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# A NOTE ON TWO COMPARABILITY GRAPHS

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The *comparability graph* of the partially ordered set (poset) $P$ is the graph whose vertex set is $P$ and such that $xy$ is an edge if and only if $x$ and $y$ are comparable elements in the poset $P$. Wolk [2] called a graph $G=(V, E)$ a $D$-*graph* if and only if for distinct $x_1, x_2, x_3, x_4 \in V$, $x_1x_2, x_2x_3, x_3x_4 \in E$ imply $x_1x_3 \in E$ or $x_2x_4 \in E$. He showed that a graph is a $D$-graph if and only if it is the comparability graph of a tree poset. Jung [1] generalized this by calling a graph $G=(V, E)$ a $D^*$-*graph* if and only if for distinct $x_1, x_2, x_3, x_4 \in V$, $x_1x_2, x_2x_3, x_3x_4 \in E$ imply $x_1x_3 \in E$ or $x_2x_4 \in E$ or $x_1x_4 \in E$. It was shown that a graph is a $D^*$-graph if and only if it is the comparability graph of a multitree.

In this note we restrict the above definitions as follows (we assume that all graphs and posets are finite and our graphs have no loops or multiple edges): A graph $G=(V, E)$ is a *strong* $D$-*graph (strong* $D^*$-*graph)* if and only if for distinct $x_1, x_2, x_3, x_4 \in V$, $x_1x_2, x_2x_3, x_3x_4 \in E$ imply $x_1x_3 \in E$ and $x_2x_4 \in E$ (imply $x_1x_4 \in E$). Clearly a strong $D$-graph is a strong $D^*$-graph. Before proving our characterizations of these graphs recall that a poset is *fan* if and only if the is a zero and every non-zero element is maximal, and a poset $P$ is a *complete bipartite poset* if and only if there exist disjoint non-empty subsets $X$ and $Y$ with $X \cup Y = P$ and $x<y$ for all $x \in X$, $y \in Y$ with no comparabilities in $X$ or in $Y$.

The *free sum* of the posets $P$ and $Q$ is the set $P \cup Q$ with $x<y$ in the free sum if and only if $x, y \in P$ and $x<y$ in $P$, or $x, y \in Q$ and $x<y$ in $Q$. That is, the Haase diagram of the free sum of $P$ and $Q$ is obtained by placing the Hasse diagrams of $P$ and $Q$ side by side.

**Theorem 1.** *A graph $G=(V, E)$ is a strong D-graph if and only if G is the comparability graph of a free sum of fans and chains.*

Proof. The comparability graph of a fan with $n+1$ elements is $K_{1,n}$ which is a strong $D$-graph, while the comparability graph of a chain is a complete graph, which is also a strong $D$-graph. Hence the comparability graph of a free sum of fans and chains is a strong $D$-graph.

Assume $G=(V, E)$ to be a strong $D$-graph. Since $G$ is a strong $D$-graph if and only if every component of $G$ is a strong $D$-graph, we assume further that $G$ is connected. It then suffices to show that $G$ is $K_{1,n}$ for some $n$ or that $G$ is complete.

From a lemma of Wolk [2, p. 108] there exists $c \in V$ such that $vc \in E$ for all $v \in V$, $v \neq c$. If $G$ is not complete, then there exist $x, y \in V \setminus \{c\}$ such that $xy \notin E$. Suppose there is vertex $z \neq c$ such that $zx \in E$. Then $zxcy$ is a path and the strong $D$-graph condition gives $xy \in E$, a contradiction. If there exist vertices $z$ and $w$ distinct from $x$, $y$ and $c$ such that $zw \in E$, then the path $wzcx$ gives $xz \in E$ and we are back in the first case. Hence $G$ is $K_{1,n}$ for some $n$.

**Theorem 2.** *A graph $G=(V, E)$ is a strong $D^*$-graph if and only if $G$ is the comparability graph of a free sum of chains and complete bipartite posets.*

Proof. The comparability graph of a chain or a complete bipartite poset is easily seen to be a strong $D^*$-graph.

As in the proof of Theorem 1, assume $G$ to be connected but not complete. Then there exist $x, y \in V$ such that $xy \notin E$. Let $A=\{z \in V: zx \in E\}$ and $B=\{w \in V: wx \notin E\}$. $A$ and $B$ are non-empty and we assert that $A$, $B$ is a bipartition of $V$. First let $z \in A$, $w \in B$. Then by connectivity, there is some path from $w$ to $x$. Taking one such shortest path and using the strong $D^*$ condition, either $wz \in E$ or we get a path $wrxz$ which gives $wz \in E$. In a similar vein one can show that there are no edges between vertices in $A$ (if $z_1z_2 \in E$ with $z_1, z_2 \in A$ apply the strong $D^*$ condition to $xz_1z_2y$) or between vertices in $B$ (if $w_1w_2 \in E$ with $w_1, w_2 \in B$ apply the strong $D^*$ condition to $w_1w_2zx$ for some $z \in A$). Thus $G$ is a complete bipartite graph. One can view $G$ as a poset $P$ by taking $z<w$ for all $z \in A$ and $w \in B$. Now $G$ is the comparability graph of the complete bipartite poset $P$.

## REFERENCES


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## ЗАМЕЧАНИЕ О ДВУХ ГРАФАХ СРАВНИМОСТИ

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### Резюме

Граф $G=(V, E)$ называется строгим $D$-графом (строгим $D^*$-графом) если для всяких его четырех различных вершин $x_1$, $x_2$, $x_3$, $x_4 \in V$ из $x_1x_2$, $x_2x_3$, $x_3x_4 \in E$ следует $x_1x_3 \in E$ и $x_2x_4 \in E$ (следует $x_1x_4 \in E$). Доказываются следующие два результата. Граф является строгим $D$-графом тогда и только тогда, когда он является графом сравнимости свободной суммы вееров и цепеи Граф является строгим $D^*$-графом тогда и только тогда, когда он является графом сравнимости свободной суммы цепей и полных двудольных частично упорядоченных множеств