f}): \bar{f} \in \bar{G}\}$, where $Z(\bar{f}) = \{x \in \mathfrak{R}: \bar{f}(x) = 0\}$.

(c) The topology induced by the l-group G on $\mathfrak{R}$ is thus identical with the topology induced by its realization $\bar{G} = (G_x: x \in \mathfrak{R})$ on $\mathfrak{R}$, since $\bar{f}(x) = 0 \equiv f \in \bigcup x$ holds.

By the Π'-*realization* of a representable l-group $G \neq \{0\}$ we mean the canonical realization corresponding to the Π'-regulator $\mathfrak{R}_{\Pi'}$. This is a realizer, and is called the Π'-*realizer*.

1.9 Theorem. *Let $G \neq \{0\}$ be an l-group. The following are equivalent.*

1. *The topological space $(\mathfrak{U}(\Pi'), \Sigma)$ is compact.*
2. *The topological space $(\mathfrak{U}(\Pi), \Sigma)$ is compact and G fulfils the condition* 1.5(a).
3. *The lattice $\Pi'(G)$ is Boolean.*
4. *The lattice $\Pi(G)$ is Boolean.*
5. $\Pi'(G) = \Pi(G)$.
6. *The topological space $(\mathfrak{R}_{\Pi'}, G)$ is compact.*

If the l-group G is representable, then the following condition 7 *is also equivalent to the preceding ones.*

7. *The topological space induced by the Π'-realization $(G_x\colon x \in \mathfrak{R}_{\Pi'})$ on $\mathfrak{R}_{\Pi'}$ is compact.*

Proof. Theorem 6 [12] states that condition 1 together with the claim $\{0\} \in \Pi'(G)$ and conditions 2 to 5 are equivalent. Conditions 1 and 6 are equivalent by 1.7, conditions 6 and 7 by 1.8(c). Thus there remains to verify that from the compactness of the space $(\mathfrak{U}(\Pi'), \Sigma)$, there follows $\{0\} \in \Pi'(G)$. A proof is given in Satz C [14], p. 108. For the sake of completeness we give a slightly modified proof. Since $\bigcup\{\mathfrak{U}f' : f \in G\} = \mathfrak{U}(\Pi')$, the set $\{\mathfrak{U}f' : f \in G\}$ is an open covering of the space $\mathfrak{U}(\Pi')$. Thus there exist $f_1, \ldots, f_n \in G$ such that $\mathfrak{U}(\Pi') = \bigcup\{\mathfrak{U}f'_i : i = 1, 2, \ldots, n\}$. It follows that $\mathfrak{U}(\Pi') = \bigcup_{i=1}^{n}\{x \in \mathfrak{U} : f'_i \in x\}$. Since x is a prime antifilter, $\bigwedge_{i=1}^{n} f'_i \in x \equiv f'_i \in x$ for some i holds, and so $\mathfrak{U}(\Pi') = \{x \in \mathfrak{U} : \bigwedge_{i=1}^{n} f'_i \in x\}$. Put $f = \bigvee_{i=1}^{n} |f_i|$. Then $\mathfrak{U}(\Pi') = \{x \in \mathfrak{U} : f' \in x\}$, whence $f' = 0$, *i.e.* $\{0\} \in \Pi'(G)$.

2. The mappings Z and Ψ.

2.1 Definition. Let G and $\mathfrak{R}$ be non-empty sets and $\bigcup : \mathfrak{R} \to \exp G$ a mapping. We define a binary relation (a polarity) $\varrho \subseteq G \times \mathfrak{R}$ as follows: $f\varrho x \equiv f \in \bigcup x$.

For $A \subseteq \mathfrak{R}$ and $P \subseteq G$ we define

$$\Psi(A) = \{f \in G\colon f\varrho x \text{ for every } x \in A\},$$
$$Z(P) = \{x \in \mathfrak{R}\colon f\varrho x \text{ for every } f \in P\}.$$

If it is necessary to express the dependence of Ψ and Z on $\mathfrak{R}$ (with a fixed G), we write $\Psi_{\mathfrak{R}}$ and $Z_{\mathfrak{R}}$, respectively. If $A = \{x\}$ or $P = \{f\}$ is a singleton, we use the notation $\Psi(x)$ or $Z(f)$ instead of $\Psi(\{x\})$ or $Z(\{f\})$, respectively. Ψ and Z are evidently dually isotone mappings between the sets $\exp \mathfrak{R}$ and $\exp G$.

The following lemmas 2.2—2.4 may be easily verified (see [3] IV §5).

2.2 Lemma. *$\Psi(A) = \bigcap\{\bigcup x : x \in A\} = \{f \in G : Z(f) \supseteq A\}$ for every $A \subseteq \mathfrak{R}$, in particular, $\Psi(x) = \bigcup x$ for every $x \in \mathfrak{R}$.*

$Z(P) = \bigcap\{Z(f) : f \in P\} = \{x \in \mathfrak{R} : \bigcup x \supseteq P\}$ for every $P \subseteq G$, in particular $Z(f) = \{x \in \mathfrak{R} : f \in \bigcup x\}$ for every $f \in G$.

2.3 *Lemma.* $A \subseteq B \subseteq \mathfrak{R} \Rightarrow \Psi(A) \supseteq \Psi(B)$; $\Psi(A) = G \equiv A \subseteq \{x \in \mathfrak{R} : \bigcup x = G\}$, $\Psi(\mathfrak{R}) = \bigcap\{\bigcup x : x \in \mathfrak{R}\}$

$P \subseteq Q \subseteq G \Rightarrow Z(P) \supseteq Z(Q)$; $Z(G) = \{x \in \mathfrak{R} : \bigcup x = G\}$; $Z(P) = \mathfrak{R} \equiv P \subseteq \bigcap\{\bigcup x : x \in \mathfrak{R}\}$.

2.4 Lemma. $Z\Psi(A) \supseteq A$, $\Psi Z \Psi(A) = \Psi(A)$ *for every* $A \subseteq \mathfrak{R}$;

$\Psi Z(P) \supseteq P$, $Z \Psi Z(P) = Z(P)$ *for every* $P \subseteq G$.

2.5 Lemma. *For $P \subseteq G$ and $A \subseteq \mathfrak{R}$ there holds*

$$P \subseteq \Psi(A) \equiv A \subseteq Z(P);$$

in particular for $f \in G$ and $x \in \mathfrak{R}$ we have

$$f \in \bigcup x \equiv x \in Z(f).$$

Proof. $P \subseteq \Psi(A) \equiv P \subseteq \{f \in G : Z(f) \supseteq A\} \equiv Z(f) \supseteq A$ for every $f \in P \equiv A \subseteq \bigcap\{Z(f) : f \in P\} \equiv A \subseteq Z(P)$.

From 2.3 and 2.4 there follows

2.6 Lemma. *The mapping $Z\Psi$: $\exp \mathfrak{R} \to \exp \mathfrak{R}$ is a closure operation in $\mathfrak{R}$. The mapping ΨZ: $\exp G \to \exp G$ is a closure operation in G.*

We call the $Z\Psi$-images of sets of $\mathfrak{R}$ *closed* under the relation ϱ, the ΨZ-images of sets of G *closed* under the relation ϱ.

Let $G \neq \{0\}$ be an l-group, $(\mathfrak{R}, \bigcup)$ a standard regulator of G, Z and Ψ the mappings corresponding to the polarity $\varrho \subseteq G \times \mathfrak{R}$, where the mapping $\bigcup$ from the definition 2.1 is realized by the mapping $\bigcup$, the second member of the symbol $(\mathfrak{R}, \bigcup)$. By 2.5 it is clear that the definition 1.1 and 2.1 of $Z(f)$ coincide. These assumptions hold for the remainder of section 2.

2.7 Lemma. *$\Psi(A)$ is a solid subgroup of G for every $A \subseteq \mathfrak{R}$.*

It follows directly from 2.2.

We shall consider connections between the closure operation $A \to Z\Psi(A)$ in $\mathfrak{R}$ (see 2.6) and the closure operation $A \to \bar{A}$ in the topological space $(\mathfrak{R}, G)$.

2.8 Lemma. $\bar{A} = Z\Psi(A)$ *for every* $A \subseteq \mathfrak{R}$.

Proof. By 2.2 we have $\bar{A} = \bigcap\{Z(f) : f \in G, Z(f) \supseteq A\} = \bigcap\{Z(f) : f \in \Psi(A)\} = Z\Psi(A)$.

Let $\mathfrak{N}(\mathfrak{R}, G)$ (briefly $\mathfrak{N}_{\mathfrak{R}}$ or $\mathfrak{N}$) be the system of all closed sets of the topological space $(\mathfrak{R}, G)$.

2.9 Lemma. $\mathfrak{R}(\mathfrak{M}, G) = \{Z(P): P \subseteq G\} = \{Z\Psi(A): A \subseteq \mathfrak{M}\} = \{\bigcap_{f \in P} Z(f): P \subseteq G\} = \{A \subseteq \mathfrak{M}: Z\Psi(A) = A\}$.

Proof. From 2.2, 2.8 and 1.2 it follows that $\mathfrak{R}(\mathfrak{M}, G) = \{\bar{A}: A \subseteq \mathfrak{M}\} = \{Z\Psi(A): A \subseteq \mathfrak{M}\} \subseteq \{Z(P): P \subseteq G\} = \{\bigcap_{f \in P} Z(f): P \subseteq G\} \subseteq \mathfrak{R}(\mathfrak{M}, G)$.

2.10 Definition. We denote by $\Omega(\mathfrak{M}, G)$ (briefly $\Omega_{\mathfrak{M}}$ or Ω) the system of all subsets of G closed under ϱ, i.e. such sets $P \subseteq G$ which fulfil $P = \Psi Z(P)$.

The elements of the set $\Omega(\mathfrak{M}, G)$ are called $\mathfrak{M}$-*subgroups* of G. The $\mathfrak{M}$-subgroups are solid subgroups of G as it follows from the following lemma.

2.11 Lemma. $\Omega(\mathfrak{M}, G) = \{\Psi Z(P): P \subseteq G\} = \{\Psi(A): A \subseteq \mathfrak{M}\} = \{\bigcap_{x \in A} \bigcup x: A \subseteq \mathfrak{M}\}$.

Proof. Evidently $\{\Psi Z(P): P \subseteq G\} \supseteq \Omega$. The converse inclusion follows from $\Psi Z(P) = \Psi Z \Psi Z(P)$ (see 2.4). Thus $\Omega(\mathfrak{M}, G) = \{\Psi Z(P): P \subseteq G\} \supseteq \{\Psi Z \Psi(A): A \subseteq \mathfrak{M}\} = \{\Psi(A): A \subseteq \mathfrak{M}\}$ (by 2.4).

The last set is on the one hand $\supseteq \Omega(\mathfrak{M}, G)$ and on the other hand $= \left\{\bigcap_{x \in A} \bigcup x: A \subseteq \mathfrak{M}\right\}$ by 2.2.

Let the restriction of Ψ on $\mathfrak{R}_{\mathfrak{M}}$ be denoted by Ψ again. Similarly for Z on $\Omega_{\mathfrak{M}}$.

2.12 Theorem. *The mappings Ψ and Z are (mutually inverse) dual isomorphisms between the sets $\Omega(\mathfrak{M}, G)$ and $\mathfrak{R}(\mathfrak{M}, G)$, ordered by inclusion.*

Proof. By 2.9 and 2.11, $\Psi|_{\mathfrak{R}}$ is a mapping onto Ω and with respect to 2.4 and 2.9, $Z|_{\Omega}$ is a mapping onto $\mathfrak{R}$. By 2.4, 2.9 and 2.11, $Z|_{\Omega}\, \Psi|_{\mathfrak{R}} = \mathrm{id}_{\mathfrak{R}}$, $\Psi|_{\mathfrak{R}}\, Z|_{\Omega} = \mathrm{id}_{\Omega}$. The dual isotony of the mappings Ψ and Z follows from 2.3.

2.13 Corollary. *$\Omega(\mathfrak{M}, G)$ (ordered by inclusion) is a complete distributive lattice; there holds*

$\bigwedge_{\alpha} P_{\alpha} = \bigcap_{\alpha} P_{\alpha}$ *for an arbitrary family* $\{P_{\alpha}\} \subseteq \Omega(\mathfrak{M}, G)$.

Proof. The first assertion follows from 2.12. The second assertion:

1. By 2.2, $Z\left(\bigcap_{\alpha} P_{\alpha}\right) = \bigcap\left\{Z(f): f \in \bigcap_{\alpha} P_{\alpha}\right\} \supseteq \bigcap_{\alpha} \bigcap\{Z(f): f \in P_{\alpha}\} = \bigcap_{\alpha} Z(P_{\alpha})$, so by 2.4, $\bigcap_{\alpha} P_{\alpha} \subseteq \Psi Z\left(\bigcap_{\alpha} P_{\alpha}\right) \subseteq \Psi\left[\bigcap_{\alpha} Z(P_{\alpha})\right]$.

2. $f \in \bigcap_{\alpha} P_{\alpha}$ implies $Z(f) \supseteq \bigcap_{\alpha} Z(P_{\alpha})$, thus by 2.4, $\Psi\left[\bigcap_{\alpha} Z(P_{\alpha})\right] \supseteq \bigcap_{\alpha} P_{\alpha}$.

It follows that $\bigcap_{\alpha} P_{\alpha} = \Psi\left[\bigcap_{\alpha} Z(P_{\alpha})\right] \in \Omega(\mathfrak{M}, G)$ and so $\bigcap_{\alpha} P_{\alpha} = \bigwedge_{\alpha} P_{\alpha}$ in $\Omega(\mathfrak{M}, G)$.

2.14 Lemma. $\Psi(A) = \Psi(\bar{A})$ *for every* $A \subseteq \mathfrak{M}$.

Proof. By 2.8 $\bar{A} = Z\Psi(A)$ and thus by 2.4 $\Psi(\bar{A}) = \Psi Z \Psi(A) = \Psi(A)$.

Consider $P, Q \subseteq G$. Recall that $P\delta Q$ means $f\delta g$ for every $f \in P$ and every $g \in Q$.

2.15 Lemma. *For every $P, Q \subseteq G$, $P \neq \emptyset \neq Q$ there holds*

$$P\delta Q \equiv Z(P) \cup Z(Q) = \mathfrak{R}.$$

Proof. There holds $Z(f) \cup Z(g) = Z(|f| \wedge |g|)$ (for evidently $Z(|f| \wedge |g|) \supseteq Z(|f|) \cup Z(|g|) = Z(f) \cup Z(g)$ and conversely by [13] III 6.3 or [4] 1.7, there holds that $x \in Z(|f| \wedge |g|)$ implies $x \in Z(f) \cup Z(g)$, thus by 2.3, $f\delta g \equiv Z(f) \cup Z(g) = \mathfrak{R}$. Now, by definition of $P\delta Q$ there holds

$$P\delta Q \equiv f\delta g (f \in P, g \in Q) \equiv Z(f) \cup Z(g) = \mathfrak{R} (f \in P, g \in Q) \equiv$$
$$\equiv \bigcap \{Z(f): f \in P\} \cup \bigcap \{Z(g): g \in Q\} = \mathfrak{R} \equiv Z(P) \cup Z(Q) = \mathfrak{R}.$$

2.16 Lemma. $[\Psi(A)]' = \Psi(\mathfrak{R} \setminus \bar{A}) = \Psi(\overline{\mathfrak{R} \setminus \bar{A}})$ *for every* $A \subseteq \mathfrak{R}$.

Proof. $\Psi(\mathfrak{R} \setminus \bar{A}) \supseteq [\Psi(A)]'$ because $g\delta\Psi(\bar{A}) \Rightarrow \mathfrak{R} = Z(g) \cup Z\Psi(\bar{A}) = Z(g) \cup \bar{A}$ (2.15 and 2.8) $\Rightarrow Z(g) \supseteq \mathfrak{R} \setminus \bar{A} \Rightarrow g \in \Psi(\mathfrak{R} \setminus \bar{A})$ (2.3 and 2.4). Conversely, for $g \in \Psi(\mathfrak{R} \setminus \bar{A})$ and $f \in \Psi(\bar{A})$ there holds

$$Z(g) \cup Z(f) \supseteq Z\Psi(\mathfrak{R} \setminus \bar{A}) \cup Z\Psi(\bar{A}) = \overline{\mathfrak{R} \setminus \bar{A}} \cup \bar{A} = \mathfrak{R},$$

which means that $g\delta f$ (2.15). Hence $[\Psi(\bar{A})]' \supseteq \Psi(\mathfrak{R} \setminus \bar{A})$.

2.17 Definition. Denote by $\mathfrak{M}(\mathfrak{R}, G)$ (briefly $\mathfrak{M}_{\mathfrak{R}}$ or $\mathfrak{M}$) the system of all regular closed sets of the topological space $(\mathfrak{R}, G)$. Similarly as before we write Z and Ψ instead of $Z|_{\Gamma}$ and $\Psi|_{\mathfrak{M}}$, respectively.

2.18 Theorem. *Z and Ψ are (mutually inverse) dual isomorphisms between the sets $\Gamma(G)$ and $\mathfrak{M}(\mathfrak{R}, G)$, ordered by inclusion.*

Proof. We shall prove $\Psi(\mathfrak{M}) \subseteq \Gamma$: $A \in \mathfrak{M} \Rightarrow [\Psi(A)]' = \Psi(\mathfrak{R} \setminus A)$ (2.16) $\Rightarrow [\Psi(A)]'' = [\Psi(\mathfrak{R} \setminus A)]' = \Psi(\overline{\mathfrak{R} \setminus \overline{\mathfrak{R} \setminus A}}) = \Psi(A) \Rightarrow \Psi(A) \in \Gamma$. Next we show $\Gamma \subseteq \Omega$. For $K \in \Gamma$ there holds $[\Psi Z(K)]' \supseteq K'$, because $g \in K' \Rightarrow Z(g) \supseteq \mathfrak{R} \setminus Z(K)$ (2.15) $\Rightarrow g \in \Psi Z(g) \subseteq \Psi(\mathfrak{R} \setminus Z(K))$ (2.3 and 2.4). The last member is equal to $[\Psi Z(K)]'$ (2.16 and 2.9). The converse inclusion $[\Psi Z(K)]' \subseteq K'$ follows from $\Psi Z(K) \supseteq K$ (2.4) and from the fact that the mapping $A \to A'$ ($A \in \exp G$) is dually isotone. Therefore $K' = [\Psi Z(K)]' = \Psi(\mathfrak{R} \setminus Z(K)) \in \Omega$ (2.11). Thus $\Gamma \subseteq \Omega$. Furthermore, from $K' = \Psi(\mathfrak{R} \setminus Z(K))$ it follows that $\Psi Z(K) = K = K'' = [\Psi(\mathfrak{R} \setminus Z(K))]' = \Psi(\overline{\mathfrak{R} \setminus \overline{\mathfrak{R} \setminus Z(K)}})$ (2.16 and 2.9). Since $\Psi|_{\mathfrak{R}}$ is one-to-one (2.12), we have $Z(K) = \overline{\mathfrak{R} \setminus \overline{\mathfrak{R} \setminus Z(K)}}$, i.e. $Z(K) \in \mathfrak{M}$. Finally $Z(\Gamma) \subseteq \mathfrak{M}$.

Now, as $Z\Psi$ is the identity mapping on $\mathfrak{N}$ (2.12) and $\mathfrak{M} \subseteq \mathfrak{N}$, summarizing the above results we obtain $\mathfrak{M} = Z\Psi(\mathfrak{M}) \subseteq Z(\Gamma) \subseteq \mathfrak{M}$, thus $Z(\Gamma) = \mathfrak{M}$. Since ΨZ is the identity on Ω and $\Gamma \subseteq \Omega$, $\Psi(\mathfrak{M}) = \Psi Z(\Gamma) = \Gamma$, completing the proof.

2.19 Corollary. (Cf. [2]) *The set $\mathfrak{M}(\mathfrak{R}, G)$ ordered by inclusion is a complete Boolean algebra. For $\{A_\alpha\} \subseteq \mathfrak{M}(\mathfrak{R}, G)$ we have*

1. $\bigwedge_{\mathfrak{M}} A_\alpha = \overline{\operatorname{Int} \overline{\bigcap_\alpha A_\alpha}}$, $\bigvee_{\mathfrak{M}} A_\alpha = \overline{\bigcup_\alpha \operatorname{Int} A_\alpha} = \overline{\bigcup_\alpha A_\alpha}$;

2. $A, A_1, A_2 \in \mathfrak{M}(\mathfrak{R}, G) \Rightarrow A_1 \vee_{\mathfrak{M}} A_2 = A_1 \cup A_2$;

the Boolean complement of A is $A' = \overline{\mathfrak{R} \backslash A}$.

Proof. Since Γ is a complete Boolean algebra, so is $\mathfrak{M}$ (2.18).

1. Meet: a) $A_\alpha = \overline{\operatorname{Int} A_\alpha} \supseteq \overline{\operatorname{Int} \bigcap_{\beta \in I} A_\beta} \in \mathfrak{M}$ $(\alpha \in I)$;

b) $A_\alpha \supseteq A \in \mathfrak{M}$ for all $\alpha \in I \Rightarrow \overline{\operatorname{Int} \bigcap_{\beta \in I} A_\beta} \supseteq \overline{\operatorname{Int} A} = A$. Join: a) $A_\alpha \subseteq \overline{\bigcup_{\beta \in I} \operatorname{Int} A_\beta}$ $\subseteq \overline{\bigcup_{\beta \in I} A_\beta}$ $(\alpha \in I)$; b) $A_\alpha \subseteq A \in \mathfrak{M}$ for all $\alpha \in I \Rightarrow \overline{\bigcup_{\beta \in I} A_\beta} \subseteq \bar{A} = A$. Complement: By 2.16 there holds $[\Psi(A)]' = \Psi(\mathfrak{R} \backslash A)$ for every $A \in \mathfrak{M}$, hence by 2.18 A' $= Z\Psi(\overline{\mathfrak{R} \backslash A}) = \overline{\mathfrak{R} \backslash A}$. The remainder of 2 follows evidently from 1.

2.20 Lemma. $\bigvee_{\mathfrak{R}} A_\alpha = \bigvee_{\mathfrak{M}} A_\alpha$, $\bigwedge_{\mathfrak{R}} A_\alpha \supseteq \bigwedge_{\mathfrak{M}} A_\alpha (\{A_\alpha\} \subseteq \mathfrak{M})$.

The first assertion follows from 2.19, the second is evident.

2.21 Lemma. *If A is an open set,* $A, B \in \mathfrak{M}$, *then* $A \wedge_{\mathfrak{M}} B = A \wedge_{\mathfrak{R}} B (= A \cap B)$.

Proof. All the following equalities except the third are evident.

$$A \wedge_{\mathfrak{R}} B = A \cap B = \overline{A \cap \overline{\operatorname{Int} B}} = \overline{A \cap \operatorname{Int} B} =$$
$$= \overline{\operatorname{Int} A \cap \operatorname{Int} B} = \overline{\operatorname{Int} (A \cap B)} = A \wedge_{\mathfrak{M}} B\,.$$

Proof of the third equality $\overline{A \cap \overline{\operatorname{Int} B}} = \overline{A \cap \operatorname{Int} B}$. If $x \in \mathfrak{R}$ is contained in the left-hand side, an arbitrary neighbourhood U of x meets $A \cap \overline{\operatorname{Int} B}$, say in an element y. There exists a neighbourhood V of y contained in $A \cap U$. The set V meets $\operatorname{Int} B$, hence U meets $A \cap \operatorname{Int} B$, i.e. $x \in \overline{A \cap \operatorname{Int} B}$.

2.22 Theorem. *The following conditions are equivalent.*

1. $A \in \mathfrak{M}(\mathfrak{R}, G) \Rightarrow A \cap A' = \emptyset$ *(A' means the Boolean complement of A in $\mathfrak{M}$).*

2. *The space $(\mathfrak{R}, G)$ is extremally disconnected (i.e. the closures of open sets are open).*

3. *The lattice $\mathfrak{M}(\mathfrak{R}, G)$ is a sublattice of the lattice $\mathfrak{N}(\mathfrak{R}, G)$.*

4. *For every $x \in \mathfrak{R}$ the set $\bigcup x$ contains at most one (and thus exactly one) from every pair of complementary polars (in $\Gamma(G)$).*

Proof. 1 $\Rightarrow$ 2. Every $A \in \mathfrak{M}$ is an open set of the space $(\mathfrak{R}, G)$, because $A \cap A' = \emptyset$ by condition 1 and $A \cup A' = \mathfrak{R}$ by 2.19. Hence 2.

2 $\Rightarrow$ 3. The implication follows immediately from 2.20 (for joins) and 2.21 (for meets).

3 $\Rightarrow$ 4. Suppose $\bigcup x = \Psi(x) \supseteq K \cup K'$ for an element $x \in \mathfrak{R}$ and a polar $K \in \Gamma$.

Then

$$x \in Z\Psi(x) \subseteq Z(K \cup K') = Z(K) \cap Z(K') = Z(K) \wedge_{\mathfrak{N}} [Z(K)]' =$$
$$= Z(K) \wedge_{\mathfrak{M}} [Z(K)]' = \emptyset,$$

a contradiction.

4 $\Rightarrow$ 1. Suppose $A \cap A' \neq \emptyset$ for an element $A \in \mathfrak{M}$. There exists $K \in \Gamma$ such that $A = Z(K)$ and thus $A' = [Z(K)]' = Z(K')$. For every $x \in A \cap A' = Z(K) \cap Z(K')$ there holds

$$\Psi(x) \supseteq \Psi(Z(K) \cap Z(K')) \supseteq \Psi Z(K) \cup \Psi Z(K') = K \cup K'$$

and $\Psi(x)$ does not fulfil condition 4.

2.23 Definition. A solid subgroup P of an l-group G is said to be a *z-subgroup* is $f \in P$ implies $f'' \subseteq P$ ([2] 3.3.8).

A regulator formed by z-subgroups is called a *z-regulator*. The set of all minimal prime subgroups of G is a *z-regulator* ($\bigcup$ being the identical mapping), [2] 3.4; see also [13] II 2.3, III 7.6.

2.24 Note. If $(\mathfrak{R}, \bigcup)$ is a standard z-regulator fulfilling the conditions of Theorem 2.22, then $\bigcup x$ is a minimal prime subgroup of G for every $x \in \mathfrak{R}$.

Indeed, if $f \in \bigcup x$, then $f'' \subseteq \bigcup x$ and by 2.22 we have $f' \not\subseteq \bigcup x$. Then the assertion follows from [13] III 7.6 (see also [2] 3.4.13).

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Katedra algebry a geometrie
Přírodovědecké fakulty UJEP
Janáčkovo nám. 2a
662 95 Brno

ТОПОЛОГИИ НА РЕГУЛЯТОРАХ СТРУКТУРНО УПОРЯДОЧЕННЫХ ГРУПП I. ТОПОЛОГИЯ, ИНДУЦИРОВАННАЯ l-ГРУППОИ

Франтишек Шик

Резюме

Пусть G и $\mathfrak{R}$ — непустые множества и $\bigcup : \mathfrak{R} \to \exp G$ отображение. Отображения Ψ и Z, определенные формулой $\Psi(A) = \bigcap\{\bigcup x : x \in A\}$ для $A \subseteq \mathfrak{R}$ и $Z(P) = \{x \in \mathfrak{R} : \bigcup x \supseteq P\}$ для $P \subseteq G$, являются дуально изотонными отображениями множества $\exp \mathfrak{R}$ и $\exp G$, упорядоченных по теоретико-множественному включению. Специально, если G – l-группа и $\bigcup$ — отображение $\mathfrak{R}$ в множество $\mathscr{P}(G)$ всех простых подгрупп в G, то пара $(\mathfrak{R}, \bigcup)$ называется регулятором в G, когда $\bigcap\{\bigcup x : x \in \mathfrak{R}\} = \{0\}$. Если регулятор $(\mathfrak{R}, \bigcup)$ является стандартным (ето значит, что $\bigcup x \neq G$ для всех $x \in \mathfrak{R}$), мы определим топологию на $\mathfrak{R}$, называемую топологией, индуцированной l-группой G на $\mathfrak{R}$. Базой замкнутых множеств для этой топологии является множество $\mathfrak{F} = \{Z(f) : f \in G\}$; соответствующее пространство обозначается $(\mathfrak{R}, G)$. Далее определены топологии на системе всех ультраантифильтров $\mathfrak{U}(\Xi)$ структуры Ξ, где $\Xi = \Gamma(G)$ (структура всех поляр в G) или $= \Pi'(G)$ (структура всех дуальных главных поляр в G) или $= \Pi(G)$ (структура всех главных поляр в G). Установлены отношения между этими топологиями и топологиями, индуцированными на $\mathfrak{R}$, при различных регуляторах данной l-группы. Показано, что топологическое пространство $\mathfrak{U}(\Pi')$ компактно тогда и только тогда, когда $\Pi'(G)$ — Булева алгебра а это эквивалентно тому, что $\Pi'(G) = \Pi(G)$ (1.9).

В абз. 2 доказывается, что отображение Ψ отображает дуально изоморфно структуру $\mathfrak{M}(\mathfrak{R}, G)$ всех регулярных замкнутых множеств пространства $(\mathfrak{R}, G)$ на структуру поляр $\Gamma(G)$ в G и структуру $\mathfrak{N}(\mathfrak{R}, G)$ всех замкнутых множеств пространства $(\mathfrak{R}, G)$ на $\Omega(\mathfrak{R}, G)$ — структуру так называемых $\mathfrak{R}$-подгрупп, $\Omega(\mathfrak{R}, G) = \{\Psi(A) : A \subseteq \mathfrak{R}\}$. Сужение Z на соответствующие множества представляет обратное отображение к сужению Ψ. Экстремальная несвязность пространства $(\mathfrak{R}, G)$ представляет необходимое и достаточное условие для того, чтобы структура $\mathfrak{M}(\mathfrak{R}, G)$ стала подструктурой структуры $\mathfrak{N}(\mathfrak{R}, G)$ (2.22). Не каждое топологическое пространство можно представить как $(\mathfrak{R}, G)$ для подходящей l-группы G и подходящего регулятора $(\mathfrak{R}, \bigcup)$ в G. К пространствам, непредставляемым таким образом, относятся T_1-пространства, которые не являются T_2-пространствами (1.3). С другой стороны, каждое вполне регулярное пространство Хаусдорфа обладает таким представлением (G — структура всех вещественных непрерывных функций на данном пространстве $\mathfrak{R}$ и $\bigcup x = \{f \in G : f(x) = 0\}$ для всех $x \in \mathfrak{R}$).